Stokes Flow and Free Boundaries

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1 Introduction

In this report we discuss the complex variable theory of two-dimensional Stokes flow as exploited and developed first by Richardson in [Ric92]. Our presentation largely follows his paper [Ric92]. We consider free boundary problems that appear in fluid dynamics, analysing in particular the free viscous incompressible flow of a finite region which is bounded by a simple, smooth, closed curve and that is driven by surface tension. We describe the time evolution of the shape of the fluid domain in terms of a time-dependent mapping of the form $z = w(\zeta, t)$ which is conformal on $|\zeta| \leq 1$.

A particular application of this kind of problem is in a process known as *viscous sintering*, which is employed in several industrial contexts and has been studied extensively by mathematicians. The sintering process involves a Newtonian fluid surrounded by a free surface with the motion driven by a constant surface tension acting on this free surface. We assume that the flow is incompressible, the Reynolds number is low and the gravitational effects are negligible. An example of its practical significance is the manufacture of optical fibres. One can think of taking a bundle of solid fibres (e.g. metal or glass) and heating it up to a sufficiently high temperature until it becomes fluid, and then allowing surface tension to drive a flow that converts it into a single cylinder with circular cross-section.

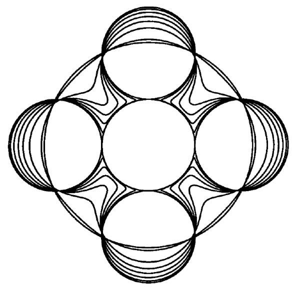


Figure 1.1: Coalescence of 5 equal circular cylinders. Figure from [Ric97].

The fundamental problem involving the coalescence of two circular cylinders of the same radius was first solved by Hopper in [Hop90] who showed that this problem admits exact solutions. In other words, the free boundary problem can be reduced to tracking the evolution of just two real parameters in a conformal map. A solution for the coalescence of two different radii circular cylinders was given by Richardson [Ric92], although this solution was quite complicated, and a simpler solution was presented by the same author in [Ric97].

This report is organized as follows: In § 2 the equations and boundary conditions are rewritten in complex variable form and the general theory developed by Richardson is explained, introducing a conformal mapping representation as in [Ric92]. Finally, in § 3 we repeat the examples that appear in [Ric92]. In particular, we present in detail the evolution of a limaçon and give an overview for the viscous sintering of two circular cylinders for both the case of equal and different radii. Details of certain calculations are given in the appendices.

2 Preliminaries in complex variable theory

We begin by outlining some basic features of the theory as developed in [Ric92].

Consider a fluid that is occupying a bounded and simply-connected domain $D(t)$ at time t in Cartesian coordinates and let the complex variable be $z = x + iy$ and its complex conjugate be denoted by $\bar{z} = x - iy$.

The flow $\boldsymbol{u} = (u(x, y), v(x, y))$ satisfies incompressibility ($\nabla \cdot \boldsymbol{u} = 0$) so we can write it in terms of a stream function $\psi(x, y)$ as

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \tag{2.1}$$

It is convenient to work with the stream function $\psi(x, y)$, which in two-dimensional slow viscous flow satisfies the biharmonic equation

$$\nabla^4 \psi = 0. \tag{2.2}$$

Next we define the stress-stream function $W(z, \overline{z}) = \phi(x, y) + i\psi(x, y)$ as it appears in [Mik14, p.179], where ϕ (stress function) and ψ (stream function) are biharmonic conjugates. According to the Goursat representation for biharmonic functions, $W(z, \overline{z})$ can be written as

$$W(z, \overline{z}) = -[\bar{z}\phi(z) + \chi(z)], \tag{2.3}$$

where $\phi(z)$ and $\chi(z)$ are analytic functions of z in the flow domain D. These functions fully specify the flow. Note that since the boundary evolves with time, both $\phi(z)$ and $\chi(z)$ depend on time, though this dependence is suppressed for purposes of brevity. The stream function therefore satisfies

$$\psi(x, y) = \mathrm{Im}\{W(z, \overline{z})\} = -\mathrm{Im}\left\{\bar{z}\phi(z) + \chi(z)\right\}. \tag{2.4}$$

It is helpful to rewrite (2.4) in the form

$$\psi = -\frac{1}{2i}\left\{\bar{z}\phi(z) + \chi(z) - z\overline{\phi(z)} - \overline{\chi(z)}\right\}, \tag{2.5}$$

using $\mathrm{Im}(z) = y = \frac{1}{2i}(z - \bar{z})$. Now we can also find the velocity vector using the chain rule, $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)$ and $\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$, to obtain

$$u + iv = \frac{\partial \psi}{\partial y} - i\frac{\partial \psi}{\partial x} = \phi(z) - z\overline{\phi'(z)} - \overline{\chi'(z)}. \tag{2.6}$$

Let the boundary of D be denoted by ∂D and suppose that it is an analytic curve. Note that the boundaries that will be considered here initially can have singularities but they are analytic for $t > 0$. Now, we want to introduce the boundary conditions that are involved in this problem.

Apart from the kinematic boundary condition which matches the velocity of the fluid at the free boundary to the velocity of the free boundary itself, there are two stress boundary conditions of the form

$$\sigma_{ij} n_j = -T\kappa n_i, \qquad \text{for } i = 1, 2 \text{ on } \partial D. \tag{2.7}$$

Here σ_{ij} is the Newtonian stress tensor, $\boldsymbol{n} = (n_i)$ is the outward normal to ∂D, T is the surface tension coefficient and κ is the curvature of the free boundary.

The stress components σ_{ij} and the pressure field are given by (see [LD64, p.187])

$$\sigma_{xx} + \sigma_{yy} \equiv -2p = 8\mu \mathrm{Re}\{\phi'(z)\}, \tag{2.8}$$

$$i(\sigma_{yy} - \sigma_{xx}) - 2\sigma_{xy} = 4\mu \left(\overline{z}\phi''(z) + \chi''(z)\right), \tag{2.9}$$

where μ represents the viscosity of the fluid.

The pressure field is thus given by

$$p = -4\mu \mathrm{Re}\{\phi'(z)\}. \tag{2.10}$$

For Stokes flow in the absence of body forces, the divergence of the stress tensor should be zero. This means [Mus13, p.108–115] that a biharmonic stress function U exists and it satisfies

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}. \tag{2.11}$$

Comparing (2.8) with (2.11) implies that the stress function, U, is of the form

$$U = 2\mu \mathrm{Re}\{\overline{z}\phi(z) + \chi(z)\} = \frac{1}{2}\left\{\overline{z}\phi(z) + z\overline{\phi(z)} + \chi(z) + \overline{\chi(z)}\right\}. \tag{2.12}$$

Now it is easily found that

$$\frac{\partial U}{\partial x} = \frac{1}{2}\left\{\overline{z}\phi'(z) + \overline{\phi(z)} + \chi'(z) + \phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)}\right\}, \tag{2.13}$$

$$\frac{\partial U}{\partial y} = \frac{1}{2} i\left\{\overline{z}\phi'(z) + \overline{\phi(z)} + \chi'(z) - \phi(z) - z\overline{\phi'(z)} - \overline{\chi'(z)}\right\}, \tag{2.14}$$

and it is convenient to calculate the following quantity

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)}. \tag{2.15}$$

Consider some arc AB in the fluid, as shown in Figure 2.1, with the positive direction being from A to B and the normal, n, pointing to the left of the arc when looking along it in the positive direction. We derive from (2.12) that the force $(X_n \,\mathrm{d}s, Y_n \,\mathrm{d}s)$ acting on an element $\mathrm{d}s$ of the arc AB exerted on the side of the positive normal, in complex form, is given by

$$(X_n + iY_n)\,\mathrm{d}s = -2i\mu\,\mathrm{d}\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right), \tag{2.16}$$

where X_n and Y_n are defined as

$$X_n = \sigma_{xx}\cos(n,x) + \sigma_{xy}\cos(n,y) = \frac{\partial^2 U}{\partial y^2}\cos(n,x) - \frac{\partial^2 U}{\partial x \partial y}\cos(n,y), \tag{2.17}$$

$$Y_n = \sigma_{yx}\cos(n,x) + \sigma_{yy}\cos(n,y) = -\frac{\partial^2 U}{\partial x \partial y}\cos(n,x) + \frac{\partial^2 U}{\partial x^2}\cos(n,y). \tag{2.18}$$

For more details, see [Mus13, p.114]. Therefore, (2.15) and (2.16) give

$$(X_n + iY_n)\,\mathrm{d}s = -2i\mu\,\mathrm{d}\left(\phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)}\right). \tag{2.19}$$

We derive the boundary conditions at the free surface as they are appear in [Ric68].

