Math UA 262

Section 1

SPRING 2023

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First-order differential equations

Section 1.1 : Introduction

A differential equation is a relationship between a function of time and its derivatives

Examples.

$$\frac{dy}{dt} = \cos(t) + 3y \qquad \text{first-order diff. eqn.}$$

$$\frac{d^2y}{dt^2} = e^{-y} + t^2 + \frac{dy}{dt} \qquad \text{second-order diff. eqn.}$$

The order of a differential equation is the order of the highest derivative of the function y that appears in the equation.

A solution of a differential equation is a continuous function $y(t)$ which together with its derivatives satisfies the relationship.

e.g. Show that $y(t) = 2\sin t - \frac{1}{3}\cos 2t$ is a solution to the equation

$$\text{LHS} = \frac{d^2y}{dt^2} + y = \cos 2t = \text{RHS}$$

Show that LHS = RHS

$$\frac{dy}{dt} = 2\cos t + \frac{2}{3}\sin 2t$$

$$\frac{d^2y}{dt^2} = -2\sin t + \frac{4}{3}\cos 2t$$

$$\text{LHS} = -\cancel{2\sin t} + \frac{4}{3}\cos 2t + \left(\cancel{2\sin t} - \frac{1}{3}\cos 2t\right)$$

$$= \cos 2t$$

$$= \text{RHS}$$

Thus $y(t) = 2\sin t - \frac{1}{3}\cos 2t$ is a solution to the given diff. eqn.

Section 1.2: First-order linear differential equations

Assume that our equation can be written as

$$\boxed{\frac{dy}{dt} = f(t,y).}$$

← Given $f(t,y)$, find all functions $y(t)$ that satisfy this diff. eqn.

Defn: The general first-order linear differential equation is

$$\boxed{\frac{dy}{dt} + a(t)\,y = b(t)}$$

This is linear because the dependent variable y appears by itself. (That is, no terms like e^{-y}, y^2, $\cos y$, etc in the equation)

e.g. $\frac{dy}{dt} = y^2 + \sin t + 2$ (nonlinear because of y^2)

$\frac{dy}{dt} = \cos(t)\,y + 3$ (linear)

$\frac{dy}{dt} = \cos y + t$ (non linear because of $\cos y$).

Defn The equation

$$\boxed{\frac{dy}{dt} + a(t)\,y = 0}$$

(so with $b(t) = 0$ from above) is called a homogeneous first-order linear differential equation, whereas when $b(t) \neq 0$ from above, it is called the nonhomogeneous first-order linear differential equation.

e.x. Solve $\frac{dy}{dt} + a(t)y = 0$

Use separation of variables:

$$\frac{dy}{dt} = -a(t)y$$

$$\int \frac{dy}{y} = \int -a(t)\,dt$$

$$\ln|y| = -\int a(t)\,dt + C$$ arbitrary constant of integration

Now taking exponentials of both sides.

$$|y| = e^{-\int a(t)dt + C} = e^{-\int a(t)dt}\,\boxed{e^{C}}$$ some constant, let's call it A

$$\Rightarrow \quad |y| = Ae^{-\int a(t)dt}$$

$$\left|ye^{\int a(t)dt}\right| = A$$

Notice that we have a continuous function of time on the LHS, i.e. $y(t)\,e^{\int a(t)dt}$ but on the RHS we have a constant

But if the absolute value of a continuous function $g(t)$ is constant then g itself must be constant. Why?

If g is <u>not</u> a constant there exist two different times t_1 and t_2 for which $g(t_1) = c$ and $g(t_2) = -c$. By the IVT g must achieve all values between $-c$ and $+c$ which is impossible if $|g(t)| = c$.

$\Rightarrow$ We get the equation $\boxed{y(t) = Ae^{-\int a(t)dt}}$

This is the general solution of the homogeneous equation.

The constant A is arbitrary. Thus, $\frac{dy}{dt} + a(t)y = 0$ has infinitely many solutions ; for each value of A we obtain a distinct solution $y(t)$.

e.g. Find the general solution to $\frac{dy}{dt} + 3ty = 0$

Here $a(t) = 3t$ and the general solution is

$$y(t) = Ae^{-\int a(t)\,dt}$$

Thus $y(t) = Ae^{-\int 3t\,dt} = Ae^{-3t^2/2}$

e.g. Determine the behavior as $t \to \infty$ of all solutions of the equation

$$\frac{dy}{dt} + ay = 0, \quad a \text{ const.}$$

The general solution is

$$y(t) = Ae^{-\int a(t)\,dt} = Ae^{-at}$$

So if $a < 0 \Rightarrow$ as $t \to \infty$, $y(t) \to \infty$ (with the exception of $y = 0$)

if $a > 0 \Rightarrow$ as $t \to \infty$, $y(t) \to 0$

Usually, we look for a SPECIFIC solution $y(t)$ which at some initial time t_0 has the value y_0. I.e.

Solve $\boxed{\frac{dy}{dt} + a(t)y = 0 \quad , \quad y(t_0) = y_0}$

This is called an <u>initial - value problem.</u>

$$\frac{dy}{dt} = -a(t)\,y$$

$$\frac{dy}{y} = -a(t)\,dt$$

Now integrate both sides between t_0 and t.

$$\int_{t_0}^{t} \frac{dy}{y} = -\int_{t_0}^{t} a(s)\,ds$$

$$\Big[\ln|y|\Big]_{t_0}^{t} = -\int_{t_0}^{t} a(s)\,ds$$

$$\underbrace{\ln|y(t)| - \ln|y(t_0)|}_{\overset{\shortparallel}{\ln\left|\frac{y(t)}{y(t_0)}\right|}} = -\int_{t_0}^{t} a(s)\,ds$$

Taking exponentials on both sides yields

$$\left|\frac{y(t)}{y(t_0)}\right| = e^{-\int_{t_0}^{t} a(s)\,ds}$$

$$\left|\frac{y(t)}{y(t_0)}\, e^{\int_{t_0}^{t} a(s)\,ds}\right| = 1$$

Q: How do we decide whether it's identically 1 or -1 ?

A: let's evaluate at $t = t_0$:

$$\underset{1}{\cancel{\frac{y(t_0)}{y(t_0)}}}\, e^{\underbrace{\int_{t_0}^{t_0} a(s)\,ds}_{\overset{\shortparallel}{0}}} = 1\cdot e^0 = 1 \quad ✓$$

$$\lrcorner$$

Hence $$\frac{y(t)}{y(t_0)}\, e^{\int_{t_0}^{t} a(s)\,ds} = 1$$

$$\Rightarrow \quad y(t) = y(t_0)\, e^{-\int_{t_0}^{t} a(s)\,ds}$$

where $y(t_0) = y_0$ and so we get $\boxed{y(t) = y_0\, e^{-\int_{t_0}^{t} a(s)\,ds}}$

__Example__: Solve the IVP:

$$\frac{dy}{dt} + (\cos t)\, y = 0 \quad \text{with } y(0) = \frac{3}{2}$$

Solution is
$$y(t) = y_0\, e^{-\int_{t_0}^{t} a(s)\,ds} = \frac{3}{2}\, e^{-\int_{t_0}^{t} \cos(s)\,ds} = \frac{3}{2}\, e^{-\sin(t) + \sin(\underset{0}{\cancel{t_0}})}$$

since $t_0 = 0$

$$\Rightarrow \quad y(t) = \frac{3}{2} e^{-\sin(t)}$$

Method of integrating factor

Now back to nonhomogeneous equations...

$$\frac{dy}{dt} + a(t)\,y = b(t).$$

Think of expressing it as $\frac{d}{dt}(\star) = b(t)$ and then integrating both sides to get the solution.

So we need to ask: What should (t) be such that its derivative w.r.t. t gives the LHS $\frac{dy}{dt} + a(t)y$?

Start with: $$\boxed{\frac{dy}{dt} + a(t)y = b(t)}$$

Multiply both sides by a cts fcn $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + \mu(t)a(t)y = \mu(t)b(t)$$

We will choose $\mu(t)$ so that $\mu(t)\frac{dy}{dt} + \mu(t)a(t)y$ will be the derivative of $\mu(t)y$ iff $\frac{d\mu(t)}{dt} = a(t)\mu(t)$

$$\frac{d}{dt}(\mu(t)y) = \boxed{\frac{d\mu(t)}{dt}}y + \mu(t)\frac{dy}{dt}$$

comparing this to $\mu(t)\frac{dy}{dt} + \boxed{\mu(t)a(t)}y$ we have that $\frac{d\mu(t)}{dt} = \mu(t)a(t)$.

But $\frac{d\mu}{dt} = \mu(t)a(t)$ is a first-order, linear homogeneous equation for $\mu(t)$ $\left(\frac{d\mu}{dt} - a(t)\mu = 0\right)$ and we know how to solve it. I.e.

$$\boxed{\mu(t) = e^{\int a(t)\,dt}}$$

INTEGRATING FACTOR

So with this $\mu(t)$ we have

$$\mu(t)\frac{dy}{dt} + \mu(t)a(t)y = \mu(t)b(t)$$

$$\frac{d}{dt}(\mu(t)y) = \mu(t)b(t)$$

and now integrate this:

$$\mu(t)y = \int \mu(t)b(t)\,dt + c$$

Equivalently, this is

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)b(t)\,dt + c\right)$$

$$= \frac{1}{e^{\int a(t)dt}}\left(\int \mu(t)b(t)\,dt + c\right)$$

$$= e^{-\int a(t)dt}\left(\int \mu(t)b(t)\,dt + c\right) \qquad (*)$$

With an initial condition, we would integrate from t_0 to t to get

$$\mu(t)y - \mu(t_0)y_0 = \int_{t_0}^{t} \mu(s)b(s)\,ds$$

$$\Rightarrow \quad y = \frac{1}{\mu(t)}\left[\mu(t_0)y_0 + \int_{t_0}^{t} \mu(s)b(s)\,ds\right] \qquad (**)$$

Note. Do <u>not</u> memorize $(*)$ and $(**)$. Instead, solve all nonhomogeneous equations by:

(1) Multiplying both sides by $\mu(t)$.

(2) Writing the new LHS as the derivative of $\mu(t)y(t)$

(3) Integrating both sides of the equation.

Examples. Find the general solution of $\frac{dy}{dt} - 2ty = t$

$$\frac{dy}{dt} + a(t)y = b(t)$$

Here $a(t) = -2t$.

Integrating factor (I.F.): $\mu(t) = e^{\int a(t)dt} = e^{\int -2t\,dt} = e^{-t^2}$

Multiply both sides of $\frac{dy}{dt} - 2ty = t$ by I.F.

$$\underbrace{e^{-t^2}\frac{dy}{dt} - e^{-t^2}2ty}_{\text{product rule and chain rule}} = e^{-t^2}t$$

$$\Rightarrow \quad \frac{d}{dt}(e^{-t^2}y) = e^{-t^2}t$$

Now integrate both sides w.r.t. t:

$$e^{-t^2}y = \int e^{-t^2}t\,dt$$

$$e^{-t^2}y = -\frac{1}{2}e^{-t^2} + c$$

$$y = e^{t^2}\left[-\frac{1}{2}e^{-t^2} + c\right]$$

$$\boxed{y = -\frac{1}{2} + ce^{t^2}}$$

Example Find the solution to the I.V.P.

$$\frac{dy}{dt} + 2ty = t \ , \quad y(1) = 2$$

I.F. $\quad \mu(t) = e^{\int 2t\,dt} = e^{t^2}$

Multiply both sides by e^{t^2}:

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = te^{t^2}$$

$$\frac{d}{dt}\left(e^{t^2}y\right) = te^{t^2}$$

Integrate both sides w.r.t. t from t_0 to t.

$$e^{t^2}y - e^{t_0^2}y_0 = \int_{t_0}^{t} se^{s^2}\,ds$$

where $t_0 = 1$, $y_0 = 2$

$$e^{t^2}y - e^{1}(2) = \int_{1}^{t} se^{s^2}\,ds$$

$$e^{t^2}y - 2e = \left[\frac{1}{2}e^{s^2}\right]_1^t$$

$$e^{t^2}y - 2e = \frac{1}{2}e^{t^2} - \frac{1}{2}e$$

$$y = e^{-t^2}\left[\frac{1}{2}e^{t^2} - \frac{1}{2}e + 2e\right]$$

$$\boxed{y = \frac{1}{2} + \frac{3}{2}e^{-t^2+1}}$$

Section 1.4: Separable equations

We have already used this in Sec. 1.2 but let's look at the general method:

Solve the general differential equation

$$\boxed{\frac{dy}{dt} = \frac{g(t)}{f(y)}}$$

where f and g are continuous functions of y and t.

Any equation which can be put into this form, is said to be <u>separable</u>.

Multiply both sides by $f(y)$:

$$f(y)\frac{dy}{dt} = g(t)$$

$$\frac{d}{dt}\left(F(y(t))\right) = g(t)$$

where $F(y)$ is an antiderivative of $f(y)$, $F(y) = \int f(y)\,dy$.

Upon integration w.r.t. t we get:

$$F(y(t)) = \int g(t)\,dt + C$$

Then solve this for $y(t)$ to find the general solution.

<u>Example</u>. Find the general solution of $\frac{dy}{dt} = \frac{t^2}{y^2}$.

$$y^2\frac{dy}{dt} = t^2$$

$$\frac{d}{dt}\left(\frac{y^3}{3}\right) = t^2$$

$$\frac{y^3}{3} = \int t^2 dt + c$$

$$\frac{y^3}{3} = \frac{t^3}{3} + c$$

$$\boxed{y = \left[\, t^3 + 3c \right]^{1/3}}$$

Example. Solve the I.V.P.

$$e^y \frac{dy}{dt} - (t+t^3) = 0 \;, \; y(1) = 1$$

Rearrange into the form $\dfrac{dy}{dt} = \dfrac{g(t)}{f(y)}$.

$$e^y \frac{dy}{dt} = t + t^3$$

$$\frac{d}{dt}\left(e^{y(t)}\right) = t + t^3$$

$$e^y = \int (t + t^3) dt$$

$$e^y = \frac{t^2}{2} + \frac{t^4}{4} + c$$

Now since $y(1) = 1$ we can determine the constant of integration c

$$e^1 = \frac{1}{2} + \frac{1}{4} + c \Rightarrow e = \frac{3}{4} + c \Rightarrow c = e - \frac{3}{4}$$

$$e^{y} = \frac{t^2}{2} + \frac{t^4}{4} + e - \frac{3}{4}$$

$$\boxed{y = \ln\left|\frac{t^2}{2} + \frac{t^4}{4} + e - \frac{3}{4}\right|}$$

<u>Example</u> Solve the I.V.P. $\frac{dy}{dt} = (1+y)t$, $y(0) = -1$

$$\frac{1}{1+y}\frac{dy}{dt} = t$$

It's clear from this that if we plug in the initial condition $y(0) = -1$ we will be dividing by 0.

<u>But</u>, we can see that $\boxed{y(t) = -1}$ is a solution of this I.V.P.

Check that it's a solution

$$LHS = \frac{dy}{dt} = \frac{d}{dt}(-1) = 0$$

$$RHS = (1+y)t = \left(\cancelto{0}{1+(-1)}\right)t = 0$$

$\Rightarrow$ $LHS = RHS$

$\therefore$ $y(t) = -1$ is a solution.

