Modelling Bed Formation in Shallow Overland Flow

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Outline

Outline

Setting the scene

Figure 1 :
https://commons.wikimedia.org/wiki/File:Rill_network_from_Tyrone,_Ireland.jpg

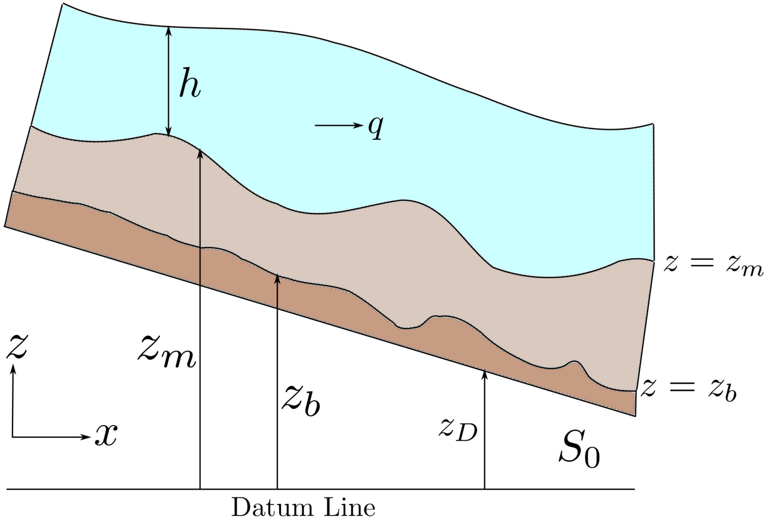
Setting the scene

http://www.youtube.com/watch?v=zgesmtodrUM&t=0m42s

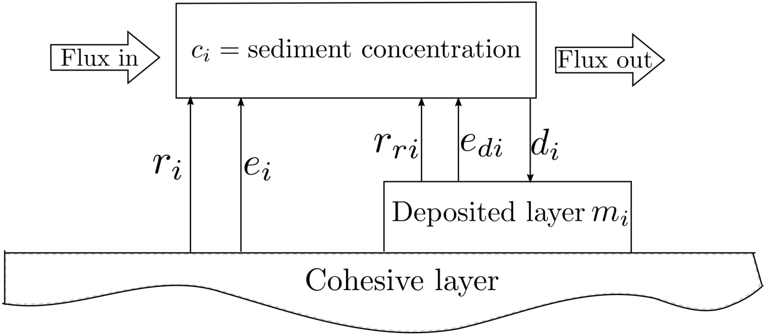
Figure 2 : https://commons.wikimedia.org/wiki/File:Ripples_mcr1.JPG

Outline

Setup



Setup



Flux in
c_i = sediment concentration
Flux out
r_i
e_i
r_{ri}
e_{di}
d_i
Deposited layer m_i
Cohesive layer

Equations

Water equations

$$\text{Mass: } \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0,$$

$$\text{Momentum: } \frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(u^2 h + \frac{g}{2} h^2) = gh(-\frac{\partial z_m}{\partial x} - S_f).$$

Equations

Sediment equations

$$\text{Suspended: } \frac{\partial}{\partial_t}(hc_i) + \frac{\partial}{\partial_x}(qc_i) = E_{ci} = r_i + r_{ri} - d_i,$$

$$\text{Bedform: } \frac{\partial m_i}{\partial_t} = E_{mi} = d_i - r_{ri}.$$

Equations

Interface equations

$$\text{Deposited:}\ \ \rho_s(1-\phi_m)\frac{\partial}{\partial t}(z_m - z_b) = \sum_{i=1}^{i_{max}} d_i - r_{ri},$$

$$\text{Cohesive:}\ \ \rho_s(1-\phi_b)\frac{\partial}{\partial t}(z_b - z_D) = -\sum_{i=1}^{i_{max}} r_i.$$