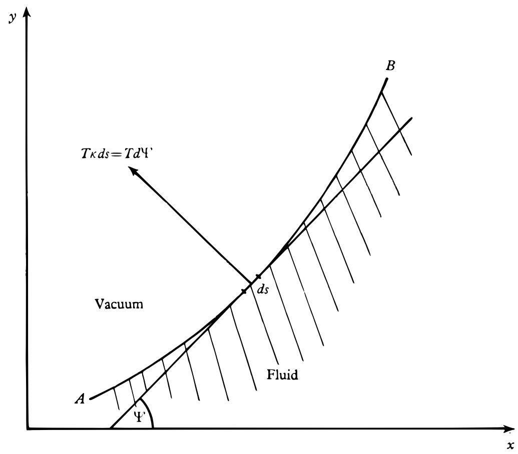


Figure 2.1: Sketch to introduce the notation for the derivation of the boundary conditions at the free boundary taken from [Ric68].

If AB is the section of the fluid surface where the surface tension, T, is acting, then $(X_n \,\mathrm{d}s, Y_n \,\mathrm{d}s)$ is a force of magnitude $T\kappa \,\mathrm{d}s$ directed along the outward normal to the surface, where $\kappa = \frac{\mathrm{d}\Psi}{\mathrm{d}s}$ is the curvature at that point. Therefore,

$$(X_n + iY_n)\,\mathrm{d}s = T\kappa\,\mathrm{d}s(-\sin\Psi + i\cos\Psi) = T\,\mathrm{d}\left[\frac{\mathrm{d}z}{\mathrm{d}s}\right]. \tag{2.20}$$

By equating (2.19) and (2.20) and traversing the boundary in the anticlockwise direction, we can write the boundary condition (2.7) as

$$\phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)} = \frac{Ti}{2\mu}\frac{\mathrm{d}z}{\mathrm{d}s} \quad \text{on} \quad \partial D. \tag{2.21}$$

Alternative derivations of this boundary condition can be found in [CT98] and [TV95].

The velocity at any point on the free surface is given by (2.6) and (2.21), which together yield

$$u + iv = 2\phi(z) - \frac{Ti}{2\mu}\frac{\mathrm{d}z}{\mathrm{d}s} \quad \text{on} \quad \partial D. \tag{2.22}$$

Note that at time t the fluid occupies a particular domain $D(t)$ and the boundary condition (2.21) determines $\phi(z)$ and $\chi(z)$ at that particular time. The velocity on the free surface (2.22) indicates how $D(t)$ is changing at time t. Our aim is to determine the evolution of $D(t)$.

First we consider the case where D is a circular domain of radius R centred at the origin. Any point on the boundary of this circular disc is thus given by $|z| = R$ which is equivalent to writing $\overline{z} = \frac{R^2}{z}$. The boundary condition (2.21) now becomes

$$\phi(z) = -z\overline{\phi}'\left(\frac{R^2}{z}\right) - \overline{\chi}'\left(\frac{R^2}{z}\right) - \frac{T}{2\mu R}z \quad \text{on} \quad |z| = R, \tag{2.23}$$

where we have used the Schwarz function notation $\overline{\phi}(z) = \overline{\phi(\overline{z})}$, so that $\overline{\phi}(z)$ and $\phi(z)$ have regions of analyticity that are reflections of each other in the real axis. Also, note that the arc length is $s = R\theta$ and any point on the boundary of the circle with radius R is thus given by $z = Re^{i\theta} = Re^{i(s/R)}$. So $\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{i}{R}Re^{i(s/R)} = \frac{i}{R}z$ and the last term in (2.23) follows.

The left-hand side of (2.23) is an analytic function for $|z| \leq R$ and the right-hand side is analytic for $|z| \geq R$. This means that $\phi(z)$ can be analytically continued into the whole plane to an entire function.

If $\phi(z)$ is analytically continued then we can use $z = \frac{R^2}{\overline{z}}$ to write (2.23) as

$$\phi\left(\frac{R^2}{\overline{z}}\right) = -\frac{R^2}{\overline{z}}\overline{\phi}'(\overline{z}) - \overline{\chi}'(\overline{z}) - \frac{T}{2\mu R}\frac{R^2}{\overline{z}} \quad \text{on} \quad |z| = R, \tag{2.24}$$

and taking the complex conjugate of this, we obtain

$$\overline{\phi}\left(\frac{R^2}{z}\right) = -\frac{R^2}{z}\phi'(z) - \chi'(z) - \frac{T}{2\mu}\frac{R}{z} \quad \text{on} \quad |z| = R. \tag{2.25}$$

Rearranging, we notice that this is identical to

$$\chi'(z) = -\overline{\phi}\left(\frac{R^2}{z}\right) - \frac{R^2}{z}\phi'(z) - \frac{T}{2\mu}\frac{R}{z} \quad \text{on} \quad |z| = R, \tag{2.26}$$

which shows that $\chi'(z)$ can also be analytically continued into the whole complex plane as an entire function.

At this point, we want to investigate what is the behaviour of (2.23) for large $|z|$. If $|z| \to \infty$ we see that (2.23) tends to a linear function of z which implies that

$$\phi(z) = a + bz, \tag{2.27}$$

where a and b are complex constants. Similarly, if we let $|z| \to \infty$ we observe that (2.26) tends to a complex constant c and so

$$\chi'(z) = c. \tag{2.28}$$

Using this form for $\phi(z)$ and $\overline{\chi'(z)} = \bar{c}$, together with $\overline{\phi'(z)} = \bar{b}$, and inserting them in (2.23) implies

$$a + bz = -\bar{b}z - \bar{c} - \frac{T}{2\mu R}z \quad \text{for all } \ z. \tag{2.29}$$

Now, $\mathcal{O}(1)$ coefficients of (2.29) give $a = -\bar{c}$ and $\mathcal{O}(z)$ coefficients give $b = -\bar{b} - \frac{T}{2\mu R}$ which is equivalent to $\mathrm{Re}(b) = -\frac{T}{4\mu R}$ by $\mathrm{Re}(b) = \frac{1}{2}(b + \bar{b})$. Therefore the boundary conditions can be written as

$$\phi(z) = a + [\mathrm{Re}(b) + i\mathrm{Im}(b)]\, z = a + \left[-\frac{T}{4\mu R} + i\mathrm{Im}(b)\right] z \quad \text{and} \quad \chi'(z) = -\bar{a}. \tag{2.30}$$

The velocity field (2.6) becomes, upon substitution of (2.30),

$$\begin{aligned} u + iv &= \phi(z) - z\overline{\phi'(z)} - \overline{\chi'(z)} \\ &= a + \left[-\frac{T}{4\mu R} + i\mathrm{Im}(b)\right] z - \left[-\frac{T}{4\mu R} - i\mathrm{Im}(b)\right] + a \\ &= 2a + 2i\mathrm{Im}(b). \end{aligned} \tag{2.31}$$

This has a physical interpretation: the first term corresponds to a translation and the second term corresponds to a rotation about the origin with angular velocity equal to $2\mathrm{Im}(b)$. This highlights an important feature of the problems under consideration. If the distribution of forces around the boundary ∂D is specified, then we can describe the motion only as translation and rotation.