Later in the class we will show that it's the only solution.

Section 1.9 Exact equations, and why we cannot solve very many differential equations

Generally, we can solve all differential equations of the form

$$\frac{d}{dt}\phi(t,y)=0$$

for some $\phi(t,y)$. To solve this we integrate both sides w.r.t. t to obtain

$$\phi(t,y) = \text{constant}.$$

Then, if possible, solve for y in terms of t.

Example. Solve $\cos(t+y) + [1+\cos(t+y)]\frac{dy}{dt} = 0$

$$\Rightarrow \frac{d}{dt}[y+\sin(t+y)]=0$$

Verification: $$\frac{dy}{dt} + \underbrace{\cos(t+y)\cdot 1 + \cos(t+y)\frac{dy}{dt}}_{\substack{\text{from } \frac{d}{dt}(\sin(t+y)) \\ \text{once diff. the } t \text{ term \& then the } y \text{ term}}} = 0$$

$$\frac{dy}{dt}[1+\cos(t+y)] + \cos(t+y) = 0$$

Thus from $\frac{d}{dt}[y+\sin(t+y)]=0$ we see that the solution is

$$y+\sin(t+y) = \text{const}.$$

But this is an implicit equation in y that cannot be solved for y explicitly in time.

Which equations can be put into the form $\frac{d}{dt}\phi(t,y)=0$?

From the chain rule: $\frac{d}{dt}\phi(t,y(t)) = \underbrace{\frac{\partial\phi}{\partial t}}_{\downarrow} + \underbrace{\frac{\partial\phi}{\partial y}}_{\searrow}\frac{dy}{dt}.$

$$= M(t,y) + N(t,y)\frac{dy}{dt}$$

So a diff. eqn. can be written in the form $\frac{d}{dt}\phi(t,y)=0$ if and only if there exists a fcn $\phi(t,y)$ s.t.

$$\boxed{M(t,y)=\frac{\partial\phi}{\partial t}} \text{ and } \boxed{N(t,y)=\frac{\partial\phi}{\partial y}}.$$

Does such a function $\phi(t,y)$ exist?

Theorem: Let $M(t,y)$ and $N(t,y)$ be continuous + have continuous partial derivatives w.r.t. t and y in the rectangle R consisting of those points (t,y) with $a<t<b$ and $c<y<d$. There exists a function $\phi(t,y)$ s.t. $M(t,y)=\frac{\partial\phi}{\partial t}$ and $N(t,y)=\frac{\partial\phi}{\partial y}$ iff

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}}$$

in R.

Proof. $M(t,y)=\frac{\partial\phi}{\partial t}$ for some $\phi(t,y)$ iff $\phi(t,y)=\int M(t,y)dt + h(y)$

↗ arbitrary function of y

Taking partial derivatives on both sides of this w.r.t. y, we get

$$\frac{\partial\phi}{\partial y} = \int \frac{\partial M}{\partial y}(t,y)\, dt + h'(y)$$

Thus, this can be equal to $N(t,y)$ iff

$$N(t,y) = \int \frac{\partial M}{\partial y}(t,y)\,dt + h'(y)$$

$$\Rightarrow \quad \underset{\text{fcn of y only}}{h'(y)} = \underbrace{N(t,y) - \int \frac{\partial M}{\partial y}(t,y)\,dt}_{\text{fcns of y and t}}$$

But this cannot be true which means that the RHS also has to be a function of y alone. I.e

$$\frac{\partial}{\partial t}\left[N(t,y) - \int \frac{\partial M}{\partial y}(t,y)\,dt \right] = \frac{\partial N}{\partial t} - \frac{\partial M}{\partial y} = 0$$

Therefore, if $\frac{\partial N}{\partial t} \neq \frac{\partial M}{\partial y}$ then there is no function $\phi(t,y)$ s.t. $M = \frac{\partial \phi}{\partial t}$, $N = \frac{\partial \phi}{\partial y}$. However, if $\frac{\partial N}{\partial t} = \frac{\partial M}{\partial y}$ then we can solve for

$$h(y) = \int \left[N(t,y) - \int \frac{\partial M}{\partial y}(t,y)\,dt \right] dy$$

This implies that $\quad M = \frac{\partial \phi}{\partial t}, \; N = \frac{\partial \phi}{\partial y}$ with

$$\phi(t,y) = \int M(t,y)\,dt + \int \left[N(t,y) - \int \frac{\partial M(t,y)}{\partial y}\,dt \right] dy$$

(Recall that $\frac{\partial \phi}{\partial y} = \int \frac{\partial M}{\partial y}dt + h'(y) \Rightarrow \phi = \int M\,dt + h(y)$)

Definition. The diff. eqn. $\boxed{M(t,y) + N(t,y)\frac{dy}{dt} = 0}$ is said to be **exact** if $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}}$.

Practically how do we choose $\phi(t,y)$?

Method 1: The equation $M(t,y) = \frac{\partial \phi}{\partial t}$ determines $\phi(t,y)$ up to an arbitrary fcn of y alone, i.e.

$$\phi(t,y) = \int M(t,y)\,dt + h(y)$$

We then take the derivative of this w.r.t. the other variable, i.e. y

$$\frac{\partial \phi}{\partial y} = \int \frac{\partial M}{\partial y}\, dt + h'(y)$$

$$\Rightarrow \quad h'(y) = \frac{\partial \phi}{\partial y} - \int \frac{\partial M}{\partial y}\, dt = N(t,y) - \int \frac{\partial M}{\partial y}\, dt$$

which means that h(y) can be determined from this equation

Method 2: If $N(t,y) = \frac{\partial \phi}{\partial y}$ then

$$\phi(t,y) = \int N(t,y)\,dy + \underbrace{k(t)}_{\text{arbitrary fcn of } t \text{ alone}}$$

Now differentiate w.r.t. the other variable, i.e. t

$$M = \frac{\partial \phi}{\partial t} = \int \frac{\partial N}{\partial t}\, dy + k'(t)$$

$$\Rightarrow \quad k'(t) = M(t,y) - \int \frac{\partial N(t,y)}{\partial t}\, dy$$

Method 3: $\frac{\partial \phi}{\partial t} = \overset{\textcircled{1}}{M(t,y)}$ and $\frac{\partial \phi}{\partial y} = \overset{\textcircled{2}}{N(t,y)}$

$$\begin{array}{l}\Rightarrow \\ \Rightarrow\end{array} \left[\begin{array}{l}\phi(t,y) = \int M(t,y)\,dt + h(y) \\ \phi(t,y) = \int N(t,y)\,dy + k(t)\end{array}\right] \begin{array}{l}\text{Integrating } \textcircled{1} \text{ w.r.t. } t \\ \text{Integrating } \textcircled{2} \text{ w.r.t. } y\end{array}$$

Then we can usually determine h(y) and k(t) by inspection.

Example

Find the general solution to

$$\underbrace{3y+e^{t}}_{M(t,y)}+\underbrace{(3t+\cos y)}_{N(t,y)}\frac{dy}{dt}=0$$

This equation is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial t}$

$$\frac{\partial M}{\partial y}=3 \quad \text{and} \quad \frac{\partial N}{\partial t}=3 \quad \text{so this equation is exact.}$$

Thus, there exist a ϕ s.t.

$$M=\frac{\partial \phi}{\partial t} \quad \text{and} \quad N=\frac{\partial \phi}{\partial y}$$

i.e.

$$\frac{\partial \phi}{\partial t}=3y+e^{t}$$

$$\frac{\partial \phi}{\partial y}=3t+\cos y$$

Let's find $\phi(t,y)$ now...

Method 1: $\frac{\partial \phi}{\partial t}=3y+e^{t} \Rightarrow \phi(t,y)=\int(3y+e^{t})dt+h(y)$

$$=3yt+e^{t}+h(y)$$

Differentiate this wrt y:

$$\frac{\partial \phi}{\partial y}=3t+\underset{\sim}{h'(y)}=3t+\underset{\sim}{\cos y} \quad \text{(from def}^{n}\text{ of N)}$$

$\Rightarrow h'(y)=\cos(y)$

$h(y)=\sin(y)$

Thus, $\phi(t,y) = 3yt + e^t + \sin y$.

Method 2 $\frac{\partial \phi}{\partial y} = 3t + \cos y$. Integrate wrt y to get

$$\phi(t,y) = 3ty + \sin y + k(t)$$

Differentiate wrt t: $\frac{\partial \phi}{\partial t} = 3y + k'(t) = 3y + e^t \ (=M)$

Thus $k'(t) = e^t \Rightarrow k(t) = e^t$

So we have $\phi(t,y) = 3ty + \sin y + e^t$ (which is the same as the answer from Method 1).

Method 3. $\phi(t,y) = e^t + 3ty + h(y)$

$\phi(t,y) = 3ty + \sin y + k(t)$

Now comparing this two it's clear that $h(y) = \sin y$ and $k(t) = e^t$

Hence, again, $\phi(t,y) = e^t + 3ty + \sin y$.

Example. Find the solution of the IVP

$$4t^3 e^{t+y} + t^4 e^{t+y} + 2t + (t^4 e^{t+y} + 2y)\frac{dy}{dt} = 0, \quad y(0) = 1$$

Verify that this is an exact equation

Is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$? $M = 4t^3 e^{t+y} + t^4 e^{t+y} + 2t$

$N = t^4 e^{t+y} + 2y$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 4t^3 e^{t+y} + t^4 e^{t+y} \\ \frac{\partial N}{\partial t} &= 4t^3 e^{t+y} + t^4 e^{t+y} \end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \Rightarrow \text{exact equation}$$

Now integrate either $N = \frac{\partial \phi}{\partial y}$ wrt y or $M = \frac{\partial \phi}{\partial t}$ wrt t:

Integrating $N = \frac{\partial \phi}{\partial y}$ wrt y is easier:

$$\phi = \int N \, dy = \int (t^4 e^{t+y} + 2y) \, dy$$

$$= t^4 e^{t+y} + y^2 + k(t)$$

Now differentiate this wrt t:

$$\frac{\partial \phi}{\partial t} = 4t^3 e^{t+y} + t^4 e^{t+y} + k'(t) = M = 4t^3 e^{t+y} + t^4 e^{t+y} + 2t$$

So comparing the two we see that $k'(t) = 2t \Rightarrow k(t) = t^2$

Thus the general solution is $\phi = t^4 e^{t+y} + y^2 + t^2 = c$ (since $\phi = c$ is the sol^n)

Now using the initial condition $y(0) = 1$ we have

$$0 + 1^2 + 0^2 = c$$

$$c = 1$$

$$\Rightarrow \boxed{t^4 e^{t+y} + y^2 + t^2 = 1}$$

Suppose that the equation now is not exact. Can we make it exact?

Yes, using a similar procedure to the integrating factor from before.

I.F. $\mu(t) = e^{\int R(t) dt}$ where $R(t) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)$

Example. Find the general solution of

$$\frac{y^2}{2}+2ye^t+(y+e^t)\frac{dy}{dt}=0$$

$$M=\frac{y^2}{2}+2ye^t=\frac{\partial\phi}{\partial t} \qquad \frac{\partial M}{\partial y}=y+2e^t$$

$$N=y+e^t=\frac{\partial\phi}{\partial y} \qquad \frac{\partial N}{\partial t}=e^t$$

$$\left.\begin{matrix}\frac{\partial M}{\partial y}=y+2e^t\\ \frac{\partial N}{\partial t}=e^t\end{matrix}\right\} \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial t}$$

$\therefore$ Not exact

So we'll now find the integrating factor:

$$R(t)=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)=\frac{1}{y+e^t}\left(y+2e^t-e^t\right)$$

$$=\frac{y+e^t}{y+e^t}$$

$$=1$$

So the I.F. is $e^{\int R(t)\,dt}=e^t$ and we multiply the diff. eqn to obtain the <u>exact</u> form of the eqn

$$e^t\frac{y^2}{2}+2ye^{2t}+(e^ty+e^{2t})\frac{dy}{dt}=0$$

Check that it's exact now:

$$\frac{\partial M}{\partial y}=e^ty+2e^{2t} \quad \checkmark$$

$$\frac{\partial N}{\partial t}=e^ty+2e^{2t} \quad \checkmark$$

$$\phi=\int M\,dt=e^t\frac{y^2}{2}+ye^{2t}+h(y)$$

$$\frac{\partial\phi}{\partial y}=e^ty+e^{2t}+h'(y)=N=e^ty+e^{2t} \Rightarrow h'(y)=0 \Rightarrow h(y)=k$$

$\phi = e^t \frac{y^2}{2} + ye^{2t} = c$ (quadratic eqn for y) ✓2

Thus the solution is

$$y(t) = \frac{-e^{2t} \pm \sqrt{(e^{2t})^2 - \cancel{4}^{2}\left(\frac{e^t}{\cancel{2}}\right)(-c)}}{\cancel{2}\left(e^t/\cancel{2}\right)}$$

$$= e^t \pm \frac{\sqrt{e^{4t} + 2ce^t}}{e^t}$$

$$= e^t \pm \frac{\sqrt{\cancel{e^{2t}}(e^{2t} + 2ce^{-t})}}{\cancel{e^t}}$$

$$= e^t \pm \sqrt{e^{2t} + 2ce^{-t}}$$

Section 1.10: The existence-uniqueness theorem; Picard iteration

Consider the IVP $\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$

Q (1) Does this IVP have solutions?
(2) How many solutions?

ALGORITHM FOR PROVING EXISTENCE OF A SOLUTION $y(t)$

(a) Construct a sequence of functions $y_n(t)$ which come closer and closer to solving the IVP.

→ (b) Show that the sequence of functions $y_n(t)$ has a limit $y(t)$ on a suitable interval $t_0 \leq t \leq t_0 + \alpha$

→ (c) Prove that $y(t)$ is a solution of the IVP on this interval.

(a) Write the IVP as $\boxed{y(t) = L(t, y(t))}$ where L may depend explicitly on y and on integrals of functions of y.

$$y' = f(t,y)$$

Now we can integrate this wrt t: $\displaystyle\int_{t_0}^{t} \frac{dy}{ds}\, ds = \int_{t_0}^{t} f(s, y(s))\, ds$

$$\Rightarrow \quad y(t) - \underset{y_0}{y(\cancel{t_0})} = \int_{t_0}^{t} f(s, y(s))\, ds$$

$$\Rightarrow L(t, y(t)) = \boxed{y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds} \quad (*)$$ Integral equation

Conversely, if $y(t)$ is continuous and satisfies this then $\frac{dy}{dt} = f(t, y(t))$

Scheme for constructing a sequence of approximate solutions $y_n(t)$.

