Introduction to vortex dynamics

Streamlines

A streamline is a line which at each instant is locally parallel to the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$.

Letting $\mathrm{d}\boldsymbol{x}$ to denote an infinitesimal section of a streamline,

$$\mathrm{d}\boldsymbol{x} = k\boldsymbol{v}, \tag{1}$$

where k may depend on $\boldsymbol{x}$ and t.

Stream function

If we have an incompressible flow in 2D, then our condition is $\nabla \cdot \boldsymbol{v} = 0$. This implies that there exists a scalar function $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \tag{2}$$

Now note that

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial x}\mathrm{d}x + \frac{\partial \psi}{\partial y}\mathrm{d}y + \frac{\partial \psi}{\partial t}\mathrm{d}t \tag{3}$$

and consider $\mathrm{d}\psi$ as we move along a streamline fixed at some instant in time. Note that time is fixed so $\mathrm{d}t = 0$. Furthermore, along a streamline $\mathrm{d}\boldsymbol{x} = k\boldsymbol{v} = k\left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$ and thus

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial x}\left(k\frac{\partial \psi}{\partial y}\right) + \frac{\partial \psi}{\partial y}\left(-k\frac{\partial \psi}{\partial x}\right) = 0. \tag{4}$$

This is equivalent to saying that ψ is constant along each streamline. Therefore, $\psi(\boldsymbol{x}, t)$ is called the *stream function* of the flow.

Vorticity

The vorticity field $\boldsymbol{\omega}$ of a flow $\boldsymbol{v}(\boldsymbol{x}, t)$ is defined by $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$.

In 2D, we therefore have

$$\boldsymbol{\omega} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ u(x,y) & v(x,y) & 0 \end{vmatrix} = \left(0, 0, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = (0, 0, \omega(\boldsymbol{x}, t)).$$

If $\boldsymbol{\omega} \equiv \mathbf{0}$, then the flow is referred to as *irrotational*.

Example

Consider a flow $(u_r, u_\theta) = (0, \gamma/(2\pi r))$, $\gamma \in \mathbb{R}$.

If we draw the streamlines for this we get concentric circles centered at the origin. As for solid body rotation, this flow is purely in the azimuthal direction.

So globally the fluid rotates about the origin. However

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ u_r & ru_\theta & u_z \end{vmatrix} = \mathbf{0},$$

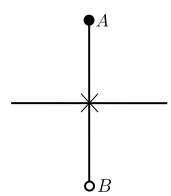
if $r \neq 0$. If $r = 0$ the flow is *singular* and there the vorticity is infinite. Therefore, we deduce that there is no local rotation about non-zero points.

Note that one can check that $\nabla \cdot \boldsymbol{v} = 0 = \dfrac{1}{r}\left(\dfrac{\partial(ru_r)}{\partial r} + \dfrac{\partial u_\theta}{\partial \theta}\right)$ and so the flow is incompressible.

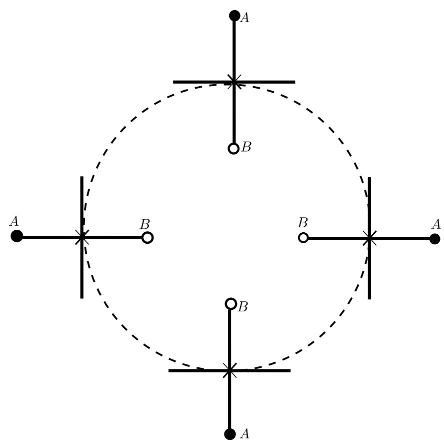
One may examine the difference between this singular flow and solid body rotation as follows:

Difference between singular flow and solid body rotation

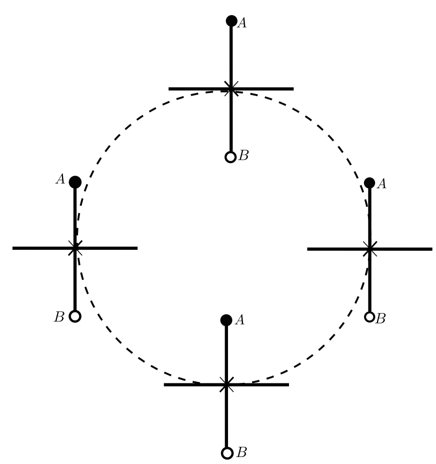
Consider a vorticity meter, as in [Ach91].



For *solid body rotation* the angular velocity is uniform (*i.e.* the same at all points) and $v \propto r$. Considering motion relative to the midpoint $\times$, we observe that there is a local rotation.



For our *singular flow*, $(u_r, u_\theta) = (0, \gamma/(2\pi r))$, however, the angular velocity is <u>not</u> uniform; it decreases as r increases ($v \propto 1/r$). In fact, it varies precisely the right way so that one observes the following:



i.e. there is no local rotation about the midpoint $\times$ (or, in fact, any other point not at the origin), which implies that the vorticity is zero. This singular flow is called a *point vortex flow*.

As a measure of global rotation of a fluid flow, we introduce the following.

Circulation

We let $C(t)$ be a closed contour in the flow domain , each point along which, $\mathrm{d}\boldsymbol{x}$ moves with the local velocity field.

The circulation, $\Gamma(t)$, around $C(t)$ is defined to be

$$\Gamma(t) = \oint_{C(t)} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x}, \tag{5}$$

and we integrate round $C(t)$ with the interior on the left. We can interpret $\Gamma(t)$ as a measure of flow around $C(t)$. Applying *Stokes' theorem*, we obtain

$$\Gamma(t) = \iint_{S(t)} (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{n} \, \mathrm{d}S = \iint_{S(t)} \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{6}$$

where $S(t)$ is any surface spanning $C(t)$ and $\boldsymbol{n}$ represents the unit norm to the surface $S(t)$. We assume that $\boldsymbol{v}$ is non-singular in $S(t)$ to apply Stokes' theorem. Note that $\Gamma(t)$ can also be thought of as the flux of vorticity through $S(t)$.

In 2D, $\boldsymbol{n} = (0, 0, 1)$ and $\boldsymbol{\omega} = (0, 0, \omega)$, therefore we obtain

$$\Gamma(t) = \iint_{S(t)} \omega \, \mathrm{d}S. \tag{7}$$

References

[Ach91] D. J. Acheson. Elementary fluid dynamics, 1991.