2.1 Symmetry and uniqueness

If the initial domain $D(0)$ is symmetric about both the x-axis and the y-axis then it is true that $D(t)$ would retain this symmetry for all time $t > 0$. Most of the examples shown in [Ric92] are symmetric about the x-axis and the non-uniqueness corresponds to a uniform translation in the x-direction. The symmetry requires that $\phi(z)$ and $\chi(z)$ are real functions on the x-axis. In what follows we assume that $z = 0$ lies within the domain $D(t)$ for all $t > 0$ and uniqueness is ensured by choosing $\phi(0) = 0$.

2.2 Mathematical formulation

Consider now a fluid that occupies a simply-connected domain $D(t)$ at time t. This domain has an analytic boundary ∂D with the origin $z = 0$ lying within this region.

Define the time-dependent conformal map of the unit disc $|\zeta| < 1$ in the ζ-plane onto the fluid domain D by

$$z = w(\zeta) = w(\zeta; t) \tag{2.32}$$

and for uniqueness of the mapping we need to insist that $w(0; t) = 0$, $w'(0; t) > 0$ for all $t > 0$. Note that such a map always exists by the Riemann mapping theorem.

In § 2.1 we mentioned that we choose D to be symmetric about the x-axis and take $w(\zeta)$ to be a real function on the x-axis, implying that $w'(0)$ is also real. Uniqueness is achieved by assuming that $w'(0) > 0$ and by specifying the orientation around the domain.

We use Green's theorem to calculate the area of the domain D in the complex plane. In particular, we have[1]

$$\text{Area}(D) = \frac{1}{2i} \oint_{\partial D} \overline{z}\, \mathrm{d}z = \frac{1}{2i} \int \overline{w(\zeta)} w'(\zeta)\, \mathrm{d}\zeta, \tag{2.33}$$

where the integral is calculated in the anti-clockwise direction around the unit disc $|\zeta| = 1$. The shape evolution of $D(t)$ does not change the area of the domain.

On the boundary ∂D we have $\frac{\mathrm{d}z}{\mathrm{d}s} = w'(\zeta)\frac{\mathrm{d}\zeta}{\mathrm{d}s}$ using the chain rule on (2.32). We also have $i\zeta\frac{w'(\zeta)}{|w'(\zeta)|} = i\zeta\left[\frac{w'(\zeta)}{\overline{w'(\zeta)}}\right]^{1/2}$, recalling that $|w'(\zeta)| = [w'(\zeta)\overline{w'(\zeta)}]^{1/2}$ and care should be taken to choose the correct branch cut[2]. Therefore, using these with $\zeta = e^{i\theta}$, we can write

$$\frac{\mathrm{d}z}{\mathrm{d}s} = w'(\zeta)\frac{\mathrm{d}\zeta}{\mathrm{d}s} = w'(e^{i\theta})\frac{\mathrm{d}(e^{i\theta})}{\mathrm{d}s} = i\zeta w'(\zeta)\frac{\mathrm{d}\theta}{\mathrm{d}s} = i\zeta\frac{w'(\zeta)}{|w'(\zeta)|}, \tag{2.34}$$

where the last equality follows from $\left|\frac{\mathrm{d}z}{\mathrm{d}s}\right| = 1$ and that $\frac{\mathrm{d}\theta}{\mathrm{d}s} \in \mathbb{R}^+$ for the anticlockwise tangent. For more details on deriving this, see Appendix B.

We also define the functions $\Phi(\zeta)$ and $X(\zeta)$ on the unit disc as

$$\Phi(\zeta) = \phi(w(\zeta)) \quad \text{and} \quad X(\zeta) = \chi'(\zeta), \tag{2.35}$$

[1] $\oint \overline{z}\mathrm{d}z = \oint_{\partial D}(x-iy)\mathrm{d}(x+iy) = \oint_{\partial D}(x\mathrm{d}x+y\mathrm{d}y)+i(x\mathrm{d}y-y\mathrm{d}x) = \iint[0+i(1+1)]\mathrm{d}x\mathrm{d}y = 2i\text{Area}(D)$.

[2] Where the branch of the square root on the right to be used is the one that ensures the whole expression gives the unit tangent to ∂D that points in the anti-clockwise direction.

and this ensures that $\Phi(0) = 0$ as required.

Notice that the boundary condition (2.21), using the conformal mapping $z = w(\zeta)$, becomes

$$\phi(w(\zeta)) + w(\zeta)\overline{\phi'(w(\zeta))} + \overline{\chi'(w(\zeta))} = \frac{Ti}{2\mu} w'(\zeta)\frac{\mathrm{d}\zeta}{\mathrm{d}s} \tag{2.36}$$

and with the new notation (2.35) for $\phi(w(\zeta))$, the boundary condition (2.21) becomes

$$\Phi(\zeta) + w(\zeta)\overline{\left(\frac{\Phi'(\zeta)}{w'(\zeta)}\right)} + \overline{X(\zeta)} = \frac{Ti}{2\mu} i\zeta \left\{\frac{w'(\zeta)}{\overline{\overline{w'(\zeta)}}}\right\}^{1/2}. \tag{2.37}$$

Using the notation $\overline{\phi}(\zeta) = \overline{\phi(\overline{\zeta})}$, the above expression can be written as

$$\Phi(\zeta) + w(\zeta)\frac{\overline{\Phi}'(1/\zeta)}{\overline{w}'(1/\zeta)} + \overline{X}(1/\zeta) = -\frac{T}{2\mu}\zeta \left\{\frac{w'(\zeta)}{\overline{w}'(1/\zeta)}\right\}^{1/2} \quad \text{on } |\zeta| = 1. \tag{2.38}$$

This means that $\Phi(\zeta)$ can be analytically continued in $|\zeta| > 1$, but this time we would expect singularities of $\Phi(\zeta)$ in $|\zeta| > 1$.

Analytic continuation is complicated by the square root branch point in $w'(\zeta;t)$ but this is dealt with by introducing the two functions $F_+(\zeta)$ and $F_-(\zeta)$ which are defined by the sum decomposition

$$\frac{1}{[w'(\zeta)\overline{w}'(1/\zeta)]^{1/2}} = F_+(\zeta) - F_-(\zeta). \tag{2.39}$$

Here $F_+(\zeta)$ is an analytic function for $|\zeta| \leq 1$ and $F_-(\zeta)$ is analytic for $|\zeta| \geq 1$. These functions are unique if we also insist that $F_-(\zeta) \to 0$ as $\zeta \to \infty$. Note that this fact will be used later on, when we consider the evolution of a limaçon.