Our guess for $y_0(t)=y_0$. To check if $y_0(t)$ is a solution of (*) we compute

$$y_1(t)=y_0+\int_{t_0}^{t} f(s,y_0(s))\,ds$$

If $y_1(t)=y_0$, then $y(t)=y_0$ is indeed a solution of (*)

If not, then we try $y_1(t)$ as our next guess. To check if that is a solution of (*) we compute

$$y_2(t)=y_0+\int_{t_0}^{t} f(s,y_1(s))\,ds$$

Thus, we define a sequence of functions $y_1(t), y_2(t), \dots,$ where

$$y_{n+1}(t)=y_0+\int_{t_0}^{t} f(s,y_n(s))\,ds$$

↑

successive approximations/

Picard iterates

These Picard iterates always converge on a suitable interval to a solution $y(t)$ of (*)

Example. Compute the Picard iterates for the IVP

$$y'=y,\ \boxed{y(0)=1}$$

$y'=f(t,y)$

and show that they converge to the solution $y(t)=e^t$.

for this example

$\boxed{f=y}$ the right-hand side

$$y=L(t,y(t)) \quad \text{where} \quad L(t,y(t))=y_0+\int_{t_0}^{t}\underline{f(s,y(s))}\,ds$$

$$\underline{y_1(t)}=1+\int_0^t \underline{y_0}\,ds$$

$$=1+\int_0^t 1\,ds$$

$$=1+t$$

$$y_2(t) = 1 + \int_0^t y_1(s)\,ds$$
$$= 1 + \int_0^t (1+s)\,ds$$
$$= 1 + \left[s + \frac{s^2}{2}\right]_0^t$$
$$= 1 + t + \frac{t^2}{2!}$$

and in general $y_n(t) = 1 + \int_0^t y_{n-1}(s)\,ds$

$$= 1 + \int_0^t \left[1 + s + \dots + \frac{s^{n-1}}{(n-1)!}\right] ds$$

$$= 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!}$$ Taylor series expansion of e^t

Since $e^t = 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!}$, the Picard iterates $y_n(t)$ converge to the solution $y(t)$ of this IVP.

HW 1 Due date Feb 6 (Monday) at 11:59pm on Gradescope.

Example Compute the Picard iterates $y_1(t), y_2(t)$ for the IVP

$$y' = 1 + y^3, \quad y(1) = 1$$

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s))\,ds$$

$$y_1(t) = 1 + \int_1^t (1+1^3)\,ds = 1 + 2t - 2 = 2t - 1$$

$$y_2(t) = 1 + \int_1^t (1 + (2s-1)^3)\,ds$$
$$= 1 + \int_1^t (\cancel{1} + 8s^3 - 12s^2 + 6s \cancel{-1})\,ds$$
$$= 1 + \left[2s^4 - 4s^3 + 3s^2\right]_1^t$$
$$= \cancel{1} + 2t^4 - 4t^3 + 3t^2 \cancel{- \overbrace{(2-4+3)}^{1}}$$
$$= 2t^4 - 4t^3 + 3t^2$$

$(2s-1)(2s-1)^2$
$= (2s-1)(4s^2 - 4s + 1)$
$= 8s^3 - 8s^2 + 2s$
$\quad - 4s^2 + 4s - 1$
$= 8s^3 - 12s^2 + 6s - 1$

(b) Convergence of Picard iterates

The solutions may not exist for all time t Thus the Picard iterates may not converge ∀ t. We try to find an interval in which all the $y_n(t)$ are uniformly bounded (i.e. $|y_n(t)| \le K$ for some constant K).

Lemma Choose any two positive numbers a and b. and let R be the rectangle $t_0 \le t \le t_0 + a$, $|y - y_0| \le b$. Compute

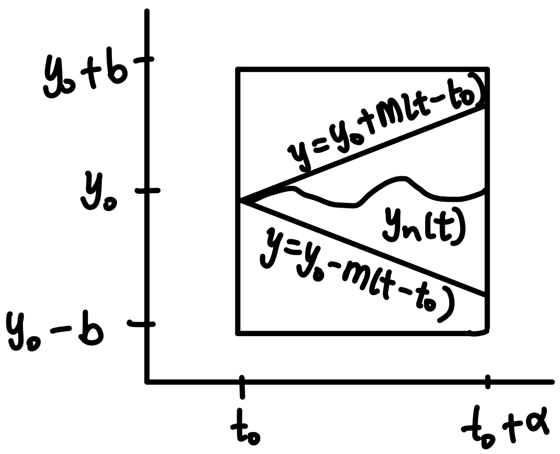
$$M = \max_{(t,y) \text{ in } R} |f(t,y)| \quad \text{and set } \alpha = \min\left(a, \frac{b}{M}\right)$$

Then

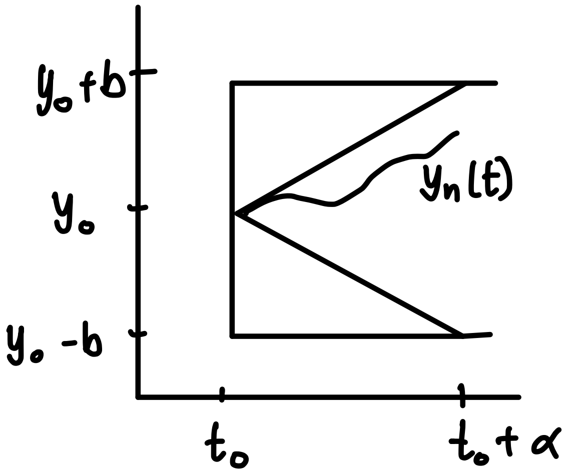
$$|y_n(t) - y_0| \le M(t - t_0) \quad \Rightarrow \quad -M(t-t_0) \le y_n(t) - y_0 \le M(t-t_0)$$

$$y_0 - M(t-t_0) \le y_n(t) \le y_0 + M(t-t_0)$$

for $t_0 \le t \le t_0 + \alpha$.



$\boxed{\alpha = a}$



$\boxed{\alpha = \frac{b}{M}}$

Proof We use induction on n

Observe that $|y_n(t) - y_0| \le M(t - t_0)$ is true for $n=0$ since

$$|y_0(t) - y_0| = |y_0 - y_0| = 0 \le M(t - t_0)$$

Next, we must show that $|y_n(t)-y_0| \le M(t-t_0)$ is true for $n=j+1$ if true for $n=j$.

Assume true for $|y_j(t)-y_0| \le M(t-t_0)$

For $n=j+1$

$$|y_{j+1}(t)-y_0| = |\underbrace{\cancel{y_0}+\int_{t_0}^{t} f(s,y_j(s))\,ds}_{\overset{\shortparallel}{y_{j+1}(t)}} - \cancel{y_0}|$$

$$= \left|\int_{t_0}^{t} f(s,y_j(s))\,ds\right|$$

$$\le \int_{t_0}^{t} |f(s,y_j(s))|\,ds$$

since $M = \max_{(t,y) \text{ in } R} |f(t,y)|$ →

$$\le M(t-t_0)$$

for $t_0 \le t \le t_0+\alpha$. Thus, $|y_n(t)-y_0| \le M(t-t_0)$ is true for all n, by induction. □

Next, we show that the Picard iterates $\{y_n(t)\}$ converge for each t in the interval $t_0 \le t \le t_0+\alpha$, if $\frac{\partial f}{\partial y}$ exists and is continuous.

Write $y_n(t)$ as follows

$$y_n(t) = \cancel{y_0(t)} + (\cancel{y_1(t)} - \cancel{y_0(t)}) + (\cancel{y_2(t)} - \cancel{y_1(t)}) + \dots + (\cancel{y_{n-1}(t)} - \cancel{y_{n-2}(t)}) + (y_n(t) - \cancel{y_{n-1}(t)})$$

So the iterates $\{y_n(t)\}$ are a partial sum for the series

$$y_0(t)+\sum_{n=1}^{\infty}\left(y_n(t)-y_{n-1}(t)\right)$$

Clearly $y_n(t)$ converges iff the infinite series

$$[y_1(t)-y_0(t)]+[y_2(t)-y_1(t)]+\ldots+[y_n(t)-y_{n-1}(t)]$$

converges. So we need to show that

$$\sum_{n=1}^{\infty}|y_n(t)-y_{n-1}(t)|<\infty$$

$$|y_n(t)-y_{n-1}(t)|=\left|\cancel{y_0}+\int_{t_0}^{t}f(s,y_{n-1}(s))ds-\cancel{y_0}-\int_{t_0}^{t}f(s,y_{n-2}(s))ds\right|$$

$$\leq\int_{t_0}^{t}|f(s,y_{n-1}(s))-f(s,y_{n-2}(s))|\,ds$$

$$=\int_{t_0}^{t}\left|\frac{\partial f}{\partial y}(s,\xi(s))\right||y_{n-1}(s)-y_{n-2}(s)|\,ds$$

where $\xi(s)$ lies between $y_{n-1}(s)$ and $y_{n-2}(s)$. <u>Note</u> We have $f(s,y_1)-f(s,y_2)=\int_{y_2}^{y_1}\frac{\partial f}{\partial y}(s,t)dt$, and so

$$|f(s,y_1)-f(s,y_2)|\leq\int_{y_2}^{y_1}\left|\frac{\partial f}{\partial y}(s,t)\right|dt\leq\left|\frac{\partial f}{\partial y}(s,\xi(s))\right||y_1-y_2|.$$

It follows from the lemma that the points $(s,\xi(s))$ lie in the rectangle R for $s<t_0+\alpha$.

$$\Rightarrow\quad|y_n(t)-y_{n-1}(t)|\leq\boxed{L}\int_{t_0}^{t}|y_{n-1}(s)-y_{n-2}(s)|ds,\quad t_0\leq t\leq t_0+\alpha$$

$$\uparrow$$

$$L=\max_{(t,y)\in R}\left|\frac{\partial f(t,y)}{\partial y}\right|$$

Setting $n=2$ gives

$$|y_2(t) - y_1(t)| \leq L\int_{t_0}^{t} |y_1(s) - y_0(s)|\, ds$$

$$\leq L\int_{t_0}^{t} M(s-t_0)\, ds \qquad \text{by the lemma}$$

$$= \frac{LM}{2}(s-t_0)^2$$

This implies that

$$|y_3(t)-y_2(t)| \leq L\int_{t_0}^{t} |y_2(s)-y_1(s)|\, ds$$

$$\leq L\int_{t_0}^{t} \frac{LM}{2}(s-t_0)^2\, ds$$

$$= \frac{L^2M}{6}(t-t_0)^3$$

Proceeding with induction, we have that

$$|y_n(t) - y_{n-1}(t)| \leq \frac{L^{n-1}M(t-t_0)^n}{n!} \quad \text{for } t_0 \leq t \leq t_0+\alpha$$

Therefore, for $t_0 \leq t \leq t_0 + \alpha$

$$|y_1(t)-y_0(t)| + |y_2(t)-y_1(t)| + \ldots \leq M(t-t_0) + \frac{LM}{2}(t-t_0)^2 + \frac{L^2M}{6}(t-t_0)^3 + \ldots$$

(since $t \leq t_0+\alpha$
$\Rightarrow t-t_0 \leq \alpha$)

$$\leq M\alpha + \frac{LM\alpha^2}{2} + \frac{L^2M\alpha^3}{3!} + \ldots$$

$$= \frac{M}{L}\left[L\alpha + \frac{L^2\alpha^2}{2!} + \frac{L^3\alpha^3}{3!} + \ldots\right]$$

$$= \frac{M}{L}\left(e^{L\alpha} - 1\right) < \infty$$

So we managed to show that the Picard iterates $y_n(t)$ converge for each t in the interval $t_0 \leq t \leq t_0+\alpha$. We denote the limit of the sequence $y_n(t)$ by $y(t)$

□

Proof that $y(t)$ satisfies the IVP $\frac{dy}{dt} = f(t,y)$, $y(t_0)=y_0$. and is cts.

The Picard iterates $y_n(t)$ are defined recursively through

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds$$

Taking limits of both sides we get

$$y(t) = y_0 + \underbrace{\lim_{n\to\infty} \int_{t_0}^{t} f(s, y_n(s))\,ds}_{\text{we want to show that this equals } \int_{t_0}^{t} f(s,y(s))ds}$$

We must show that $\left| \int_{t_0}^{t} f(s,y(s))ds - \int_{t_0}^{t} f(s,y_n(s))ds \right|$ approaches zero as $n \to \infty$.

$$\left| \int_{t_0}^{t} f(s,y(s))ds - \int_{t_0}^{t} f(s,y_n(s))\,ds \right| \leq \int_{t_0}^{t} |f(s,y(s)) - f(s, y_n(s))|\,ds$$

$$\leq \boxed{L} \int_{t_0}^{t} |y(s) - y_n(s)|\,ds$$

$L = \max_{(t,y)\in R} \left| \frac{\partial f(t,y)}{\partial y} \right|$ as before

→ Estimate $|y(s) - y_n(s)|$

$$y(s) - y_n(s) = \sum_{j=n+1}^{\infty} [y_j(s) - y_{j-1}(s)]$$

since $y(s) = y_0 + \sum_{j=1}^{\infty} [y_j(s) - y_{j-1}(s)]$ and $y_n(s) = y_0 + \sum_{j=1}^{n} [y_j(s) - y_{j-1}(s)]$

$$|y(s) - y_n(s)| = \left| \sum_{j=n+1}^{\infty} [y_j(s) - y_{j-1}(s)] \right|$$

$$\le \sum_{j=n+1}^{\infty} |y_j(s) - y_{j-1}(s)|$$

and previously we showed $\rightarrow |y_n(t) - y_{n-1}(t)| \le \frac{L^{n-1}M(t-t_0)^n}{n!}$

$$\le \sum_{j=n+1}^{\infty} \frac{L^{j-1}M}{j!}(s-t_0)^j$$

$$\le \sum_{j=n+1}^{\infty} \frac{L^{j-1}M}{j!}\alpha^j$$

since $t \le t_0 + \alpha \Rightarrow t - t_0 \le \alpha$.

$$= \frac{M}{L} \sum_{j=n+1}^{\infty} \frac{(L\alpha)^j}{j!}$$

$$\le \frac{M}{L} \frac{(L\alpha)^{n+1}}{(n+1)!} \boxed{\sum_{p=0}^{\infty} \frac{(L\alpha)^p}{p!}} = e^{L\alpha}$$

$$= \frac{M}{L} \frac{(L\alpha)^{n+1}}{(n+1)!} e^{L\alpha}$$

$\rightarrow 0$ as $n \rightarrow \infty$ by ratio test

Ratio test $\lim_{n\to\infty} \left| \frac{\cancel{\frac{M}{L}} \frac{(L\alpha)^{\cancel{n+1}}}{(n+1)!} \cancel{e^{L\alpha}}}{\cancel{\frac{M}{L}} \frac{\cancel{(L\alpha)}^{n}}{n!} \cancel{e^{L\alpha}}} \right| = \lim_{n\to\infty} \frac{(L\alpha)\cancel{n!}}{(n+1)!} = \lim_{n\to\infty} \frac{L\alpha}{n+1}$

$\hookrightarrow (n+1)\cancel{n!}$

$= 0 < 1$

Therefore $\lim_{n\to\infty} \int_{t_0}^{t} f(s, y_n(s))ds = \int_{t_0}^{t} f(s, y(s))ds$ where $y(t)$ satisfies the integral equation $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$.

Recall this is what we wanted to show:

We must show that $\left| \int_{t_0}^{t} f(s, y(s))ds - \int_{t_0}^{t} f(s, y_n(s))ds \right|$ approaches zero as $n \rightarrow \infty$.