Summary

Summary of dimensional HR model equations

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \text{ Water mass cons.}$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{h} + \frac{g}{2}h^2\right) = gh(-\frac{\partial z_m}{\partial x} - S_f) \text{ Water mom. cons.}$$

$$\rho_s(1-\phi_m)\frac{\partial}{\partial t}(z_m - z_b) = \sum_i v_i c_i - \frac{F}{g}\frac{H}{h}\frac{\rho_s}{\rho_s - \rho}(\Omega - \Omega_{\mathrm{cr}})\frac{m_i}{m_t} \text{ Interface: deposited}$$

$$\rho_s(1-\phi_b)\frac{\partial}{\partial t}(z_b - z_D) = -\sum_i \frac{F}{J}p_i(1-H)(\Omega - \Omega_{\mathrm{cr}}) \text{ Interface: cohesive}$$

$$\frac{\partial(hc_i)}{\partial t} + \frac{\partial(qc_i)}{\partial x} = \frac{F}{J}p_i(1-H)(\Omega - \Omega_{\mathrm{cr}}) + \frac{F}{g}\frac{H}{h}\frac{\rho_s}{\rho_s - \rho}(\Omega - \Omega_{\mathrm{cr}})\frac{m_i}{m_t} - v_i c_i \text{ Susp. sed.}$$

$$\frac{\partial m_i}{\partial t} = v_i c_i - \frac{F}{g}\frac{H}{h}\frac{\rho_s}{\rho_s - \rho}(\Omega - \Omega_{\mathrm{cr}})\frac{m_i}{m_t}. \text{ Bed. sed.}$$

Outline

Non-dimensionalisation

Define scales: $\bar{x} = \frac{x}{x_0}$, $\bar{t} = \frac{t}{t_0}$, $\bar{c} = \frac{c}{c_0}$, etc.

Summary of dimensionless HR model equations

$$\varepsilon \frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \text{ Water mass cons.}$$

$$\varepsilon Fr^2 \frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(Fr^2 \frac{q^2}{h} + \frac{h^2}{2}\right) = h\left(-\frac{\partial z_m}{\partial x} + \delta \frac{u^2}{h^k}\right) \text{ Water mom. cons.}$$

$$\frac{\partial}{\partial t}(z_m - z_b) = \sum_i \left[v_i c_i - \frac{H}{h}(\Omega - \Omega_{\mathrm{cr}})\frac{m_i}{m_t}\right] \text{ Interface: deposited}$$

$$\beta \frac{\partial z_b - z_D}{\partial t} = -\sum_i Ap_i(1-H)(\Omega - \Omega_{\mathrm{cr}}) \text{ Interface: cohesive}$$

$$\varepsilon \frac{\partial (hc_i)}{\partial t} + \frac{\partial (qc_i)}{\partial x} = Ap_i(1-H)(\Omega - \Omega_{\mathrm{cr}}) + \frac{H}{h}(\Omega - \Omega_{\mathrm{cr}})\frac{m_i}{m_t} - v_i c_i \text{ Susp. sed.}$$

$$\frac{\partial m_i}{\partial t} = v_i c_i - \frac{H}{h}(\Omega - \Omega_{\mathrm{cr}})\frac{m_i}{m_t} \text{ Bed. sed.}$$

Non-dimensionalisation

The scales chosen are:

$t_0 = \frac{z_0 \rho_s (1-\phi_m)}{v_0 c_0}$, $h_0 = \frac{F}{g} \frac{\Omega_0}{v_0 c_0} \frac{\rho_s}{\rho_s - \rho}$, $x_0 = \frac{q_0}{v_0}$, $c_0 = \frac{F}{g} \frac{\Omega_0}{h_0 v_0} \frac{\rho_s}{\rho_s - \rho}$,
$m_0 = h_0 \rho_s (1 - \phi_m)$

With parameters:

$\varepsilon = \frac{h_0}{t_0 v_0}$, $A = \frac{F \Omega_0}{J v_0 c_0}$, $Fr^2 = \frac{u_0^2}{g h_0} = \frac{q_0^2}{g h_0^3}$, $\beta = \frac{1-\phi_D}{1-\phi_m}$, $\delta = \frac{C_r u_0^3}{h_0^k v_0}$

where the k is chosen depending on what friction law you use!