By Cauchy's integral formula we can write

$$F_\pm(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{F_+(\zeta) - F_-(\zeta)}{\tau - \zeta}\,\mathrm{d}\tau = \frac{1}{2\pi i}\int_\Gamma \frac{1}{[w'(\zeta)\overline{w}'(1/\zeta)]^{1/2}(\tau - \zeta)}, \tag{2.40}$$

for $|\zeta| < 1$ and $|\zeta| > 1$, respectively, where Γ is the unit circle $|\tau| = 1$ and the orientation is in the anti-clockwise direction. We can analytically continue this by deforming the contour of integration. The function on the left-hand side of (2.40) is real on the boundary of the unit disc $|\zeta| = 1$ where $\zeta = \frac{1}{\overline{\zeta}}$. This means that the sum decomposition (2.39) can be rewritten as

$$\frac{1}{[w'(1/\overline{\zeta})\overline{w}'(\overline{\zeta})]^{1/2}} = F_+(1/\overline{\zeta}) - F_-(1/\overline{\zeta}), \tag{2.41}$$

and if we take the complex conjugate of this together with the notation $\overline{\phi}(\zeta) = \overline{\phi(\overline{\zeta})}$, then the sum decomposition (2.39) becomes

$$\begin{aligned}\frac{1}{[\overline{w}'(1/\zeta)w'(\zeta)]^{1/2}} &= \overline{F}_+(1/\zeta) - \overline{F}_-(1/\zeta) \\ &= [-\overline{F}_-(1/\zeta) + \overline{F_+(0)}] - [-\overline{F}_+(1/\zeta) + \overline{F_+(0)}]. \end{aligned} \tag{2.42}$$

This sum decomposition is unique and so by comparing (2.39) and (2.42) we can deduce that

$$F_+(\zeta) = -\overline{F}_-(1/\zeta) + \overline{F_+(0)} \quad \text{and} \quad F_-(\zeta) = -\overline{F}_+(1/\zeta) + \overline{F_+(0)}. \tag{2.43}$$

Also note that if we use $\zeta = \frac{1}{\overline{\zeta}}$ into the first equation of (2.43) we obtain $F_+(1/\overline{\zeta}) = -\overline{F}_-(\overline{\zeta}) + \overline{F_+(0)}$ which is $F_+(\zeta) = -\overline{F}_-(\overline{\zeta}) + \overline{F_+(0)}$. If we now take the conjugate of the second equation of (2.43) we will obtain $\overline{F_-(\zeta)} = -\overline{\overline{F}_+(1/\zeta)} + F_+(0)$ which implies that $\overline{F}_-(\overline{\zeta}) = -\overline{\overline{F_+(1/\overline{\zeta})}} + F_+(0)$. This is the same as writing $\overline{F}_-(\overline{\zeta}) = -F_+(\zeta) + F_+(0)$. Therefore, if we compare

$$F_+(\zeta) = -\overline{F}_-(\overline{\zeta}) + \overline{F_+(0)} \quad \text{and} \quad \overline{F}_-(\overline{\zeta}) = -F_+(\zeta) + F_+(0), \tag{2.44}$$

we notice that they both hold only when $\overline{F_+(0)} = F_+(0)$, which is only possible if $F_+(0)$ is real.

Now that we have shown that $F_+(0)$ is real and using $\overline{F}_+(1/\zeta) = \overline{F_+(1/\overline{\zeta})} = \overline{F_+(\zeta)}$, it is evident that (2.42) becomes

$$\frac{1}{[\overline{w}'(1/\zeta)w'(\zeta)]^{1/2}} = F_+(\zeta) + \overline{F}_+(\zeta) - F_+(0) \quad \text{on } |\zeta| = 1. \tag{2.45}$$

If we multiply (2.39) by $w'(\zeta)$ we get $\left\{\frac{w'(\zeta)}{\overline{w}'(1/\zeta)}\right\}^{1/2} = F_+(\zeta)w'(\zeta) - F_-(\zeta)w'(\zeta)$ and substituting this into (2.38) and rearranging, yields

$$\Phi(\zeta) + \frac{T}{2\mu}\zeta F_+(\zeta)w'(\zeta) = -w(\zeta)\frac{\overline{\Phi}'(1/\zeta)}{\overline{w}'(1/\zeta)} - \overline{X}(1/\zeta) + \frac{T}{2\mu}\zeta F_-(\zeta)w'(\zeta) \quad \text{on } |\zeta| = 1. \tag{2.46}$$

The expression on the left-hand side here is analytic for $|\zeta| \leq 1$ and vanishes at $\zeta = 0$. The boundary condition indicates that it can be analytically continued into the region $|\zeta| > 1$. The singularities of the right-hand side of (2.46) in $|\zeta| > 1$ arise only from those of $w(\zeta)$ in $|\zeta| > 1$. Note also that by definition of a conformal map, $\overline{w}'(1/\zeta) \neq 0$ for $|\zeta| > 1$ because $w(\zeta)$ is a conformal map of $|\zeta| < 1$. This means that $w'(\zeta) \neq 0$ for $|\zeta| < 1$. Therefore, by analytic continuation, if $w(\zeta)$ is a polynomial then the left-hand side of (2.46) is also a polynomial and the only possible singularity of the right-hand side is a pole at infinity. More generally, if $w(\zeta)$ is a rational function then the left-hand side of (2.46) must also be a rational function when analytically continued. The poles of $w(\zeta)$ must be at the same points as the poles of the left-hand side. Thus, we can write down its form with a priori unknown coefficients.

The aim is to determine the shape evolution of the domain $D(t)$ with time. We have defined $z = w(\zeta; t)$ and so if we now let $\zeta = e^{i\theta}$, we have that the particle that was at position $w(e^{i\theta}; t)$ at time t on the free boundary, after the time increment δt, (at time $t + \delta t$) is at

$$w(e^{i\theta}; t) + (u + iv)\delta t + \mathcal{O}(\delta t^2). \tag{2.47}$$

The shape evolution is assumed to be smooth in time, so $w'(e^{i\theta})$ is a continuous function. Similarly, we have that $w(e^{i(\theta+\delta\theta)}; t + \delta t)$ (not necessarily the original point on the unit circle in the ζ-plane) and so this implies that

$$w(e^{i\theta}; t) + w'(e^{i\theta}; t)ie^{i\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\delta t + \frac{\partial w}{\partial t}(e^{i\theta}; t)\delta t + \mathcal{O}(\delta t^2). \tag{2.48}$$

If we now equate the $\mathcal{O}(\delta t)$ terms of (2.47) and (2.48) we get

$$u + iv = i\zeta w'(\zeta; t)\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial w}{\partial t}(\zeta; t) \quad \text{on } |\zeta| = 1. \tag{2.49}$$

We have two expressions for $u + iv$ on ∂D given by (2.22) and (2.49). Equating them and dividing throughout by $\zeta w'(\zeta; t)$, yields

$$\frac{1}{\zeta w'(\zeta; t)}\left[2\phi(z) - \frac{Ti}{2\mu}\frac{\mathrm{d}z}{\mathrm{d}s} - \frac{\partial w}{\partial t}(\zeta; t)\right] = i\frac{\mathrm{d}\theta}{\mathrm{d}t}. \tag{2.50}$$

Furthermore, $\Phi(\zeta) = \phi(w(\zeta)) = \phi(z)$ and $\frac{\mathrm{d}z}{\mathrm{d}s} = i\zeta\left[\frac{w'(\zeta)}{\overline{w'(\zeta)}}\right]^{1/2}$ together, show that (2.50) can be written as

$$\frac{1}{\zeta w'(\zeta; t)}\left[2\Phi(\zeta) - \frac{\partial w}{\partial t}(\zeta; t)\right] + \frac{T}{2\mu}\frac{1}{[\overline{w}'(1/\zeta)w'(\zeta)]^{1/2}} = i\frac{\mathrm{d}\theta}{\mathrm{d}t} \quad \text{on } |\zeta| = 1. \tag{2.51}$$