Now we show that the limit $y(t)$ is continuous. I.e we must show that for all $\varepsilon > 0$ $\exists$ a $\delta > 0$ s.t. $|y(t+h) - y(t)| < \varepsilon$ if $|h| < \delta$.

We do <u>NOT</u> know $y(t)$ explicitly. So... we choose a large $N \in \mathbb{Z}$ and observe that

$$\begin{aligned} y(t+h) - y(t) &= [y(t+h) - y_N(t+h)] \\ &\quad + [y_N(t+h) - y_N(t)] \\ &\quad + [y_N(t) - y(t)] \end{aligned}$$

We choose the integer N large enough s.t

$$\frac{M}{L} \sum_{j=N+1}^{\infty} \frac{(L\alpha)^j}{j!} < \frac{\varepsilon}{3}$$

Then from what we showed before, i.e. that

$$|y(s) - y_n(s)| \le \frac{M}{L} \sum_{j=n+1}^{\infty} \frac{(L\alpha)^j}{j!} \quad (\ast)$$

we have that

$$\begin{aligned} |y(t+h) - y(t)| &\le |y(t+h) - y_N(t+h)| \leftarrow \text{from } (\ast) \text{ this is } < \tfrac{\varepsilon}{3} \\ &\quad + |y_N(t+h) - y_N(t)| \quad (+) \\ &\quad + |y_N(t) - y(t)| \leftarrow \text{from } (\ast) \text{ this is } < \tfrac{\varepsilon}{3} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for $|h| < \delta$.

Regarding $(+)$, we construct $y_N(t)$ by N repeated integrations of continuous functions so it's itself continuous. This implies that we choose $\delta > 0$ so small that $|y_N(t+h) - y_N(t)| < \frac{\varepsilon}{3}$ for $|h| < \delta$.

Thus $y(t)$ is a continuous solution of the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$$

and this finishes our proof that $y(t)$ satisfies the IVP.

□

We just proved the following theorem:

Theorem: Let f and $\frac{\partial f}{\partial y}$ be continuous in the rectangle $R: t_0 \le t \le t_0 + a$, $|y - y_0| \le b$. Compute $M = \max_{(t,y)\in R} |f(t,y)|$ and set $\alpha = \min\left(a, \frac{b}{M}\right)$. Then the IVP $y' = f(t,y), y(t_0) = y_0$ has at least one solution $y(t)$ on the interval $t_0 \le t \le t_0 + \alpha$.

Uniquess of solutions of $y' = f(t,y), y(t_0) = y_0$

Consider $y' = \sin(2t) y^{1/3}, y(0) = 0$

Note that $\boxed{y(t) = 0}$ is a solution.

If we ignore the I.C. $y(0) = 0$ then the general solution is found using separation of variables

$$\int y^{-1/3}\,dy = \int \sin(2t)\,dt$$

$$\frac{3}{2} y^{2/3} = -\frac{1}{2}\cos(2t) + C$$

So then if $y(0) = 0$, we get $0 = -\frac{1}{2} + C = C = \frac{1}{2}$

$$\Rightarrow \quad \frac{3}{2} y^{2/3} = \frac{1}{2} - \frac{1}{2}\cos(2t) = \sin^2(2t)$$

$$\Rightarrow \quad y^{2/3} = \frac{2}{3}\sin^2(2t)$$

$$\Rightarrow \quad y = \pm\sqrt{\frac{8}{27}}\sin^3 t$$

So why are there multiple solutions to this IVP? $y' = \overbrace{\sin(2t)y^{1/3}}^{f(t,y)}$

But this RHS does not have a $\frac{\partial f}{\partial y}$ at $y=0$.

(Note $\frac{\partial f}{\partial y} = \frac{1}{3}\sin(2t)\frac{1}{y^{2/3}}$)

<u>New theorem</u> Let f and $\frac{\partial f}{\partial y}$ be continuous in $R: t_0 \le t \le t_0 + a, |y-y_0| \le b$

Compute $M = \max_{(t,y)\in R} |f(t,y)|$, and set $\alpha = \min(a, \frac{b}{M})$. Then the IVP

$$\frac{dy}{dt} = f(t,y), \; y(t_0) = y_0$$

has a unique solution $y(t)$ on the interval $t_0 \le t \le t_0 + \alpha$. I.e. if $y(t)$ & $z(t)$ are two solutions of the IVP then $y(t) = z(t)$ for $t_0 \le t \le t_0 + \alpha$.

<u>Proof</u>. By the previous theorem, there exists <u>at least one solution</u> $y(t)$

Suppose $z(t)$ is a second solution. Then both satisfy

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$$

$$z(t) = y_0 + \int_{t_0}^{t} f(s, z(s))\,ds$$

Now, if we subtract the two we get

$$|y(t) - z(t)| = \left| \cancel{y_0} + \int_{t_0}^{t} f(s,y(s))\,ds - \cancel{y_0} - \int_{t_0}^{t} f(s,z(s))\,ds \right|$$

$$\le \int_{t_0}^{t} |f(s,y(s)) - f(s,z(s))|\,ds$$

$$\le \underset{\uparrow \atop \max_{(t,y)\in R} \left|\frac{\partial f}{\partial y}\right|}{L} \int_{t_0}^{t} |y(s) - z(s)|\,ds$$

$\Rightarrow$ $y(t)=z(t)$ So the IVP has a unique solution $y(t)$.

Why is $|y(t)-z(t)| \le L\int_{t_0}^{t} |y(s)-z(s)|\,ds$?

Lemma Let $w(t)$ be a non-negative function with

$$w(t) \le L\int_{t_0}^{t} w(s)\,ds. \qquad (\dagger)$$

Then $w(t)$ is identically zero.

Example. Show that the solution $y(t)$ of the IVP

$$\frac{dy}{dt} = e^{-t^2}+y^3, \; y(0)=1$$

exists for $0\le t\le 1/9$, and in this interval, $0\le y\le 2$.

$t_0=0$ and $a=\frac{1}{9}$

$\rightarrow$ Let R be the rectangle $t_0\le t\le t_0+a$, in this case $0\le t\le\frac{1}{9}$

$|y-y_0|\le b$ $\qquad 0\le y\le 2$

$|y-1|\le b \Rightarrow -b+1\le y\le b+1$ $\boxed{(\text{so } b=1)}$

Compute $M=\max\limits_{(t,y)\in R}|f(t,y)| = \max\limits_{\substack{0\le t\le\frac{1}{9}\\ 0\le y\le 2}} |e^{-t^2}+y^3| = |e^0+2^3| = 9$

We see that $y(t)$ exists for $0\le t\le \min\left(\frac{1}{9},\frac{1}{9}\right)$ and in this interval

$0\le y\le 2$.

$t_0\le t\le t_0+\alpha$

where $\alpha=\min\left(a,\frac{b}{M}\right)$

$=\min\left(\frac{1}{9},\frac{1}{9}\right)$

$=\frac{1}{9}$

Example Show that the solution $y(t)$ of the IVP

$$\frac{dy}{dt} = t^2 + e^{-y^2}, \quad y(0) = 0$$

exists for $0 \le t \le \frac{1}{2}$ and in this interval $|y(t)| \le 1$.

→ let R be the rectangle s.t. $t_0 \le t \le t_0 + a$ where $t_0 = 0$ and $a = \frac{1}{2}$

$$|y - y_0| \le b \Rightarrow |y - 0| \le b \Rightarrow b = 1$$

Compute

$$M = \max_{(t,y)\in R} |f(t,y)| = \max_{\substack{0 \le t \le \frac{1}{2} \\ -1 \le y \le 1}} |t^2 + e^{-y^2}| = \left(\frac{1}{2}\right)^2 + e^0$$

$$= \frac{1}{4} + 1 = \frac{5}{4}$$

Thus we have $\alpha = \min\left(a, \frac{b}{M}\right) = \min\left(\frac{1}{2}, \frac{1}{(5/4)}\right) = \min\left(\frac{1}{2}, \frac{4}{5}\right) = \frac{1}{2}$

$\Rightarrow$ $t_0 \le t \le t_0 + \alpha$ is $0 \le t \le \frac{1}{2}$ and in this interval $|y(t)| \le 1$.

Section 1.13: Numerical approximations; Euler's method

As we've already discussed, oftentimes it is not possible to write down an analytical solution to the IVP $\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$.

In this section, we'll learn numerical methods to compute accurate approximations of the solution $y(t)$.

We'll compute approximate values $y_1, \dots, y_N$ of $y(t)$ at a finite number of points $t_1, t_2, \dots, t_N$.

The simplest approximation at some point $t > t_0$ is to use the Taylor series approximation:

$$y(t) \approx y(t_0) + y'(t_0)\cdot(t - t_0)$$

Using the information available, we have

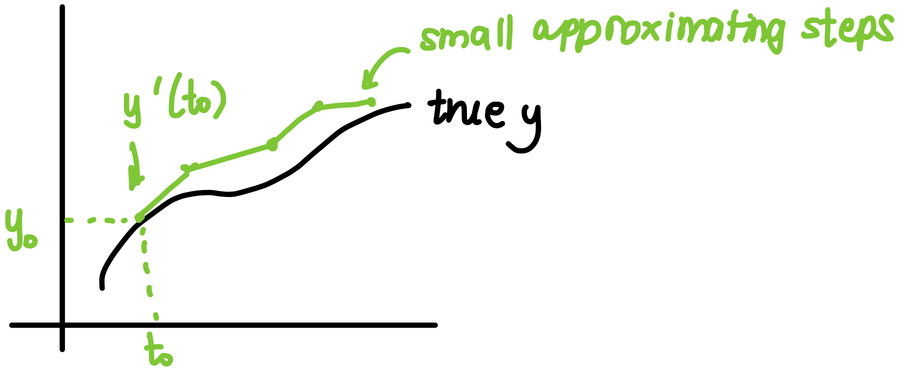
$$y(t) \approx y(t_0) + f(t_0, y_0)\cdot(t - t_0)$$

$$\frac{dy}{dt} \approx \frac{y(t_1) - y(t_0)}{\underbrace{t_1 - t_0}_{=h}} = f(t_0, y_0)$$

If we look at $t_0, t_1, \dots$ and $\boxed{t_{\ell+1} - t_\ell = h}$ then

$$y(t_1) \approx y(t_0) + h f(t_0, y_0)$$ ← this is known as EXPLICIT EULER method

Pictorially:



To summarize: $y_\ell \approx y(t_\ell)$

Define $y_{\ell+1} = y_\ell + h\underbrace{f(t_\ell, y_\ell)}_{\text{approximation to } y'(t_\ell)}$ (since we do not know the true y_ℓ)

Example $\quad y'(t) = 1+(y-t)^2, \quad y(t_0)=y_0$

Explicit Euler: $\quad y_{\ell+1} = y_\ell + h\left(1+(y_\ell-t_\ell)^2\right)$

Error analysis

Recall the Taylor series:

$$y(t) = y(t_0) + y'(t_0)(t-t_0) + \frac{y''(t_0)}{2!}(t-t_0)^2 + \dots$$

Taylor's theorem says that if we truncate this, then

$$y(t) \underset{\substack{\uparrow \\ \text{EQUALS}}}{=} y(t_0) + y'(t_0)(t-t_0) + \frac{y''(\xi)}{2!}(t-t_0)^2$$

$\uparrow\!\!\lfloor\ \xi$ is some number in the interval $[t_0, t]$.

To find the error in Euler's method we examine $\underbrace{y_{\ell+1}}_{\text{approx.}} - \underbrace{y(t_{\ell+1})}_{\text{true value}}$

Euler: $\quad y_{\ell+1} = y_\ell + hf(t_\ell, y_\ell)$

Taylor: $\quad y(t_{\ell+1}) = y(t_\ell) + y'(t_\ell)h + \frac{y''(\xi_\ell)}{2!}h^2 \qquad (-)$

$$y_{l+1} - y(t_{l+1}) = y_l - y(t_l) + h[f(t_l, y_l) - y'(t_l)] - \frac{y''(\xi_l)}{2!} h^2$$

Note that $$f(t_l, y_l) - f(t_l, y(t_l)) = \underbrace{\frac{f(t_l, y_l) - f(t_l, y(t_l))}{y_l - y(t_l)}}_{\frac{\partial f}{\partial y}(t_l, \eta_l)} (y_l - y(t_l))$$

$$\uparrow \text{some } \eta_l$$

$$\Rightarrow \quad |y_{l+1} - y(t_{l+1})| \le |y_l - y(t_l)| + h\left|\frac{\partial f}{\partial y}(t_l, \eta_l)\right| \cdot |y_l - y(t_l)| + \frac{|y''(\xi_l)|}{2} h^2$$

Set $\boxed{\epsilon_l = |y_l - y(t_l)|}$ ← error

$$\Rightarrow \quad \epsilon_{l+1} \le \epsilon_l + \left|\frac{\partial f}{\partial y}(t_l, \eta_l)\right| \epsilon_l h + \frac{|y''(\xi_l)|}{2} h^2$$

$$= \left(1 + h\left|\frac{\partial f}{\partial y}(t_l, \eta_l)\right|\right) \epsilon_l + \frac{|y''(\xi_l)|}{2} h^2$$

$$\le (1 + hL)\epsilon_l + \frac{D}{2} h^2$$

with $\boxed{L = \max\left|\frac{\partial f}{\partial y}\right|, \quad D = \max|y''|}$

and note $y'' = \frac{d}{dt} y' = \frac{d}{dt} f(t, y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f.$

To summarize:

$$\boxed{\epsilon_{l+1} \le (1 + hL)\epsilon_l + \frac{D}{2} h^2} \qquad \text{for } l = 0, 1, \cdots, N-1$$

Note that $\epsilon_0 = 0$ since $y(t_0) = y_0$. So if $\epsilon_{l+1} \le A\epsilon_l + B$, $\epsilon_0 = 0$

with $\boxed{A = 1+hL}$ and $\boxed{B = \frac{Dh^2}{2}}$ then can we say anything about ϵ_l independent of ϵ_{l-1}?

If $\boxed{\epsilon_{k+1} \le A\epsilon_k + B}$, then we can show that

$$\boxed{\epsilon_k \le \frac{B}{A-1}\left(A^k - 1\right)}$$

for how this follows see pages 92-93 of your textbook

$$= \frac{D}{2} h^{\cancel{2}} \frac{1}{\cancel{1}+\cancel{h}L\cancel{-1}} \left((1+hL)^k - 1\right)$$

$$= \underset{\substack{\downarrow \\ 0 \\ \text{as } h \to 0}}{\frac{D}{2L} h} \underbrace{\left((1+hL)^k - 1\right)}_{\to 0 \text{ as } h \to 0}$$

We can also obtain an estimate for ϵ_k that is independent of k

Note that $\boxed{(1+hL) \le e^{hL}}$ since from the Taylor series expansion of the exponential function we have

$$e^{hL} = \boxed{1 + hL} + \frac{(hL)^2}{2!} + \frac{(hL)^3}{3!} + \dots \ge 1 + hL$$

positive

Therefore, $\epsilon_k \le \frac{D}{2L} h\left(\left(e^{hL}\right)^k - 1\right) = \frac{D}{2L} h\left(e^{hLk} - 1\right)$

Since $hk \le \alpha$, we have $\epsilon_k \le \frac{Dh}{2L}\left(e^{\alpha L} - 1\right)$ where α is the one from the existence and uniquess theorem.