Typical values:

$\varepsilon = 0.043$, $A = 0.12$, $\beta = 1.5$, $Fr = 1.37$

Rankine-Hugoniot Conditions

$$\frac{dx}{dt} = \frac{[q]_-^+}{[\varepsilon h]_-^+} \tag{1}$$

$$\frac{dx}{dt} = \frac{\left[Fr^2 \frac{q^2}{h} + \frac{h^2}{2}\right]_-^+}{[\varepsilon Fr^2 q]_-^+} \tag{2}$$

$$\frac{dx}{dt} = \frac{[qc_i]_-^+}{[\varepsilon hc_i]_-^+} \tag{3}$$

Derivation of travelling wave solution

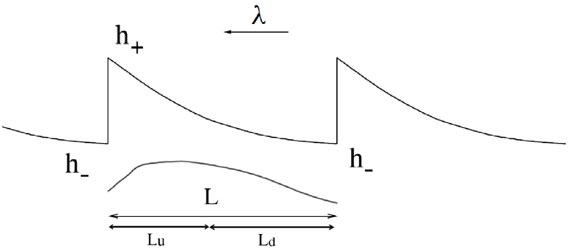
Consider the change of variables

$$\xi = x + \lambda t \tag{4}$$

$$\frac{\partial}{\partial t} = \lambda \frac{d}{d\xi}$$

$$\frac{\partial}{\partial x} = \frac{d}{d\xi}$$

and transform the dimensionless equations to be functions of ξ only



Derivation of travelling wave solution

And let $\varepsilon \to 0$ and $A \to 0$ to obtain the leading order system.

Leading order system

$$q = uh = 1$$

$$\frac{d}{d\xi}\left(\frac{Fr^2}{h} + \frac{h^2}{2}\right) = -h\left(\delta + \frac{1}{\lambda}\frac{dc}{d\xi} - \frac{\delta}{h^{10/3}}\right)$$

$$\frac{dc_i}{d\xi} = \Omega \frac{H}{h}\frac{m_i}{m_t} - v_i c_i$$

$$\lambda \frac{dm_i}{d\xi} = -\frac{dc_i}{d\xi}$$

$$\frac{dz_m}{d\xi} = \sum_{i=1}^{i_{\max}} \frac{dm_i}{d\xi}$$

Derivation of travelling wave solution

Rearranging the leading order St.Venant momentum equation we get

$$\frac{dh}{d\xi} = \frac{h^3}{h^3 - Fr^2}\left(\delta + \frac{1}{\lambda}\frac{dc}{d\xi} - \frac{\delta}{h^{10/3}}\right) \tag{4}$$

Problem!

Singularity develops as $h^3 \to Fr^2$

Derivation of travelling wave solution

Rearranging the leading order St.Venant momentum equation we get

$$\frac{dh}{d\xi} = \frac{h^3}{h^3 - Fr^2}\left(\delta + \frac{1}{\lambda}\frac{dc}{d\xi} - \frac{\delta}{h^{10/3}}\right) \tag{4}$$

Problem!

Singularity develops as $h^3 \to Fr^2$

Solution!

$$\left(\delta + \frac{1}{\lambda}\frac{dc}{d\xi} - \frac{\delta}{h^{10/3}}\right)\Bigg|_{h=h_c=Fr^{2/3}} = 0$$

Recap

- Non-dimensionalise
- Transform to travelling wave coordinates
- Find leading order system
- Remove the singularity from the St.Venant momentum equation

Rankine-Hugoniot conditions

- $c(\xi = 0) = c(\xi = L)$
- $\left(\frac{Fr^2}{h} + \frac{h^2}{2}\right)\Big|_{\xi=0} = \left(\frac{Fr^2}{h} + \frac{h^2}{2}\right)\Big|_{\xi=L}$

Main equations:

$$\frac{dh}{d\xi} = \frac{1}{\lambda}\frac{h^3}{h^3 - h_c^3}\left(\lambda\delta - \lambda\delta h^{-10/3} + h^{-13/3} - c\right),$$
$$\frac{dc}{d\xi} = h^{-13/3} - c\,.$$