We also had $\frac{1}{[\overline{w}'(1/\zeta)w'(\zeta)]^{1/2}} = F_+(\zeta) - F_-(\zeta) = F_+(\zeta) + \overline{F_+(\zeta)} - F_+(0)$ and so the real part of (2.51) becomes

$$\mathrm{Re}\left\{\frac{1}{\zeta w'(\zeta; t)}\left[2\Phi(\zeta) - \frac{\partial w}{\partial t}(\zeta; t)\right] + \frac{T}{\mu}F_+(\zeta)\right\} = \frac{T}{2\mu}F_+(0) \quad \text{on } |\zeta| = 1. \tag{2.52}$$

Note that if the problem we are considering is symmetric about the x-axis then the analytic function in the curly brackets is real for $\zeta = 0$ and it must be equal to the real constant $\frac{T}{2\mu}F_+(0)$. Therefore, solving (2.52) for $\frac{\partial w}{\partial t}$ yields

$$\frac{\partial w}{\partial t}(\zeta; t) = 2\Phi(\zeta) + \frac{T}{2\mu}[2F_+(\zeta) - F_+(0)]\zeta w'(\zeta; t). \tag{2.53}$$

2.3 Obtaining our solutions

We observe that the left-hand side of (2.46) appears in the right-hand side of (2.53) but has twice its value. There is an extra term $-\frac{T}{2\mu}F_+(0)\zeta w'(\zeta;t) = -\frac{T}{\mu}\frac{K(b)}{\pi a}F_+(0)\zeta w'(\zeta;t)$. But we have already established that if $w(\zeta)$ is a rational function then this combination of functions will also be rational.

Solutions are determined in the following way: Let $w(\zeta)$ be a rational function of ζ with time-dependent coefficients. We must have $w(0;t) = 0$ and that there are no poles in $|\zeta| \leq 1$. As mentioned before, we use (2.46) to determine the rational function equal to its left-hand side, with the coefficients being dependent upon the coefficients of $w(\zeta)$. The final step is to substitute into (2.53)and equate the singularities on each side. We then obtain first order differential equations for the coefficients in $w(\zeta)$. By solving them with appropriate initial conditions, we can find $w(\zeta)$ and thus the corresponding shape evolution of $D(t)$ in time.

In the following section, we make this solution procedure clearer by considering two particular examples.

3 Examples

In this section, we follow the examples that appear in [Ric92] and [CH99b] using the methods described in the preceding sections. Several time-dependent problems have been solved exactly using conformal maps and we outline similar problems here. Some of these include:

1. The evolution of domains described by polynomial maps of the form

$$w(\zeta;t) = a(t)\left(\zeta - \frac{1}{n}b(t)\zeta^n\right), \tag{3.1}$$

   for any integer $n \geq 2$, under the action of both surface tension (Stokes flow) and a point sink (Hele-Shaw flow). Note that we will expand on the case $n = 2$ which is the evolution of a limaçon under surface tension in § 3.1.

2. The coalescence of two circular cylinders of fluid under surface tension. In § 3.2, we will discuss the different approaches to this case for both equal and unequal circular cylinders. Several authors either generalised this theory, e.g. [Ric97], or developed new theories, for instance using algebraic curves (see [Cro02]), to tackle the problem for n touching circular cylinders.

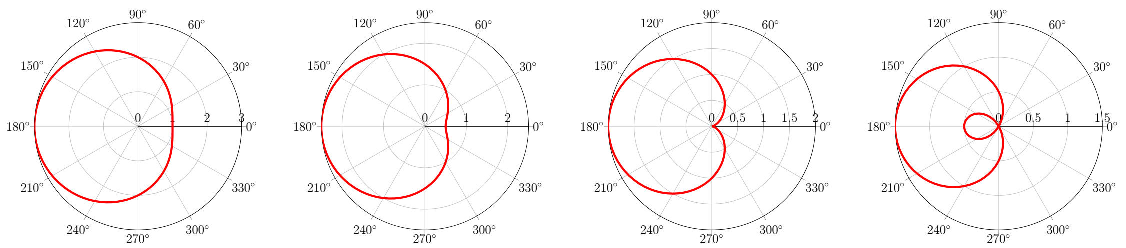
3.1 Evolution of a limaçon

Here we consider probably the simplest non-trivial case; if $w(\zeta;t)$ is a linear function of ζ then $D(t)$ is a circular disc for all t. We take the conformal mapping to be

$$w(\zeta;t) = a(t)\left(\zeta - \frac{1}{2}b(t)\zeta^2\right), \tag{3.2}$$

a quadratic function which corresponds to a limaçon, with $w(0;t) = 0$. In Appendix A we show that (3.2) with $b = 1$ is a conformal map from the unit disc to a cardioid.

It is required that the mapping $w(\zeta;t)$ is *univalent* on the unit disc; meaning that $w(\zeta;t)$ has to be analytic and injective, with non-vanishing derivative (conformal), so that the physical fields remain single-valued in the evolving domain. Note that if $w(\zeta;t)$ ceases to be single-valued at some point on the unit disc then the fluid domain is starting to overlap itself. At this point we need to modify the model to account for this change in topology. If $w'(\zeta)$ becomes zero at some point on the unit disc then there is a cusp in the free boundary, which might be smoothed off by the action of positive surface tension. We refer forward to Figures 3.1c and 3.1d for examples of cusped and overlapping flow domains, respectively.



(a) $r = 2 - \cos\theta$ (b) $r = 1.5 - \cos\theta$ (c) $r = 1 - \cos\theta$ (d) $r = 0.5 - \cos\theta$

Figure 3.1: Examples of fluid domains in the shape of limaçons.

In addition, without loss of generality, we assume that $a(t) \in \mathbb{R}^+$ and $b(t) \in \mathbb{R}_0^+$ for all time. Note that the fact that $b(t) \in \mathbb{R}$ corresponds to the assumption that $D(t)$ is symmetric about the x-axis. Also, for (3.2) to be a univalent map in $|\zeta| < 1$, as required, we need $b(t) \in [0, 1]$. Note that as $b \to 0^+$ the limaçon tends to a circle and as $b \to 1^-$ the limaçon tends to a cardioid.

Given the form of the conformal mapping (3.2) and by invoking analytic continuation as in (2.46) we have

$$\Phi(\zeta) + \frac{T}{2\mu}F_+(\zeta)\zeta w'(\zeta) = \alpha\zeta + \beta\zeta^2, \tag{3.3}$$

where α and β are real.

In what follows we must determine the right-hand side of (2.46) for $|\zeta| \to \infty$. Even though this involves $F_-(\zeta)$ for large $|\zeta|$, by (2.43b), we see that this is equivalent to observing the behaviour of $F_+(\zeta)$ for small $|\zeta|$. The final result will involve only the term $F_+(0)$ and this motivates the following calculation.

We have the explicit representation

$$F_+(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\theta}{|w'(e^{i\theta})|} \tag{3.4}$$

and since $w'(e^{i\theta}) = a(1 - be^{i\theta}) = a(1 - b\cos\theta - ib\sin\theta)$ this means that $|w'(e^{i\theta})| = a[1 - 2b\cos\theta + b^2]$, giving

$$F_+(0) = \frac{1}{\pi a} \int_0^{\pi} \frac{\mathrm{d}\theta}{[1 - 2b\cos\theta + b^2]^{1/2}} = \frac{2}{\pi a} K(b), \tag{3.5}$$

with $K(b)$ representing the complete elliptic integral of the first kind, defined by

$$K(b) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{[1 - b^2\sin^2\theta]^{1/2}}. \tag{3.6}$$

It is mentioned in [Ric92] that the last equality in (3.5) is due to Landen's transformation and the reader can find this in [GR14, equation (3.674-1)].