$\Rightarrow$ Euler's scheme is FIRST-ORDER CONVERGENT. I.e if $h \to h/2$ then $\epsilon_k \to \epsilon_k/2$.

Example Consider $\frac{dy}{dt} = \frac{t^2+y^2}{2}$, $y(0)=0$

(i) Show that $y(t)$ exist at least for $0 \le t \le 1$ and that in this interval $-1 \le y(t) \le 1$

let R be the rectangle $0 \le t \le 1, -1 \le y \le 1$.

$$M = \max_{(t,y)\in R} |f(t,y)| = \max_{\substack{0\le t\le 1 \\ -1\le y\le 1}} \left|\frac{t^2+y^2}{2}\right| = \frac{1+1}{2} = 1$$

$$\alpha = \min\left(a, \frac{b}{M}\right) = \min\left(1, \frac{1}{1}\right) = 1$$

Hence by the existence-and-uniqueness theorem, $y(t)$ exists at least for

$$t_0 \le t \le t_0+\alpha \Rightarrow \quad 0 \le t \le 1$$

and in this interval $-1 \le y \le 1$

(b) Let N be a large positive integer. Set up Euler's scheme to find approximate values of y at the points $t_k = k/N$, $k=0,1,\dots,N$.

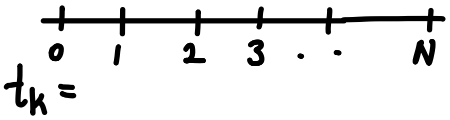
Euler's scheme:

$$y_{k+1} = y_k + hf(t_k, y_k)$$

since $\frac{dy}{dt} = \frac{t^2+y^2}{2}$

$$y_{k+1} = y_k + h\left(\frac{t_k^2+y_k^2}{2}\right)$$

$$= y_k + \frac{1}{2N}\left[\left(\frac{k}{N}\right)^2 + y_k^2\right]$$



with $k=0,1,\dots,N-1$ and $y_0=0$

since $y(0)=0$

(c) Determine the stepsize $\boxed{h=\frac{1}{N}}$ so that the error we make in approximating $y(t_k)$ by y_k does not exceed 10^{-4}.

In this example $f(t,y) = \frac{t^2+y^2}{2}$ and so $\frac{\partial f}{\partial y} = y$, $\frac{\partial f}{\partial t} = t$

Recall that $y'' = \frac{d}{dt}y' = \frac{d}{dt}f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\overset{=f}{\frac{dy}{dt}} = \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} = t + \left(\frac{t^2+y^2}{2}\right)y$

So we have $|y(t_k) - y_k| \le \frac{Dh}{2L}(e^L - 1)$ where

$$L = \max\left|\frac{\partial f}{\partial y}\right| = 1$$

$$D = \max|y''| = 1 + \frac{|t|}{2} = 2$$

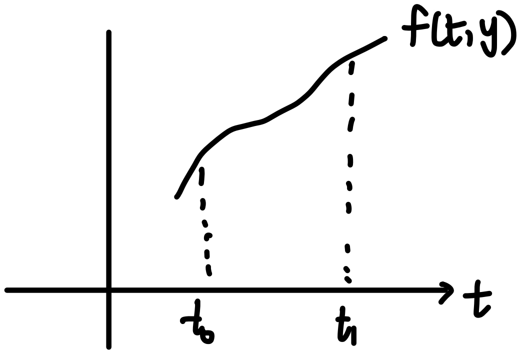
Hence $|y(t_k) - y_k| \le \frac{\cancel{2}h}{\cancel{2(1)}}(e^1 - 1) = h(e-1) \lesssim 10^{-4}$

So the stepsize must be $\boxed{h \le \frac{10^{-4}}{e-1}}$

Interpreted in terms of the exact solution:

IVP: $y'(t) = f(t,y)$, $y(t_0) = y_0$

Integrate both sides of the diff. eqn.



$$\Rightarrow \quad y(t) = y_0 + \int_{t_0}^{t} f(s,y)\,ds$$

$$\approx y_0 + (t - t_0)\, f(t_0, y(t_0))$$

$$= \underbrace{y_0 + hf(t_0, y_0)}$$

Euler's method obtained by approximating this integral

In this case, the value of f at t_0 was used. Alternatively we could have used the value of t :

$$y(t_1) = y_0 + \int_{t_0}^{t_1} f(s, y(s))\,ds$$

$$\approx y_0 + (t_1 - t_0)\, f(t_1, y(t_1))$$

Now, the equation $y_1 = y_0 + hf(t_1, y_1)$ must be solved for the value of y_0. This is known as IMPLICIT EULER. The error is similar, but the stability is better.

STABILITY OF EULER

Examine the model problem $y' = -\lambda y$, with $\lambda > 0$.

$y_{l+1} = y_l + hf(t_l, y_l)$

<u>Explicit Euler</u>: $y_{l+1} = y_l - h\lambda y_l = (1-h\lambda) y_l$

The true solution is $y = ce^{-\lambda t}$, and $y(t) \to 0$ as $t \to \infty$ (since $\lambda > 0$)

In order for $y_l \to 0$, we require

$$y_1 = (1-h\lambda) y_0$$
$$y_2 = (1-h\lambda) y_1 = (1-h\lambda)\left[(1-h\lambda) y_0\right] = (1-h\lambda)^2 y_0$$

that $|1-h\lambda| < 1$ and therefore since $\lambda > 0, h > 0$, we require

$$-1 < 1-h\lambda < 1 \Rightarrow \quad -2 < -h\lambda < 0$$
$$0 < h\lambda < 2$$
$$\boxed{0 < h < \frac{2}{\lambda}}$$

This means that the <u>stepsize h</u> must be in this interval to ensure stability.

IMPLICIT EULER

$$y_{l+1} = y_l - h\lambda y_{l+1}$$

Solve for y_{l+1} to obtain $\quad y_{l+1}(1+h\lambda) = y_l \Rightarrow y_{l+1} = \left(\frac{1}{1+h\lambda}\right) y_l$

$$= \frac{1}{(1+h\lambda)^{l+1}} y_0$$

The factor $\frac{1}{1+h\lambda}$ is <u>always</u> <1 if $h > 0, \lambda > 0$, and therefore implicit Euler is <u>A-stable</u>.

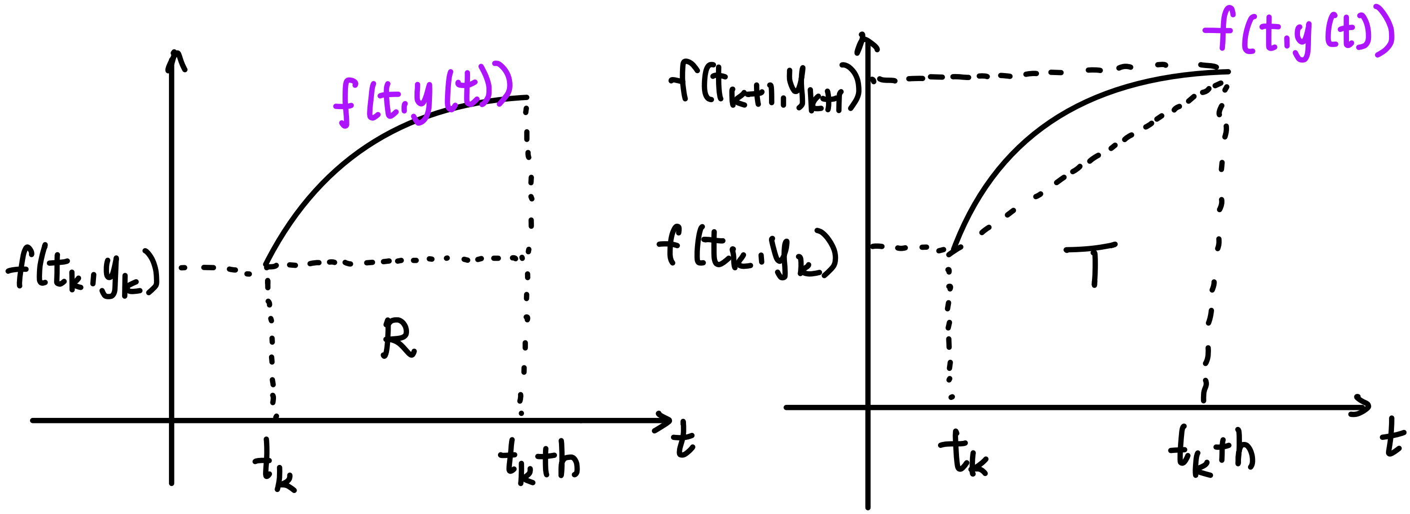
Section 1.15 Improved Euler method

Consider the IVP $y'(t) = f(t,y)$, $y(t_0) = y_0$.

Integrating the diff. eqn. between t_k and t_k+h gives:

$$y(t_{k+1}) = y(t_k) + \underbrace{\int_{t_k}^{t_k+h} f(t,y(t))\,dt}_{\substack{\text{we must approx.}\\ \text{the area under the}\\ \text{curve } f(t,y) \text{ bet}^n\ t_k \text{ and } t_k+h}}$$

Pictorially



$$\text{Area } R = f(t_k,y_k)(\cancel{t_k}+h-\cancel{t_k}) = h\,f(t_k,y_k)$$

$$\text{Area } T = \frac{h}{2}\left(f(t_k,y_k) + f(t_{k+1},y_{k+1})\right)$$

The area of the trapezoid T is a much better approximation of the area under the curve compared to the area of the rectangle R

So if we replace the integral in $y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_k+h} f(t,y(t))\,dt$ with the area under the trapezoid, we get the following numerical scheme:

$$(*) \qquad y_{k+1} = y_k + \frac{h}{2}\left[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\right]$$

we cannot determine y_{k+1} from y_k
because y_{k+1} also appears on the RHS.

On the RHS we can then use Euler's method. I.e.

$$y_{k+1} = y_k + hf(t_k, y_k)$$

Thus $(*)$ becomes

$$\boxed{y_{k+1} = y_k + \frac{h}{2}\Big[f(t_k, y_k) + f\big(\underbrace{t_{k+1}}_{t_k + h}, y_k + hf(t_k, y_k)\big)\Big]}, \quad y_0 = y(t_0)$$

↑ this is called IMPROVED EULER METHOD

Example. Write down the improved Euler method to approximate the solution $y(t)$ to the IVP

$$y' = 1 + (y-t)^2, \quad y(0) = \frac{1}{2}$$

at points $t_k = \frac{k}{N}$ with $k = 1, \dots, N$.

→ Improved Euler method:

$$y_{k+1} = y_k + \frac{h}{2}\Big\{\underbrace{1 + (y_k - t_k)^2}_{f(t_k, y_k)} + \underbrace{1 + \Big[y_k + h\big(1 + (y_k - t_k)^2\big) - t_{k+1}\Big]^2}_{f(t_{k+1}, y_k + hf(t_k, y_k))}\Big\}$$

with $h = \frac{1}{N}$, $y_0 = \frac{1}{2}$. The integer $k = 0, \dots, N-1$.

Section 1.16: The Runge-Kutta method

$$y_{k+1} = y_k + \underbrace{\frac{h}{6}[L_{k,1} + 2L_{k,2} + 2L_{k,3} + L_{k,4}]}_{\text{think of this as an average slope}}, \quad k=0,1,\dots,N-1$$

where $y_0 = y(t_0)$ and

$$L_{k,1} = f(t_k, y_k)$$
$$L_{k,2} = f(t_k + \tfrac{1}{2}h,\ y_k + \tfrac{1}{2}hL_{k,1})$$
$$L_{k,3} = f(t_k + \tfrac{1}{2}h,\ y_k + \tfrac{1}{2}hL_{k,2})$$
$$L_{k,4} = f(t_k + h,\ y_k + hL_{k,3})$$

The Runge-Kutta method is much more accurate than Euler's method and the improved Euler method.

Note from above that there are 4 functional evaluations at each step for Runge-Kutta whereas in the Euler method we perform only one functional evaluation at each step. However, the Runge-Kutta method is still much more accurate.

SUMMARY

First-order accurate methods

Forward (explicit) Euler. $y_{k+1} = y_k + hf(t_k, y_k)$

Backward (implicit) Euler: $y_{k+1} = y_k + hf(t_{k+1}, y_{k+1})$

Second-order accurate method

Improved Euler: $y_{k+1} = y_k + \frac{h}{2}\left[f(t_k, y_k) + f\left(t_{k+1}, y_k + hf(t_k, y_k)\right)\right]$

Fourth-order accurate method

Runge-Kutta: $y_{k+1} = y_k + \frac{h}{6}[L_{k,1} + 2L_{k,2} + 2L_{k,3} + L_{k,4}]$ with $L_{k,1}, L_{k,2}, L_{k,3}, L_{k,4}$ from above

Let's say we have 3 numerical methods that have an error

$$3h, \quad 11h^2, 42h^4$$

If we require 8 decimal places accuracy, then the step sizes h_1, h_2, h_3 of these three schemes must satisfy

$$\text{error}_1 = 3h_1 \leq 10^{-8} \quad \Rightarrow \quad 3\left(\frac{1}{N_1}\right) \leq 10^{-8} \Rightarrow N_1 \geqslant 3\times10^{8} = 300 \text{ million}$$

$$\text{error}_2 = 11h_2^2 \leq 10^{-8} \qquad \Rightarrow N_2 \geqslant \sqrt{11} \times 10^4 \approx 34\,000$$

$$\text{error}_3 = 42h_3^4 \leq 10^{-8} \qquad \Rightarrow \underbrace{N_3 \geqslant \sqrt[4]{42} \times 10^2 \approx 260}_{\text{number of iterations to reach 8 d.p. of accuracy.}}$$

Chapter 2: Second-order linear differential equations

A 2nd-order differential eqn is of the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

If this is an IVP then the I.C.s are of the form

$$\boxed{\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y'_0 \end{aligned}}$$

We'll learn to solve a second-order <u>linear</u> differential equation. This is of the form

$$\boxed{\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)}$$

Linear because both y and $\frac{dy}{dt}$ appear by themselves.

e.g. $\frac{d^2y}{dt^2} + e^t\frac{dy}{dt} + 2y = 1$ Linear

$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + (\sin t)^2 y = e^t$ linear

$\frac{d^2y}{dt^2} + 5\left(\underline{\frac{dy}{dt}}\right)^2 = 1$ nonlinear

$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + \underline{\sin y} = t^3$ nonlinear

We start with the homogeneous case:

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=\underset{\substack{\uparrow\\ g(t)=0}}{0}, \quad y(t_0)=y_0, \quad y'(t_0)=y_0'$$

First we want to know if a solution exists.

Existence-uniqueness theorem

Let $p(t)$ and $q(t)$ be continuous functions in the interval $\alpha<t<\beta$. Then there exists one and only one function $y(t)$ satisfying

$$y''+p(t)y'+q(t)y=0$$

on the entire interval $\alpha<t<\beta$ and the prescribed I.C. $y(t_0)=y_0$, $y'(t_0)=y_0'$. Note that any solution $y=y(t)$ which satisfies the IVP with $y(t_0)=0$ and $y'(t_0)=0$ at some time $t=t_0$ must be identically 0.