Outline

Overview: The Problem at Hand

After all simplifications, the model reads (Manning's Law):

$$\frac{dh}{d\xi} = \frac{1}{\lambda}\frac{h^3}{h^3 - h_c^3}\left(\lambda\delta - \lambda\delta h^{-10/3} + h^{-13/3} - c\right), \tag{5}$$

$$\frac{dc}{d\xi} = h^{-13/3} - c\,. \tag{6}$$

Initial condition: $h(0) = h_c, c(0) = c_c$,

$$c_c = \lambda\delta(1 - h_c^{-10/3}) + h_c^{-13/3}\,. \tag{7}$$

Around the singularity, we use an analytic expansion, otherwise we integrate using ODE45. We are interested in one period only.

Form of the Solution

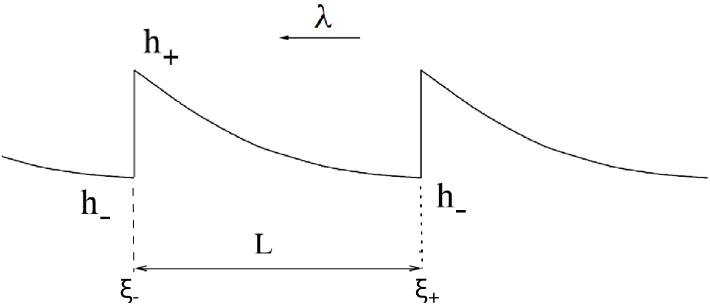


Figure 3 : Profile for $h(\xi)$

Challenges

We don't know λ, ξ_- and ξ_+. We have two conditions from the shock:

$$\text{Hydraulic jump condition} \qquad \left(\frac{Fr^2}{h} + \frac{h^2}{2}\right)\Big|_{\xi-}^{\xi+} = 0\,, \tag{8}$$

$$\text{Concentration condition} \qquad c(\xi_-) = c(\xi_+)\,. \tag{9}$$

However, we need one more condition to get a unique solution. Define $L = \xi_+ - \xi_-$

$$\text{Sediment Flux Mass Conservation} \qquad \frac{1}{L}\int_{\xi_-}^{\xi_+} c\, d\xi = 1 \tag{10}$$

Numerical Procedure

Idea: iterative procedure:

1. fix λ,
2. produce data: $h(\xi)$ and $c(\xi)$, analytic expansion + numerical integration,
3. check all three conditions, find best values of ξ_- and ξ_+ for given λ, record error.

Repeat steps 1 - 3 for a given range of λ, compare the final errors and chose the value of λ with the smallest error.

Output

lambda	tRight	tLeft	cDiff	QDiff	L2Error
5	0.2975	-0.265	0.05703	0.24357	0.25015
5.3333	0.2975	-0.2625	0.060432	0.23882	0.24635
5.6667	12.692	-0.7925	0.0073752	0.0013565	0.0074989
6	10.697	-1.025	0.016182	0.052391	0.054833
6.3333	0.2975	-0.2625	0.072135	0.22749	0.23865
6.6667	0.2975	-0.2625	0.076016	0.2237	0.23626
7	0.2975	-0.2625	0.079891	0.2199	0.23397
7.3333	0.2975	-0.2625	0.08376	0.2161	0.23177
7.6667	0.2975	-0.2625	0.087625	0.2123	0.22967
8	0.2975	-0.2625	0.091485	0.20849	0.22768

```
Best value of lambda found: 5.667, with an L2 error of 0.00750 and a time span from -0.79 to 12.69.
 Computation time needed: 4.9 s.
>>
```

Figure 4 : Sample output

Outline

The numerical experiment

Free parameters for the numerical model:

1. S_0: the average bed-slope
2. q_0: the flow rate

We fix one parameter and vary the other.