Rearranging (3.3), we find that the form of $\Phi(\zeta)$ is

$$\Phi(\zeta) = \alpha\zeta + \beta\zeta^2 - \frac{T}{2\mu} F_+(\zeta)\zeta w'(\zeta) = \alpha\zeta + \beta\zeta^2 - \frac{T}{2\mu} F_+(\zeta)\zeta[a(1 - b\zeta)], \tag{3.7}$$

and if we differentiate this with respect to ζ, we obtain

$$\Phi'(\zeta) = \alpha + 2\beta\zeta - \frac{aT}{2\mu}\left\{F_+'(\zeta)\zeta(1 - b\zeta) + F_+(\zeta)(1 - b\zeta) - bF_+(\zeta)\zeta\right\}. \tag{3.8}$$

At this point let LHS $= \Phi(\zeta) + \frac{T}{2\mu}\zeta F_+(\zeta)w'(\zeta)$. Using this notation, we show that (2.46) can be written as

$$\begin{aligned}
\text{LHS} &= -\frac{w(\zeta)}{\overline{w}'(1/\zeta)}\overline{\Phi}'(1/\zeta) + \frac{T}{2\mu}F_-(\zeta)\zeta w'(\zeta) + \mathcal{O}(1) \\
&= -\frac{(\zeta - \frac{1}{2}b\zeta^2)}{(1 - b\zeta^{-1})}\Bigg[\alpha + 2\beta\zeta^{-1} - \frac{aT}{2\mu}\big\{F_+'(1/\zeta)\zeta^{-1}(1 - b\zeta^{-1}) + F_+(1/\zeta)(1 - b\zeta^{-1}) \\
&\qquad\qquad -b\zeta^{-1}F_+(1/\zeta)\big\}\Bigg] + \frac{T}{2\mu}F_-(\zeta)\zeta[a(1 - b\zeta)] + \mathcal{O}(1).
\end{aligned} \tag{3.9}$$

In order to simplify (3.9) we need to take the following into account. Recall that we are interested in the behaviour of the right-hand side of (2.46) as $\zeta \to \infty$. Firstly, note that we include the term $\overline{X}(1/\zeta)$ in the $\mathcal{O}(1)$ terms of (3.9). We also use the binomial expansion $1-b\zeta^{-1} = 1+b\zeta^{-1}+\frac{1}{2}b^2\zeta^{-2}+\mathcal{O}\left(\zeta^{-3}\right)$ for $\zeta^{-1}\to 0$. Finally, $F_-(\zeta)\zeta(1-b\zeta)\to 0$ as $\zeta\to\infty$ because $F_-(\zeta)$ vanishes as $\zeta\to\infty$. So (3.9) reduces to

$$\text{LHS} = -\left(\zeta-\frac{1}{2}b\zeta^2\right)\left(1+\frac{b}{\zeta}+\frac{b^2}{2\zeta^2}\right)\left[\alpha+2\beta\zeta^{-1}-\frac{aT}{2\mu}\left\{F_+(1/\zeta)(1-2b\zeta^{-1})\right\}\right]+\mathcal{O}(1). \tag{3.10}$$

At this point we note that $F_+(1/\zeta) = F_+(0)+\mathcal{O}(1/\zeta) = \frac{2}{\pi a}K(b)+\mathcal{O}(1/\zeta)$ as $\zeta\to\infty$ using (3.5). This also implies that $F_+'(1/\zeta)\to 0$ as $\zeta\to\infty$ since $F_+(1/\zeta)\to\frac{2}{\pi a}K(b)$ as $\zeta\to\infty$ which is a constant.

It then follows that the right-hand side of (2.46) as $\zeta\to\infty$ behaves like

$$\frac{1}{2}b\left(\alpha-\frac{T}{\pi\mu}K(b)\right)\zeta^2+\left[\frac{1}{2}\alpha b^2-\alpha+\beta b+\frac{T}{\pi\mu}\left(1+\frac{1}{2}b^2\right)K(b)\right]\zeta+\mathcal{O}(1). \tag{3.11}$$

Now that we have determined how the right-hand side of (2.46) behaves for $\zeta\to\infty$, we can finally determine the values of α and β. For that, we equate coefficients of powers of ζ from (3.3) and (3.11) to obtain the following

$$\mathcal{O}(\zeta): \qquad \alpha = \frac{1}{2}\alpha b^2-\alpha+\beta b+\frac{T}{\pi\mu}\left(1+\frac{1}{2}b^2\right)K(b), \tag{3.12}$$

$$\mathcal{O}(\zeta^2): \qquad \beta = \frac{1}{2}b\left(\alpha-\frac{T}{\pi\mu}K(b)\right). \tag{3.13}$$

Thus, substituting (3.13) into (3.12) we find α as

$$\alpha = \frac{T}{\pi\mu}\left(\frac{1}{2-b^2}\right)K(b) \tag{3.14}$$

and this determines β which is given by

$$\beta = -\frac{Tb}{2\pi\mu}\left(\frac{1-b^2}{2-b^2}\right)K(b). \tag{3.15}$$

The task now is to determine the shape time evolution. For that recall that we had (2.53), and also that $\Phi(\zeta) = \alpha\zeta+\beta\zeta^2-\frac{T}{2\mu}F_+(\zeta)\zeta w'(\zeta)$. So combining these together, yields

$$\frac{\partial w}{\partial t} = 2\left(\alpha\zeta+\beta\zeta^2\right)-\frac{T}{2\mu}F_+(0)\zeta w'(\zeta). \tag{3.16}$$

Finally, note that the derivative of (3.2) with respect to ζ gives $w'(\zeta) = a(1-b\zeta)$. So, if we substitute this with (3.14), (3.15) and $F_+(0) = \frac{2}{\pi a}K(b)$ in (3.16), we obtain the following formula for the shape evolution of the fluid domain

$$\frac{\partial w}{\partial t} = \frac{T}{\pi\mu}\frac{b^2}{2-b^2}K(b)\zeta+\frac{T}{\pi\mu}\frac{b}{2-b^2}K(b)\zeta^2. \tag{3.17}$$

However, recall that we have the mapping (3.2) with a and b both functions of t and so we can find an alternative form of the time derivative of $w(\zeta;t)$ given by

$$\frac{\partial w}{\partial t} = \frac{\mathrm{d}a}{\mathrm{d}t}\zeta - \frac{1}{2}\frac{\mathrm{d}(ab)}{\mathrm{d}t}\zeta^2. \tag{3.18}$$

Equating coefficients of powers of ζ in (3.17) and (3.18) we get

$$\begin{aligned} \mathcal{O}(\zeta): \qquad && \frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{T}{\pi\mu}\frac{b^2}{2-b^2}K(b), && (3.19)\\ \mathcal{O}(\zeta^2): \qquad && \frac{\mathrm{d}(ab)}{\mathrm{d}t} &= -\frac{2T}{\pi\mu}\frac{b}{2-b^2}K(b). && (3.20) \end{aligned}$$

By eliminating the right-hand side of (3.19) and (3.20) we obtain

$$2\frac{\mathrm{d}a}{\mathrm{d}t} + b\frac{\mathrm{d}(ab)}{\mathrm{d}t} = 0, \tag{3.21}$$

and by expanding and collecting like terms, (3.21) becomes

$$(2+b^2)\frac{\mathrm{d}a}{\mathrm{d}t} + ab\frac{\mathrm{d}b}{\mathrm{d}t} = 0. \tag{3.22}$$

This is a first order differential equation which can be solved using separation of variables to give

$$a^2\left(1+\frac{1}{2}b^2\right) = \text{constant}. \tag{3.23}$$

For a detailed derivation of (3.23), see Appendix C.