* * *

Now we will view the differential equation through operators L. We use the relation

$$\boxed{L[y](t)= y''(t)+p(t)y'(t)+q(t)y(t)}$$

where L is an operator which operates on functions. i.e. it associates each function y to a new function $L[y]$.

Example. If $p(t)=0$, $q(t)=t$ then

$$L[y](t)=y''(t)+ty(t).$$

If $y(t) = \cos t$ then $L[y](t) = -\cos t + t\cos t$

If $y(t) = t^3$ then $L[y](t) = t^4 + 6t$

"function of a function"

Properties

1. $L[cy] = cL[y]$ for any constant c
2. $L[y_1 + y_2] = L[y_1] + L[y_2]$

Proofs

1.
$$\begin{aligned} L[cy](t) &= (cy)''(t) + p(t)(cy)'(t) + q(t)(cy)(t) \\ &= cy''(t) + cp(t)y'(t) + cq(t)y(t) \\ &= c\left[y''(t) + p(t)y'(t) + q(t)y(t)\right] \\ &= cL[y](t). \end{aligned}$$

2.
$$\begin{aligned} L[y_1 + y_2](t) &= (y_1 + y_2)''(t) + p(t)(y_1 + y_2)'(t) + q(t)(y_1 + y_2)(t) \\ &= y_1'' + y_2'' + p(t)y_1' + p(t)y_2' + q(t)y_1 + q(t)y_2 \\ &= \left[y_1'' + p(t)y_1' + q(t)y_1\right] + \left[y_2'' + p(t)y_2' + q(t)y_2\right] \\ &= L[y_1](t) + L[y_2](t) \end{aligned}$$

Definition An operator L which assigns functions to functions and satisfies properties 1 and 2 is called a *linear operator*.

All others are nonlinear.

Eg. $L[y](t) = y'' - 2t[y]^4$

This operator assigns to $y = \frac{1}{t}$ the function

$$L[y](t) = \frac{2}{t^3} - 2t\left(\frac{1}{t}\right)^4 = 0$$

but to $y = c \cdot \frac{1}{t}$ it assigns

$$L[cy](t) = \frac{2c}{t^3} - \frac{2c^4}{t^3} = \frac{2c(1-c^3)}{t^3}$$

Thus for $c \neq 0, 1$ and $y(t) = \frac{1}{t}$ we see that $L[cy](t) \neq cL[y](t)$ so this operator is nonlinear.

Why are Properties 1 and 2 useful?

The solutions $y(t)$ to $y'' + p(t)y' + q(t)y = 0$ are exactly those functions y for which

$$L[y](t) = y'' + p(t)y' + q(t)y = 0$$

i.e. the solutions $y(t)$ are exactly those functions y to which the operator L assigns the zero function.

- So if $y(t)$ is a solution by property 1 then so is $cy(t)$ since

$$L[cy](t) = cL[y](t) = 0.$$

- By property 2 if $y_1(t)$ and $y_2(t)$ are both solutions of the diff. eqn. then $y_1(t) + y_2(t)$ is also a solution since

$$\begin{aligned} L[y_1 + y_2](t) &= L[y_1](t) + L[y_2](t) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

The two properties together imply that all <u>linear combinations</u>

$$c_1 y_1(t) + c_2 y_2(t)$$

of solutions of the diff. eqn are again solutions.

$\Rightarrow$ We can generate infinitely many other solutions.

<u>e.g.</u> Consider $\frac{d^2y}{dt^2} + y = 0.$

Two solutions are $\left.\begin{array}{l} y_1(t) = \cos t \\ y_2(t) = \sin t \end{array}\right] \Rightarrow y(t) = c_1 \cos t + c_2 \sin t$

is also a solution for every choice of c_1 and c_2

By the existence-uniqueness theorem, $y(t)$ exists for all t.

Let $y(0) = y_0$, $y'(0) = y_0'$ and consider

$$\phi(t) = y_0 \cos t + y_0' \sin t \longleftarrow$$

solution since it's a linear combination of solutions

and

$\phi(0) = y_0$

$\phi'(0) = y_0'$

Thus $y(t)$ and $\phi(t)$ satisfy the same 2^{nd}-order linear diff. eqn and the same I.C.s.

<u>Theorem 2</u> (from textbook): Let $y_1(t)$ and $y_2(t)$ be two solutions of $y'' + p(t)y' + q(t)y = 0$ on the interval $\alpha < t < \beta$ with

$$\boxed{y_1(t) y_2'(t) - y_1'(t) y_2(t) \neq 0}$$

in this interval. Then $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is the general solution of the diff. eqn.

Definition The quantity $y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is called the Wronskian of y_1, y_2 and is denoted by $W(t) = W[y_1,y_2](t)$.

$$\boxed{W[y_1,y_2](t) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1y_2' - y_1'y_2}$$

Theorem 3 (from textbook): Let $p(t)$ and $q(t)$ be continuous in the interval $\alpha<t<\beta$ and let $y_1(t)$ and $y_2(t)$ be two solutions of

$$y'' + p(t)y' + q(t) = 0$$

Then $W[y_1,y_2](t)$ is either identically zero, or is never zero, on the interval $\alpha<t<\beta$.

Note. Let $y_1(t)$ and $y_2(t)$ be two solutions of the linear 2nd order diff. eqn. $y''+p(t)y'+q(t)y=0$. Then, their Wronskian

$$W(t) = W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

satisfies the 1st-order diff. eqn.

$$\boxed{W'(t) + p(t)W(t) = 0}$$

Note We can solve this 1st order diff. eqn. using separation of variables

$$\int \frac{dW}{W} = \int -p(t)\,dt$$

$$\Rightarrow W(t) = Ae^{-\int p(t)dt}$$

Why does the Wronskian satisfy $W'(t)+p(t)W(t)=0$?

$$W'(t) = \frac{d}{dt}(y_1 y_2' - y_1' y_2)$$
$$= \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'} \quad \text{(by product rule)}$$
$$= y_1 y_2'' - y_1'' y_2$$

Since y_1 and y_2 are both solutions of $y''+p(t)y'+q(t)y=0$ they must satisfy

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \Rightarrow y_1'' = -p(t)y_1' - q(t)y_1$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \Rightarrow y_2'' = -p(t)y_2' - q(t)y_2$$

Plugging these into $W'(t) = y_1 y_2'' - y_1'' y_2$ we obtain

$$W'(t) = y_1(-p(t)y_2' - q(t)y_2) - (-p(t)y_1' - q(t)y_1)y_2$$
$$= -p(t)y_1y_2' - \cancel{q(t)y_1y_2} + p(t)y_1'y_2 + \cancel{q(t)y_1y_2}$$
$$= -p(t)(\underbrace{y_1y_2' - y_1'y_2}_{W(t)})$$

$$\Rightarrow W'(t) + p(t)W(t) = 0$$

□

Proof of theorem 3: Choose any t_0 in the interval $\alpha<t<\beta$

Then from $W'(t)+p(t)W(t)=0$ we have

$$W[y_1, y_2](t) = W[y_1,y_2](t_0)e^{-\int_{t_0}^{t} p(s)ds}$$

from separation of variables

But $e^{-\int_{t_0}^{t} p(s)\,ds} \neq 0$ for $\alpha<t<\beta$. Thus, $W[y_1, y_2](t)$ is either identically zero, or is never zero. □

Note The Wraskian of two functions y_1, y_2 vanishes identically if one of the functions is a constant multiple of the other. If $\boxed{y_2 = cy_1}$

$$W[y_1, y_2](t) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} y_1 & cy_1 \\ y_1' & cy_1' \end{pmatrix}$$
$$= cy_1y_1' - cy_1y_1'$$
$$= 0$$

Theorem 4 Let $y_1(t)$ and $y_2(t)$ be two solutions of $y''+p(t)y'+q(t)y=0$ on the interval $\alpha<t<\beta$ and suppose $W[y_1, y_2](t_0)=0$ for some t_0 in this interval. Then one of these solutions is a constant multiple of the other.

Proof of theorem 2 Let $y(t)$ be any solution of

$$y''+p(t)y'+q(t)y=0, \quad y(t_0)=y_0, \quad y'(t_0)=y_0'.$$

We must find constants c_1, c_2 s.t. $y(t)=c_1y_1(t)+c_2y_2(t)$, for to in $\alpha<t<\beta$

$$\left.\begin{aligned} (\times\, y_2'(t_0)) \quad & c_1y_1(t_0)+c_2y_2(t_0)=y_0 \\ (\times\, y_2(t_0)) \quad & c_1y_1'(t_0)+c_2y_2'(t_0)=y_0' \end{aligned}\right] \text{ satisfies the I.C.}$$

$$\begin{aligned} & c_1y_1(t_0)y_2'(t_0)+\cancel{c_2y_2(t_0)y_2'(t_0)} = y_0y_2'(t_0) \\ - \quad & c_1y_2(t_0)y_1'(t_0) + \cancel{c_2y_2(t_0)y_2'(t_0)} = y_2(t_0)y_0' \\ \hline & c_1\left[y_1(t_0)y_2'(t_0)-y_2(t_0)y_1'(t_0)\right] = y_0y_2'(t_0)-y_2(t_0)y_0' \end{aligned}$$

$$C_1 = \frac{y_0 y_2'(t_0) - y_2(t_0) y_0'}{y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0)}$$

$$(\times y_1'(t_0)) \quad c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$

$$(\times y_1(t_0)) \quad c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

$$\begin{array}{rl} & c_1 \cancel{y_1(t_0) y_1'}(t_0) + c_2 y_2(t_0) y_1'(t_0) = y_0 y_1'(t_0) \\ - & c_1 \cancel{y_1(t_0)} y_1'(t_0) + c_2 y_1(t_0) y_2'(t_0) = y_0' y_2(t_0) \\ \hline \end{array}$$

$$c_2 [y_2(t_0) y_1'(t_0) - y_1(t_0) y_2'(t_0)] = y_0 y_1'(t_0) - y_2(t_0) y_0'$$

$$C_2 = \frac{y_0 y_1'(t_0) - y_2(t_0) y_0'}{y_2(t_0) y_1'(t_0) - y_1(t_0) y_2'(t_0)}$$

c_1 and c_2 exist if $y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0) \neq 0$.

Now, let $\phi(t) = c_1 y_1(t) + c_2 y_2(t)$ for this choice of c_1, c_2. Since it's a linear combination of solutions $\phi(t)$ is a solution too. By construction, $\phi(t_0) = y_0$ $\phi'(t_0) = y_0'$. Thus $y(t)$ and $\phi(t)$ satisfy the same 2nd-order linear homogeneous eqn and the same initial conditions. So by the uniqueness theorem, $y(t) \equiv \phi(t)$, that is

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \ \alpha < t < \beta. \quad \square$$

Proof of theorem 4: Suppose that $W[y_1,y_2](t_0)=0$. Then by theorem 3 $W[y_1,y_2](t)$ is identically zero. Assume $y_1(t)y_2(t)\neq 0$ for $\alpha<t<\beta$. Then dividing both sides of the equation

$$y_1(t)y_2'(t)-y_1'(t)y_2(t)=0$$

by $y_1(t)y_2(t)$ gives

$$\frac{y_2'(t)}{y_2(t)}-\frac{y_1'(t)}{y_1(t)}=0$$

Solving it gives: $\ln(y_2(t))=\ln(y_1(t))+\tilde{C}$

$\boxed{y_2(t)=cy_1(t)}$ for some constant c.

Definition: Two functions $y_1(t)$ and $y_2(t)$ are said to be *linearly dependent* on an interval I if one of these functions is a constant multiple of the other on I.

Corollary Two solutions $y_1(t)$ and $y_2(t)$ of $y''+p(t)y'+q(t)y=0$ are *linearly independent* on the interval $\alpha<t<\beta$ iff $\boxed{W[y_1,y_2](t)\neq 0}$ on this interval. So two solutions $y_1(t)$ and $y_2(t)$ form a *fundamental set* of solutions of the diff. eqn on $\alpha<t<\beta$ iff they are linearly independent on this interval.

Section 2.2: Linear equations with constant coefficients

Homogeneous, linear second-order equation with constant coefficients

$$\boxed{L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0} \quad (*)$$

with a, b, c constants and $a \neq 0$.

From the previous section, we know that we need only find two independent solutions y_1, y_2 and all other solutions are obtained by taking linear combinations of y_1 and y_2.

Ansatz (=educated guess):

$$y(t) = e^{rt}, \quad \text{for } r \text{ a constant}$$

$$L[e^{rt}] = ar^2e^{rt} + bre^{rt} + ce^{rt}$$
$$= e^{rt}(ar^2 + br + c)$$

$y = e^{rt}$ is a solution iff $\boxed{ar^2 + br + c = 0}$ since $e^{rt} \neq 0$.

characteristic equation of (*)

Solving the characteristic equation we see that the two roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac > 0$ then r_1, r_2 are real and distinct

$$\Rightarrow y_1 = e^{r_1t}, \quad y_2 = e^{r_2t}$$

(linearly independent on any interval I)

To show this we can also compute the Wronskian through

$$W[y_1, y_2](t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0$$

when $r_1 \neq r_2$.

Example: Find the general solution of $y'' + 5y' + 4y = 0$

The characteristic equation is $r^2 + 5r + 4 = 0$

$$(r+4)(r+1) = 0$$

$$r = -4, \; r = -1$$

$y_1(t) = e^{-4t}$, $y_2(t) = e^{-t}$ (form the fundamental set of solutions)

Thus the general solution is

$$y(t) = c_1 e^{-4t} + c_2 e^{-t}$$

for some constants c_1 and c_2

Example Solve the IVP:

$$2y'' + y' - 10y = 0, \quad y(1) = 5, \; y'(1) = 2$$

Characteristic equation: $2r^2 + r - 10 = 0$

$$(2r+5)(r-2) = 0$$

$$r = -\frac{5}{2} \quad r = 2$$

$$y(t) = c_1 e^{-\frac{5}{2}t} + c_2 e^{2t}$$

$$y'(t) = -\frac{5}{2} c_1 e^{-\frac{5}{2}t} + 2c_2 e^{2t}$$

Using I.Cs: $y(1) = c_1 e^{-\frac{5}{2}} + c_2 e^{2} = 5$

$$y'(1) = -\cancel{\frac{5}{2} c_1} e^{-\frac{5}{2}} + 2c_2 e^2 = 2$$

Multiply by $\frac{5}{2}$: $\cancel{\frac{5}{2} c_1 e^{-\frac{5}{2}}} + \frac{5}{2} c_2 e^2 = \frac{25}{2}$

Adding the two: $$\frac{9}{\cancel{2}} C_2 e^2 = \frac{29}{\cancel{2}}$$

$$\boxed{C_2 = \frac{29}{9e^2}}$$

Using $c_1 e^{-\frac{5}{2}} + C_2 e^2 = 5$ and $C_2 = \frac{29}{9e^2}$ gives us C_1 as

$$c_1 e^{-\frac{5}{2}} + \frac{29}{9\cancel{e^2}} \cancel{e^2} = 5$$

$$c_1 = (5 - 29/9) e^{5/2}$$

$$\boxed{C_1 = \frac{16}{9} e^{5/2}}$$

Thus the solution is $y(t) = \frac{16}{9} e^{\frac{5}{2}} e^{-\frac{5}{2}t} + \frac{29}{9e^2} e^{2t}$

$$\Rightarrow \boxed{y(t) = \frac{16}{9} e^{-\frac{5}{2}(t-1)} + \frac{29}{9} e^{2(t-1)}}$$

Remark: Observe from this example that $e^{r(t-t_0)}$ is also a solution of $ay''+by'+cy=0$ if $ar^2+br+C=0$. So to find the solution to the IVP

$$ay''+by'+cy=0, \quad y(t_0)=y_0, \quad y'(t_0)=y_0'$$

we would write $\boxed{y(t) = c_1 e^{r_1(t-t_0)} + c_2 e^{r_2(t-t_0)}}$

and solve for c_1 and c_2 from the initial conditions.