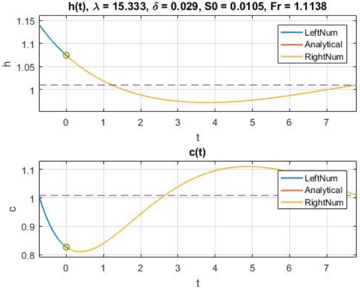
Two Parameter Family

Two parameter family of solution. Two of the most important parameters (using Manning's law) are:

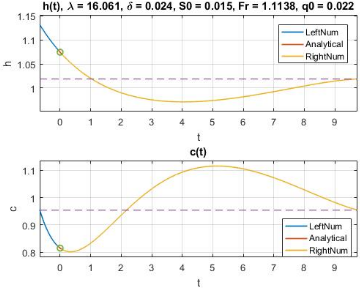
1. The Froude number (Fr): $Fr^2 = \frac{S_0^{9/10} q_0^{1/5}}{g n^{9/5}}$
2. Delta (related to roughness): $\delta = \frac{S_0^{13/10} q_0^{2/5}}{v_0 g n^{3/5}}$

Results

Results: 1 Left:$S_0 = 0.0105$ Right:$q_0 = 0.022m^3/s$

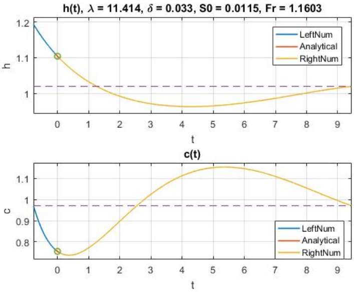


(a) Above: wave-height against ξ; Below: Concentration against ξ



(b) Above: wave-height against ξ; Below: Concentration against ξ

Results: 2 Left:$S_0 = 0.0115$ Right:$q_0 = 0.033m^3/s$



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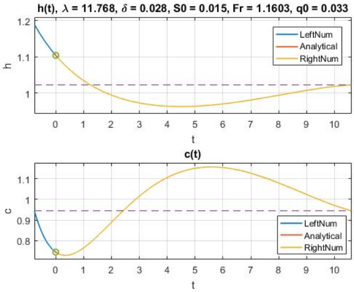
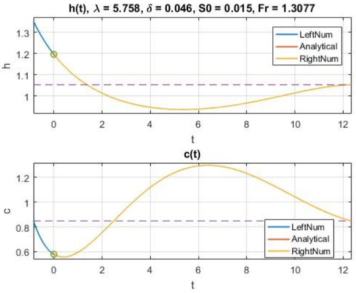
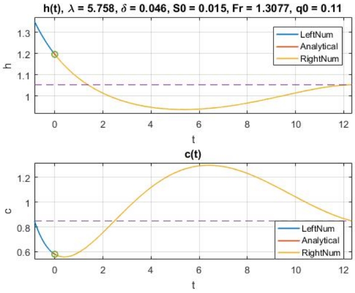


(b) Above: wave-height against ξ; Below: Concentration against ξ

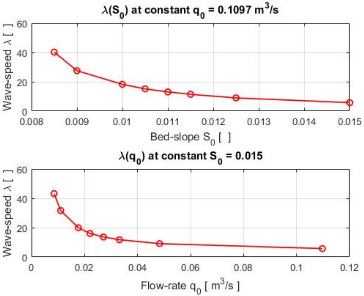
Results: 3 Left:$S_0 = 0.015$ Right:$q_0 = 0.1097 m^3/s$



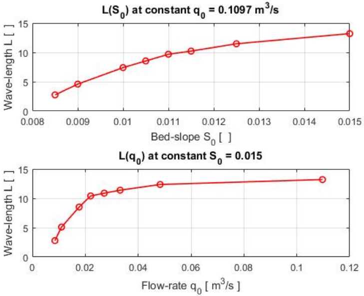
(a) Above: wave-height against ξ; Below: Concentration against ξ



(b) Above: wave-height against ξ; Below: Concentration against ξ



(a) Figure above: $\lambda(S_0)$; Figure below: $\lambda(q_0)$;



(b) Figure above: $L(S_0)$; Figure below: $L(q_0)$;

Conclusions and Further Work

Project summary:

1. Modelled the bedform evolution process

Further Work:

1. Multiple particle size classes

Thank you for your attention

Thank you! Any questions are welcome.