According to [HR95], for univalent maps of the form (3.1) the area of the image domain is given by $A = a^2\left(1+\frac{1}{n}b^2\right)\pi$ and so the expected area conservation condition for the $n=2$ case considered in this subsection, is given by

$$a^2\left(1+\frac{1}{2}b^2\right) = \frac{A}{\pi}. \tag{3.24}$$

A derivation of this can also be found in [CHK99].

If we solve (3.24) for a we get $a = \sqrt{\frac{A}{\pi}}\left(1+\frac{1}{2}b^2\right)^{-1/2}$. Now we can substitute this into (3.20) and use the product rule to get the following expression for the time derivative of b

$$\frac{\mathrm{d}b}{\mathrm{d}t} = -\frac{T}{\mu\sqrt{A\pi}}\frac{\left(1+\frac{1}{2}b^2\right)^{3/2}}{\left(1-\frac{1}{2}b^2\right)}bK(b). \tag{3.25}$$

If $t=0$ corresponds to a cardioid with $b=1$, we can integrate this to relate b and t explicitly as

$$t = -\frac{\mu\sqrt{A\pi}}{T}\int_1^b \frac{\left(1-\frac{1}{2}k^2\right)}{\left(1+\frac{1}{2}k^2\right)^{3/2}}\frac{1}{kK(k)}\,\mathrm{d}k. \tag{3.26}$$

Note that this step is only required in the case we want to see what shape $D(t)$ has at a particular time.

Since the mapping function is $w(\zeta;t) = a\left(\zeta - \frac{1}{2}b\zeta^2\right)$ and $a = \sqrt{\frac{A}{\pi\left(1+\frac{1}{2}b^2\right)}}$, we can therefore write

$$w(\zeta;t) = \sqrt{\frac{A}{\pi\left(1+\frac{1}{2}b^2\right)}}\left(\zeta - \frac{1}{2}b\zeta^2\right). \tag{3.27}$$

With this we can observe that as $t \to \infty$ we have $b \to 0$; which means that we started off with a cardioid fluid domain and as time goes to infinity, this evolves through the limaçon family towards a circular shape. (Recall that for (3.2) as $b \to 0^+$ then it tends to a circle and as $b \to 1^-$ it tends to a cardioid.)

For the evolution of an initial shape in the form of a limaçon under a source or a sink at the origin, see [How17]. This is an example of the canonical injection problem from a point source/sink into a two-dimensional Hele-Shaw cell and is tackled using the Polubarinova-Galin equation. For a comparison between the theory of Hele-Shaw flows and Stokes flows, see [CHK99].

3.2 Coalescence of two circular cylinders

Flows with an initial state corresponding to an array of n touching circular cylinders are of particular concern. In that case, $w(\zeta;t)$ must be of the form

$$w(\zeta;t) = \zeta\sum_{j=1}^{n}\frac{\overline{\beta}_j}{1-\overline{\gamma}_j\zeta}, \tag{3.28}$$

known as a *partial fraction map* [Hop90]. In this subsection we only consider the case of two circular cylinders and the reader is referred to [Ric97] for $n > 2$.

3.2.1 Hopper's example: two equal coalescing cylinders

Hopper derived an exact solution for the coalescence of two equal circular cylinders in a parametric ζ-plane [Hop90]. He described the free-boundary evolution using a conformal map $w(\zeta;t)$ from the unit disc in the ζ-plane to the flow domain $D(t)$. This map is of the form

$$w(\zeta;t) = \frac{\zeta A(t)}{\zeta^2 - a^2(t)}, \tag{3.29}$$

where $A(t)$ and $a(t)$ are two real time-evolving parameters for which Hopper gives the appropriate evolution equations.

An illustration of the problem in hand is shown in Figure 3.2.

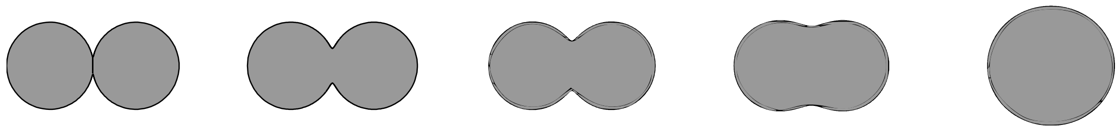


Figure 3.2: Shape evolution of two equal cylinders of fluid under surface tension taken from [BC05].

3.2.2 Richardson's example: two unequal coalescing cylinders

Richardson in [Ric92] considers the case where $w(\zeta;t)$ has two simple poles and the mapping is of the form

$$w(\zeta;t) = \zeta\left(\frac{C(t)a(t)}{1-a(t)\zeta} + \frac{D(t)b(t)}{1+b(t)\zeta}\right), \tag{3.30}$$

with $w(0;t) = 0$. With regards to § 2.1, we restrict our attention only to those maps which are symmetric about the x-axis and which have poles that lie on the real axis. It is assumed without loss of generality that $a(t), b(t) \in [0,1)$ and $C(t), D(t) > 0$. Moreover, we notice that if $a(t) = b(t)$ and $C(t) = D(t)$ then the map is also symmetric about the y-axis.

Here we give a brief overview of two circular cylinders of fluid, with unequal radii, that are initially touching as in § 4 of [Ric92]. The cylinders are allowed to coalesce and evolve to form a single circular cylinder under the action of surface tension. Richardson [Ric92] used a direct approach of combining the kinematic boundary condition and the stress condition and adjusted the time evolution of the parameters in the map $w(\zeta;t)$ to give the required analyticity properties of $G(\zeta;t)$ in the unit circle; where $G(\zeta) = \Phi(\zeta) + \frac{T}{2\mu}\zeta F_+(\zeta)w'(\zeta)$. Such a solution was obtained by making appropriate choices of initial conditions. After extensive algebraic manipulation, Richardson also deduced the existence of two invariant quantities. He also found two evolution equations for the poles of the mapping. However, since this approach was rather complicated, he reviewed his work, publishing a simplified version in [Ric97].

The two equations required are (2.46) and (2.53). As explained before, the rationale of this approach is that if $w(\zeta)$ is a rational function or a polynomial at some time, then the same holds for the combination of functions on the right-hand side of (2.46) at that time. The same combination of functions is found in the right-hand side of (2.53) and so $w(\zeta)$ must remain a rational function or a polynomial as the geometry of the fluid domain changes. The function $\Phi(\zeta)$ is no longer required if we are only interested in the evolution of the shape of the domain. Thus, in [Ric97] and

[CHK99], the result is expressed in terms of the function

$$G_{+}(\zeta) = F_{+}(\zeta) - \frac{1}{2}F_{+}(0), \tag{3.31}$$

where $F(\zeta)$ is the function introduced in [[Hop90], equation (14)]. The function $G_{+}(\zeta)$ can also be represented as

$$G_{+}(\zeta) = \frac{1}{2\pi i}\oint_{|\tau|=1} \frac{1}{|w'(\tau,t)|}\frac{\tau+\zeta}{\tau-\zeta}\frac{\mathrm{d}\tau}{\tau}, \tag{3.32}$$

with integration in the anti-clockwise direction.