- If $b^2-4ac<0$ then the characteristic equation $ar^2+br+c=0$ has complex roots

$$r_1 = \frac{-b+i\sqrt{4ac-b^2}}{2a}, \quad r_2 = \frac{-b-i\sqrt{4ac-b^2}}{2a}$$

Assume that $\boxed{y(t)=u(t)+iv(t)}$ is a complex-valued solution of

$$ay''+by'+cy=0$$

This means that it satisfies the diff. eqn. and so

$$a\left[u''(t)+iv''(t)\right]+b\left[u'(t)+iv'(t)\right]+c\left[u(t)+iv(t)\right]=0$$

$$\Rightarrow \left(au''(t)+bu'(t)+cu(t)\right)+i\left(av''(t)+bv'(t)+cv(t)\right)=0$$

Both the real and the imaginary parts must be zero.

$$\Rightarrow \left[\begin{array}{l} au''(t)+bu'(t)+cu(t)=0 \\ av''(t)+bv'(t)+cv(t)=0 \end{array}\right]$$

AND

Lemma 1 Let $y(t)=u(t)+iv(t)$ be a complex-valued solution of $ay''+by'+cy=0$ with a, b, c real. Then $y_1(t)=u(t)$ and $y_2(t)=v(t)$ are two real-valued solutions. I.e. both the real and imaginary parts of a complex-valued solution of $ay''+by'+cy=0$ are its solutions.

Q. What is e^{rt} for r complex?

A: Let $r=\underset{\substack{\uparrow \\ \text{real}}}{\alpha}+i\underset{\substack{\uparrow \\ \text{real}}}{\beta}$, $e^{rt}=e^{(\alpha+i\beta)t}=e^{\alpha t}e^{i\beta t}=e^{\alpha t}(\cos\beta t+\sin\beta t)$

The solution of $ay''+by'+cy=0$ is a complex-valued function if $b^2-4ac<0$. Recall:

$$r_1=\frac{-b+i\sqrt{4ac-b^2}}{2a}, \quad r_2=\frac{-b-i\sqrt{4ac-b^2}}{2a}$$

so by lemma 1,

$$y_1(t)=e^{r_1 t}=e^{-\frac{b}{2a}t}\cos\beta t$$

$$y_2(t)=e^{r_2 t}=e^{-\frac{b}{2a}t}\sin\beta t$$

for $\beta=\dfrac{\sqrt{4ac-b^2}}{2a}$ are real-valued solutions of the diff. eqn.

Check that these two solutions are linearly independent by showing that their Wroskian is never zero. Thus, the general solution for $b^2-4ac<0$ is

$$y(t) = e^{-\frac{bt}{2a}}\left(c_1\cos\beta t + c_2\sin\beta t\right), \quad \beta = \frac{\sqrt{4ac-b^2}}{2a}$$

Remark. We must verify that $\frac{d}{dt}e^{rt} = re^{rt}$ is true for r complex before we can say that e^{r_1t} and e^{r_2t} are complex-valued solutions of the diff. eqn. $ay''+by'+cy=0$.

$$\begin{aligned}
\frac{d}{dt}e^{(\alpha+i\beta)t} &= \frac{d}{dt}\left(e^{\alpha t}(\cos\beta t + i\sin\beta t)\right)\\
&= e^{\alpha t}\left[(\alpha\cos\beta t - \beta\sin\beta t) + i(\alpha\sin\beta t + \beta\cos\beta t)\right]\\
&= e^{\alpha t}\left[\alpha(\cos\beta t + i\sin\beta t) + i\beta(\cos\beta t + i\sin\beta t)\right]\\
&= e^{\alpha t}\left(\underbrace{\cos\beta t + i\sin\beta t}_{e^{i\beta t}}\right)(\alpha+i\beta)\\
&= e^{(\alpha+i\beta)t}(\alpha+i\beta)\\
&= re^{rt}
\end{aligned}$$

//

Example Find two linearly independent real-valued solutions of

$$4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 0$$

Characteristic eqn: $4r^2+4r+5=0$

$$r = \frac{-4\pm\sqrt{16-4(4)(5)}}{2(4)} = \frac{-4\pm\sqrt{-64}}{8} = \frac{-4}{8} \pm i\,\frac{8}{8} = -\frac{1}{2} \pm i$$

$\Rightarrow r_1 = -\frac{1}{2}+i, \quad r_2 = -\frac{1}{2}-i$

Thus $e^{r_1 t} = e^{(-\frac{1}{2}+i)t} = e^{-\frac{1}{2}t}e^{it} = e^{-\frac{1}{2}t}(\cos t + i\sin t)$

By <u>Lemma 1</u> $y_1(t) = Re\{e^{r_1 t}\} = e^{-\frac{1}{2}t}\cos t$

$y_2(t) = Im\{e^{r_1 t}\} = e^{-\frac{1}{2}t}\sin t$

are two linearly independent real-valued solutions of $4\frac{d^2y}{dt^2}+4\frac{dy}{dt}+5y=0.$

<u>Example</u> Solve the IVP $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+4y=0$; $y(0)=1$, $y'(0)=1$

Characteristic eqn. $r^2+2r+4=0$

$$r = \frac{-2\pm\sqrt{4-4(4)}}{2} = -1\pm\frac{\sqrt{-12}}{2} = -1\pm\frac{i\cancel{2}\sqrt{3}}{\cancel{2}} = -1\pm i\sqrt{3}$$

$$\Rightarrow e^{r_1 t} = e^{(-1+i\sqrt{3})t} = e^{-t}(\cos(\sqrt{3}t)+i\sin(\sqrt{3}t))$$

$y_1(t) = e^{-t}\cos(\sqrt{3}t)$

$y_2(t) = e^{-t}\sin(\sqrt{3}t)$

and the general solution is $y(t) = c_1y_1(t)+c_2y_2(t)$

$$= e^{-t}[c_1\cos(\sqrt{3}t)+c_2\sin(\sqrt{3}t)]$$

Now use the initial conditions to find c_1 and c_2:

$$y(0)=1 \Rightarrow 1 = \underset{1}{e^{\cancel{0}}}[c_1\underset{1}{\cos(\cancel{0})}+c_2\underset{0}{\sin\cancel{0}}]$$

$$\Rightarrow \boxed{1 = c_1}$$

$$y'(t) = -e^{-t}(c_1\cos(\sqrt{3}t)+c_2\sin(\sqrt{3}t)) + e^{-t}(-\sqrt{3}c_1\sin(\sqrt{3}t)+\sqrt{3}c_2\cos(\sqrt{3}t))$$

$$y'(0)=1 \Rightarrow 1 = -\underset{1 \text{ from above}}{\cancel{c_1}}+\sqrt{3}c_2$$

$$\Rightarrow 2 = \sqrt{3}c_2$$

$$\Rightarrow \boxed{c_2 = \frac{2}{\sqrt{3}}}$$

Thus the solution is $y(t) = e^{-t}\left[\cos(\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)\right]$

- If $b^2-4ac=0$ then the characteristic equation $ar^2+br+c=0$ has equal roots $r_1=r_2=-b/2a$

We get only one solution $\boxed{y_1(t) = e^{-\frac{bt}{2a}}}$ of $ay''+by'+cy=0$

METHOD OF REDUCTION OF ORDER

Q: How do we find a 2nd solution which is independent of y_1?

A: Let's define a new dependent variable v through

$$\boxed{y(t)=y_1(t)\cdot v(t)}$$

Then by the product rule $\frac{dy}{dt} = \frac{dy_1}{dt}v(t) + y_1(t)\frac{dv}{dt}$

$$\frac{d^2y}{dt^2} = \frac{d^2y_1}{dt^2}v(t) + \frac{dy_1}{dt}\frac{dv}{dt} + \frac{dy_1}{dt}\frac{dv}{dt} + y_1(t)\frac{d^2v}{dt^2}$$

$$= \frac{d^2y_1}{dt^2}v + 2\frac{dy_1}{dt}\frac{dv}{dt} + y_1\frac{d^2v}{dt^2}$$

Thus for the case of a linear 2nd order diff. eqn. (not necessarily w/ constant coefficients) we have

$$L[y](t) = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

$$= \frac{d^2y_1}{dt^2}v + 2\frac{dy_1}{dt}\frac{dv}{dt} + y_1\frac{d^2v}{dt^2} + p(t)\left[\frac{dy_1}{dt}v + y_1\frac{dv}{dt}\right] + q(t)y_1v$$

$$= y_1\frac{d^2v}{dt^2} + \frac{dv}{dt}\left[2\frac{dy_1}{dt} + p(t)y_1\right] + v\underbrace{\left[\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right]}_{\overset{\shortparallel}{0}\text{ since } y_1 \text{ is a solution } L[y_1](t)=0}$$

$$= y_1\frac{d^2v}{dt^2} + \left[2\frac{dy_1}{dt} + p(t)y_1\right]\frac{dv}{dt}.$$

This implies that $y(t) = y_1(t)v(t)$ is a solution if v satisfies

$$y_1 \underbrace{\frac{d^2v}{dt^2}}_{} + \left[2\frac{dy_1}{dt} + p(t)y_1\right]\underbrace{\frac{dv}{dt}}_{} = 0$$

If $\boxed{u = \frac{dv}{dt}}$ this becomes $y_1\frac{du}{dt} + \left[2\frac{dy_1}{dt} + p(t)y_1\right]u = 0$

which is a first order diff. equation for which we can use the integrating factor

Rewrite: $\frac{du}{dt} + \left[\frac{2}{y_1}y_1' + p(t)\right]u = 0$

$$\frac{y_1'(t)}{y_1(t)}$$

I.F. $\mu(t) = e^{\int\left[\frac{2}{y_1}y_1' + p(t)\right]dt} = e^{2\int\left(\frac{y_1'}{y_1}\right)dt}\, e^{\int p(t)dt}$

$$= e^{2\cancel{\int}\cancel{\frac{d}{dt}}\left(\ln(y_1(t))\right)dt}\, e^{\int p(t)dt}$$

$$= e^{2\ln(y_1(t))}\, e^{\int p(t)dt}$$

$$= e^{\ln(y_1(t)^2)}\, e^{\int p(t)dt}$$

$$= y_1^2\, e^{\int p(t)dt}$$

Now $\mu(t)\frac{du}{dt} + \mu(t)\left[2\frac{y_1'}{y_1} + p(t)\right]u = 0$

$$\frac{d}{dt}\left[\mu(t)u\right] = 0$$

$\mu(t)u = c$ for some constant c

$$u = \frac{c}{\mu(t)} = \frac{c}{y_1^2}e^{-\int p(t)dt}$$

But $u = \frac{dv}{dt}$ and so $v = \frac{dv}{dt} = \frac{ce^{-\int p(t)dt}}{y_1^2}$

wlog can take $\boxed{c=1}$ ↓

If we integrate again wrt t we obtain $v(t) = \int u(t)dt$ with $\boxed{u = \frac{ce^{-\int p(t)dt}}{y_1^2}}$

and thus the 2nd solution which is linearly independent to $y_1(t)$ is

$$y_2(t) = y_1(t)v(t)$$

$$\Rightarrow \boxed{y_2(t) = y_1(t)\int u(t)\,dt}$$

$y_2 \neq ky_1$ (not a constant multiple because $u = \frac{ce^{-\int p(t)dt}}{y^2}$ is never zero)

Remark: This is known as the method of reduction of order because the substitution we used; $y(t) = y_1(t)v(t)$ reduces the problem from a 2nd order diff. eqn. to a 1st order diff. eqn.

APPLICATION TO EQUAL ROOTS: We found $\boxed{y_1(t) = e^{-\frac{bt}{2a}}}$ as one solution of

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

We first want to write this in the form $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$, so where the coefficient of $\frac{d^2y}{dt^2}$ is one.

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{b}{a}\frac{dy}{dt} + \frac{c}{a}y = 0.$$

comparing the two: $\boxed{p(t) = \frac{b}{a}}$

and so we get

$$u(t) = \frac{dv}{dt} = \frac{e^{-\int p(t)dt}}{y_1^2(t)} = \frac{e^{-\int \frac{b}{a}dt}}{\left(e^{-\frac{bt}{2a}}\right)^2} = \frac{e^{-\frac{bt}{a}}}{e^{-\frac{bt}{a}}} = 1$$

Therefore $y_2(t) = y_1(t)\int u(t)\,dt = e^{-\frac{bt}{2a}}\int 1\,dt = te^{-\frac{bt}{2a}}$ is a second solution of the diff. eqn.

$y_1 = e^{-\frac{bt}{2a}}$ and $y_2 = te^{-\frac{bt}{2a}}$ are linearly independent on the interval $-\infty < t < \infty$

∴ The general solution is $y(t) = c_1y_1(t) + c_2y_2(t)$

$$\Rightarrow \boxed{y(t) = [c_1 + c_2t]\,e^{-\frac{bt}{2a}}}$$

in the case of equal roots.

Example Solve the IVP $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + y = 0;\ y(0)=1,\ y'(0)=0$

Characteristic equation: $9r^2+6r+1=0$

$$(3r+1)^2=0$$

$$r=-\frac{1}{3}\ \text{(twice)}$$

Hence the general solution is $y(t)=c_1e^{-\frac{1}{3}t}+c_2te^{-\frac{1}{3}t}.$

Now use $y(0)=1$: $\boxed{1=c_1}$

$y'(0)=0$: $y'(t) = -\frac{1}{3}c_1e^{-\frac{1}{3}t}+c_2e^{-\frac{1}{3}t}-\frac{1}{3}c_2te^{-\frac{1}{3}t}$

$$0 = -\frac{1}{3}\underset{1}{\cancel{c_1}}+c_2 \Rightarrow \boxed{\frac{1}{3}=c_2}$$

Thus, the solution to the IVP is $\boxed{y(t)=e^{-\frac{1}{3}t}+\frac{1}{3}te^{-\frac{1}{3}t}}$

Example (method of reduction of order)

Solve the IVP $(1-t^2)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0,\quad y(0)=3,\ y'(0)=-4$ on the interval $-1<t<1$, given one of the solutions is $y_1(t)=t$.