The elimination of $\Phi(\zeta)$ between (2.46) and (2.53) yields

$$\frac{\partial}{\partial t}\left[w'(\zeta)\overline{w}'(1/\zeta)\right] + 2X(\zeta)w'(\zeta) = \frac{T}{\mu}\frac{\partial}{\partial\zeta}\left[\zeta w'(\zeta)\overline{w}(1/\zeta)G_{+}(\zeta)\right]. \tag{3.33}$$

It is evident that the singularities of (3.28) in $|\zeta| < 1$ are simple poles at $\zeta = \frac{1}{\overline{\gamma}_j} = \gamma_j$ and the residues there must vanish.

Howell adopted Richardson's method to solve the more general problem with the kinematic condition in [How94]. As opposed to Richardson's analysis which resulted in ordinary differential equations for the time-dependent coefficients of the conformal map, Howell found partial differential equations for the coefficients which depended on both time and distance.

4 Conclusion

In this report, the viscous incompressible flow in a finite region, bounded initially by a simple, smooth, closed curve and driven solely by surface tension was analysed. The shape evolution of the boundary was described by a time-dependent conformal mapping $z = w(\zeta;t)$ and an equation determining the time evolution of $w(\zeta;t)$ was also derived as in [Ric92].

The two examples discussed in this report show that in the case of an initial fluid domain that is described by a rational map of the unit disc, Richardson's method as explained in the first sections of this report, gives an effective solution procedure. Although it has been shown that most of the work can be carried out analytically, some of the first order differential equations arising, might need to be solved numerically. The objective was to determine the time evolution of the shape of the fluid domain. Although we have outlined the theory for Stokes flow, we have not mentioned the complications that arise when we consider multiply-connected fluid domains.

In summary, complex variable methods are very useful for free boundary problems governed by biharmonic equations. The main idea is to transform to a known, fixed domain (e.g. a unit disc) and recast the moving boundary aspect of the problem as a time-dependent conformal mapping whose coefficients have to be determined. Note that the boundary conditions will provide constraints on the conformal map and so the correct class of mapping functions that will give a solution should be chosen.

In this report, we follow closely previous literature on two-dimensional viscous sintering. This topic has received a lot of attention especially during the period 1990–2000 because of the remarkable exact solutions that can be found using complex variable methods. Although in practice a purely two-dimensional flow does not exist, Cummings and Howell in [CH99a] have shown how similar ideas and techniques can be applied to a more physically realistic three-dimensional geometry. A possible application of their work is quantifying three-dimensional effects in experiments on the sintering of finite cylinders.

Acknowledgments. Many thanks to Professor Peter Howell for suggesting this topic.

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Appendix A Map from unit disc to cardioid

In this Appendix we show that the map (3.2) for $b \to 1-$ tends to a cardioid, i.e. the mapping function

$$w(\zeta;t) = a\left(\zeta - \frac{1}{2}\zeta^2\right) \tag{A.1}$$

maps $|\zeta| < 1$ onto the cardioid. In particular, we show that the boundary of the unit disc maps to the cardioid curve. First let $z = e^{i\theta}$. Then we have

$$w(e^{i\theta}) = a\left(re^{i\theta} - \frac{1}{2}e^{2i\theta}\right) = a\left(\cos\theta + i\sin\theta - \frac{1}{2}\cos 2\theta - \frac{1}{2}i\sin 2\theta\right). \tag{A.2}$$

At this point use the double angle formulae

$$\cos 2\theta = 2\cos^2\theta \quad \text{and} \quad \sin 2\theta = 2\sin\theta\cos\theta. \tag{A.3}$$

Thus, the map becomes

$$\begin{aligned} w(e^{i\theta}) &= a\left(\cos\theta + i\sin\theta - \frac{1}{2}(2\cos^2\theta - 1) - i\sin\theta\cos\theta\right) \\ &= a(\cos\theta - \cos^2\theta) + \frac{a}{2} + ai\sin\theta(1-\cos\theta) \\ &= (1-\cos\theta)(a\cos\theta + ai\sin\theta) + \frac{a}{2} \\ &= ae^{i\theta}(1-\cos\theta) + \frac{a}{2}. \end{aligned} \tag{A.4}$$

This is the polar form for an equation of a cardioid but shifted by $\frac{a}{2}$ to the right.

Appendix B The boundary is a streamline

This means we must have $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ along ∂D where $\boldsymbol{n}$ is the normal to ∂D. See Figure B.1 for a schematic diagram of the problem at hand. To impose this, we first find $\boldsymbol{n}$ in terms of complex variables.

The tangent is given by $\frac{\mathrm{d}z}{\mathrm{d}s}$, where $s =$ arc length $= |\mathrm{d}z|$, and the normal is given by $\pm i\frac{\mathrm{d}z}{\mathrm{d}s}$. In terms of parameterisation via our conformal mapping $z(\zeta)$ we have the following by chain rule $\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\mathrm{d}z}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}s}$. Next we note that

$$\mathrm{d}s = |\mathrm{d}z| = \sqrt{\mathrm{d}z\mathrm{d}\overline{z}} = \sqrt{\frac{\mathrm{d}z}{\mathrm{d}\zeta}\mathrm{d}\zeta\overline{\left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)\mathrm{d}\zeta}} = \sqrt{|z'(\zeta)|^2\left(-\frac{1}{\zeta^2}\left(\mathrm{d}\zeta\right)^2\right)} = \pm i\frac{|z'(\zeta)|}{\zeta}\mathrm{d}\zeta. \tag{B.1}$$

This implies that

$$\frac{\mathrm{d}\zeta}{\mathrm{d}s} = \pm i\frac{\zeta}{|z'(\zeta)|} \tag{B.2}$$

and therefore the normal is

$$\text{normal} = \pm i\frac{\mathrm{d}z}{\mathrm{d}s} = \pm iz'(\zeta)\frac{\mathrm{d}\zeta}{\mathrm{d}s} = \pm\zeta\frac{z'(\zeta)}{|z'(\zeta)|}. \tag{B.3}$$

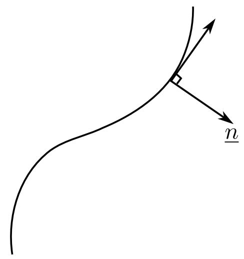

Figure B.1: Multiply by $-i$ to rotate it by 90°.

Appendix C First order ODE

In this Appendix we wish to solve a separable first order differential equation of the form

$$(2 + b^2)\frac{\mathrm{d}a}{\mathrm{d}t} + ab\frac{\mathrm{d}b}{\mathrm{d}t} = 0. \tag{C.1}$$

Separating the variables yields

$$-\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{b}{2 + b^2}\frac{\mathrm{d}b}{\mathrm{d}t}$$

and if we integrate this we obtain

$$-\log a = \frac{1}{2}\log(2 + b^2) + C, \tag{C.2}$$

where C is a constant.

Solving for the constant gives

$$\begin{aligned} -C &= \log a + \frac{1}{2}\log(2 + b^2) = \log a + \frac{1}{2}\log\left[2\left(1 + \frac{b^2}{2}\right)\right] \\ &= \log a + \log\left[2\left(1 + \frac{b^2}{2}\right)\right]^{1/2} = \log\left[a\sqrt{2}\sqrt{1 + \frac{b^2}{2}}\right]. \end{aligned}$$

This is equivalent to

$$\frac{e^{-C}}{\sqrt{2}} = a\sqrt{1 + \frac{b^2}{2}} \tag{C.3}$$

and if we now square both sides we obtain

$$a^2\left(1 + \frac{1}{2}b^2\right) = \text{constant}, \tag{C.4}$$

as required.