Using the method of reduction of order we have that a second solution $y_2(t)$ is found by $u(t)=\frac{e^{-\int p(t)\,dt}}{y_1^2(t)}$.

First we rewrite the equation such that the coeff. of y'' is 1, i.e.

$$\frac{d^2y}{dt^2}+\underset{\overset{\shortparallel}{p(t)}}{\boxed{\frac{2t}{1-t^2}}}\frac{dy}{dt}-\frac{2}{1-t^2}y=0$$

$$u(t) = \frac{e^{-\int \frac{2t}{1-t^2}dt}}{y_1^2} = \frac{e^{-(-\ln(1-t^2))}}{t^2} = \frac{e^{\ln(1-t^2)}}{t^2} = \frac{1-t^2}{t^2}$$

and $y_2(t) = y_1(t)\int u(t)\,dt = t\int \frac{1-t^2}{t^2}\,dt = t\int\left(\frac{1}{t^2}-1\right)dt = t\left(-\frac{1}{t}-t\right)$

$= -1-t^2$

Thus $y(t) = c_1y_1(t) + c_2y_2(t) = c_1t - c_2(1+t^2)$

Using the I.C. $y(0)=3$, $y'(0) = -4$, we get the values of c_1 and c_2.

$$y'(t) = c_1 - c_2(2t)$$

$y(0) = 3 \Rightarrow \quad -c_2 = 3 \Rightarrow \boxed{c_2 = -3}$

$y'(0) = -4 \Rightarrow \quad \boxed{c_1 = -4}$

Thus $\boxed{y(t) = -4t + 3(1+t^2)}$ is the solution to the diff. equ.

Section 2.3 : The nonhomogeneous equation

Consider now $L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = g(t)$ $(*)$

Continuous on $\alpha<t<\beta$

Theorem 5 (from textbook) : let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of the homogeneous equation

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = 0$$

and let $\psi(t)$ be a particular solution of the nonhomogeneous eqn $(*)$

Then, every solution $y(t)$ of $(*)$ must be of the form

$$y(t) = \underbrace{c_1y_1(t) + c_2y_2(t)}_{\text{from solving the homogeneous problem}} + \underbrace{\psi(t)}_{\text{Particular solution of nonhomogeneous equation}}$$

for some choices c_1, c_2.

Lemma The difference of any two solutions of the nonhomogeneous equation $(*)$ is a solution of the homogeneous eqn.

Proof let $\psi_1(t)$ and $\psi_2(t)$ be two solutions of $(*)$. By linearity of L

$$\begin{aligned} L[\psi_1-\psi_2](t) &= L[\psi_1](t) - L[\psi_2](t) \\ &= g(t) - g(t) \\ &= 0 \end{aligned}$$

↳ R.H.S. of L since it's a nonhomogeneous problem

So $\psi_1(t)-\psi_2(t)$ is a solution of the homogeneous problem

($L[y](t)=0 \Rightarrow y(t)$ is a solution of the homogeneous problem &
$L[\psi_1-\psi_2](t)=0$ for $y(t)=\psi_1(t)-\psi_2(t)$) $\square$

Proof of theorem 5: Let $y(t)$ be any solution of $(*)$. By the lemma, $\phi(t) = y(t) - \psi(t)$ is a solution of the homogeneous problem $y''+p(t)y'+q(t)y=0$. But every solution $\phi(t)$ of the homogeneous equation is of the form

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t)$$

for constants c_1, c_2, so

$$\begin{aligned} y(t) &= \phi(t) + \psi(t) \\ &= c_1 y_1(t) + c_2 y_2(t) + \psi(t). \end{aligned}$$

□

Theorem 5 is useful because it tells us we can find two solutions of the homogeneous problem & one solution of the nonhomogeneous problem instead of all solutions of $(*)$.

Example Three solutions of a specific 2nd order nonhomogeneous linear eqn are $\psi_1(t) = t$, $\psi_2(t) = t + e^t$, $\psi_3(t) = 1 + t + e^t$. Find the general solution.

By the lemma:

$$y_1(t) = \psi_2 - \psi_1 = \cancel{t} + e^t - \cancel{t} = e^t$$

$$y_2(t) = \psi_3 - \psi_2 = 1 + \cancel{t} + \cancel{e^t} - (\cancel{t} + \cancel{e^t}) = 1$$

these are two solutions of the homogeneous problem. They are also linearly independent. By theorem 5, every solution is of the form

$$\begin{aligned} y(t) &= c_1 y_1 + c_2 y_2 + \psi(t) \\ &= c_1 e^t + c_2 + t. \end{aligned}$$

Section 2.4: The method of variation of parameters

Q: How do we find a particular solution $\psi(t)$ of the nonhomogeneous eqn

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = g(t)$$

once we know the solutions of the homogeneous eqn?

A: let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of the homogeneous eqn $L[y] = y'' + p(t)y' + q(t)y = 0$. we'll try to find a P.S. (particular solution) $\psi(t)$ of the nonhomogeneous eqn, of the form

$$\boxed{\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t)}$$

i.e. We'll try to find functions $u_1(t)$ and $u_2(t)$ so that the linear combination $u_1(t)y_1(t) + u_2(t)y_2(t)$ is a solution. We compute

$$\begin{aligned}\frac{d}{dt}[\psi(t)] &= \frac{d}{dt}[u_1(t)y_1(t) + u_2(t)y_2(t)]\\ &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'\\ &= [u_1'y_1 + u_2'y_2] + [u_1y_1' + u_2y_2']\end{aligned}$$

We want to simplify this problem to finding solutions $u_1(t)$ and $u_2(t)$ of two very simple first-order equations

We see that $\frac{d^2}{dt}[\psi(t)]$ will have no 2nd order derivatives of u_1 and u_2 if

$$\boxed{u_1'y_1 + u_2'y_2 = 0}$$

So we want to impose this condition on the fcns $u_1(t)$ and $u_2(t)$.

$$\begin{aligned}L[\psi](t) &= \psi'' + p(t)\psi' + q(t)\psi\\ &= [u_1y_1' + u_2y_2']' + p(t)[u_1y_1' + u_2y_2'] + q(t)[u_1y_1 + u_2y_2]\\ &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p(t)u_1y_1' + p(t)u_2y_2' + q(t)u_1y_1 + q(t)u_2y_2\\ &= u_1'y_1' + u_2'y_2' + u_1[\underbrace{y_1'' + p(t)y_1' + q(t)y_1}_{=0}] + u_2[\underbrace{y_2'' + p(t)y_2' + q(t)y_2}_{=0}]\\ &= u_1'y_1' + u_2'y_2'\end{aligned}$$

since y_1 and y_2 are solutions of the homogeneous equation $L[y]=0$.

$\Rightarrow$ $\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ is a solution of the nonhomogeneous eqn. if $u_1(t)$ and $u_2(t)$ satisfy

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}}$$

multiply by y_2'

multiply by y_2 , and subtract

$$u_1' y_1 y_2' + u_2' \cancel{y_2 y_2'} = 0$$

$$u_1' y_1' y_2 + u_2' \cancel{y_2' y_2} = g(t) y_2 \quad (-)$$

$$u_1' \underbrace{(y_1 y_2' - y_1' y_2)}_{W[y_1, y_2](t)} = -g(t) y_2$$

$$\Rightarrow \boxed{u_1'(t) = \frac{-g(t)\, y_2(t)}{W[y_1, y_2](t)}}$$

Similarly,

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}}$$

multiply by y_1'

multiply by y_1 , and subtract

$$\cancel{u_1' y_1 y_1'} + u_2' y_2 y_1' = 0$$

$$\cancel{u_1' y_1' y_1} + u_2' y_2' y_1 = g(t) y_1 \quad (-)$$

$$u_2' \underbrace{(y_2 y_1' - y_2' y_1)}_{-W[y_1, y_2](t)} = -g(t) y_1$$

$$\Rightarrow \boxed{u_2'(t) = \frac{g(t)\, y_1(t)}{W[y_1, y_2](t)}}$$

To obtain $u_1(t)$ & $u_2(t)$ integrate both w.r.t. t.

Note: The general solution of the homogeneous eqn is

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

In what we did above we used $y(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$ so we are essentially allowing the constants c_1 and c_2 to vary with time. That's why this method is known as the method of variation of parameters.

Example: (a) Find a particular solution $\psi(t)$ of the equation

$$y'' + y = \tan t$$

on the interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

(b) Find the solution to the same diff. eqn. but w/ initial conditions $y(0) = 1$, $y'(0) = 1$.

Characteristic eqn: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$$y_1(t) = Re\{e^{r_1 t}\} = \cos t$$

$$y_2(t) = Im\{e^{r_1 t}\} = \sin t$$

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = \cos t \cos t - (-\sin t)\sin t = 1 \neq 0$$

$\therefore$ y_1 & y_2 are linearly independent.

From the method of variation of parameters we have

$$u_1'(t) = \frac{-g(t) y_2}{W[y_1, y_2](t)}, \quad u_2'(t) = \frac{g(t) y_1}{W[y_1, y_2](t)}$$

Here,
$g(t) = \tan t$ and $W[y_1, y_2](t) = 1$ so

$$u_1(t) = \int -\frac{\tan t \cdot \sin t}{1}\, dt = -\int \frac{\sin t}{\cos t} \sin t\, dt = -\int \frac{\sin^2 t}{\cos t}\, dt$$

$$= -\int \frac{1-\cos^2 t}{\cos t}\,dt = -\int\left(\frac{1}{\cos t} - \cos t\right)dt$$

$$= \int(\cos t - \sec t)\,dt = \sin t - \ln|\sec t + \tan t|, \quad -\frac{\pi}{2}<t<\frac{\pi}{2}$$

and $u_2(t) = \int \frac{\tan t \cos t}{1}\,dt = \int \frac{\sin t}{\cancel{\cos t}}\cdot\cancel{\cos t}\,dt = -\cos t$

Thus $\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$

$$= \left(\sin t - \ln|\sec t + \tan t|\right)\cos t - \cos t\sin t$$

$$= \cancel{\sin t\cos t} - \ln|\sec t + \tan t|\cos t - \cancel{\cos t\sin t}$$

$$= -\ln|\sec t + \tan t|\cos t$$

This is the particular solution of $y'' + y = \tan t$ on the interval $-\frac{\pi}{2}<t<\frac{\pi}{2}$.

(b) For the IVP: $y(0)=1$ and $y'(0)=1$

The general solution is $y(t) = c_1 y_1 + c_2 y_2 + \psi(t)$

$$\Rightarrow \boxed{y(t) = c_1\cos t + c_2\sin t - \ln|\sec t + \tan t|\cos t.}$$

for constants c_1 and c_2.

$$y'(t) = -c_1\sin t + c_2\cos t - \left(\cancelto{\sec(t)}{\frac{\sec(t)\tan(t) + \sec^2(t)}{\sec(t) + \tan(t)}}\right)\cos t$$

$$+ \ln|\sec t + \tan t|\sin t$$

$$= -c_1\sin t + c_2\cos t - \sec t\cos t + \ln|\sec t + \tan t|\sin t$$

$y(0)=1 \Rightarrow \quad 1 = c_1 - \cancelto{0}{\ln|1|} \quad \Rightarrow \quad \boxed{c_1 = 1}$

$y'(0)=1 \Rightarrow \quad 1 = c_2 - 1 + \cancelto{0}{\ln(1)}\cdot 0 \quad \Rightarrow \quad \boxed{c_2 = 2}$

Thus the solution to the IVP is

$$\boxed{y(t) = \cos t + 2\sin t - \ln|\sec t + \tan t|\cos t}$$

Section 2.8: Series solutions

Homogeneous linear 2nd order eqn: $L[y] = P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0$

$\uparrow$ (under $P(t)$) $\neq 0$ in $\alpha < t < \beta$

We already showed that every solution is of the form $y(t) = c_1 y_1(t) + c_2 y_2(t)$ for $y_1(t)$ and $y_2(t)$ linearly independent.

Previously, $P(t), Q(t), R(t)$ were all constants. Now we consider the case where they are <u>polynomials</u>. We can determine a polynomial solution $y(t)$ by setting the sums of the coefficients of like powers of t in $L[y](t)$ equal to zero.

<u>Example</u>. Find two linearly independent solutions of

$$L[y](t) = y'' - 2ty' - 2y = 0.$$

We set $\boxed{y(t) = \sum_{n=0}^{\infty} a_n t^n} = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$

$\Rightarrow \quad y'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \ldots = \sum_{n=0}^{\infty} n a_n t^{n-1}$

$\Rightarrow \quad y''(t) = 2a_2 + 6a_3 t + \ldots = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$

Plugging them into $L[y](t) = y'' - 2ty' - 2y$ gives us

$$L[y](t) = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - 2t\sum_{n=0}^{\infty} na_n t^{n-1} - 2\sum_{n=0}^{\infty} a_n t^n$$

$$= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - 2\sum_{n=0}^{\infty} na_n t^n - 2\sum_{n=0}^{\infty} a_n t^n$$

$$= 0$$

Next, we rewrite the first summation $\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2}$ such that the exponent of t is n instead of $n-2$ so that it matches the other two summations.

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} \rightarrow \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2} t^n$$

$$\downarrow$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n$$

(since the contribution to this sum from $n=-2$, $n=-1$ is zero since the factor $(n+2)(n+1)$ vanishes in both of these instances)

Therefore, $L[y](t) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - 2\sum_{n=0}^{\infty} na_n t^n - 2\sum_{n=0}^{\infty} a_n t^n = 0$

Setting the coefficients of like powers in t equal to zero gives

t^n: $(n+2)(n+1)a_{n+2} - 2na_n - 2a_n = 0$

$$a_{n+2} = \frac{2\cancel{(n+1)}a_n}{(n+2)\cancel{(n+1)}} = \frac{2a_n}{n+2}.$$ ← recurrence formula for the coefficients a_n

So once a_0 and a_1 are prescribed, all the coefficients are determined uniquely. The values of a_0 and a_1 are arbitrary unless we are given specific initial conditions.

To find two solutions of the diff. eqn. we choose two sets of a_0, a_1.

(1) $a_0 = 1, \; a_1 = 0$

(2) $a_0 = 0, \; a_1 = 1$

→ (1) $\boxed{a_0 = 1, \; a_1 = 0}$

Recall $a_{n+2} = \frac{2a_n}{n+2}$

$n=0$: $a_2 = \frac{2a_0}{2} = 1$

$n=1$: $a_3 = \frac{2a_1}{3} = 0$

$n=2$: $a_4 = \frac{2a_2}{4} = \frac{1}{2}(1) = \frac{1}{2}$

$n=3$: $a_5 = \frac{2a_3}{5} = 0$

$n=4$: $a_6 = \frac{2a_4}{6} = \frac{2\left(\frac{1}{2}\right)}{6} = \frac{1}{6} = \frac{1}{2 \cdot 3}$

⋮

- All odd coefficients are zero since they all depend on a_1 originally which here is set as zero.
- The even coefficients are found through

$$a_{2n} = \frac{1}{2 \cdot 3 \cdots n} = \frac{1}{n!}$$

Therefore, $y_1(t) = a_0 + \cancelto{0}{a_1} t + a_2 t^2 + \cancelto{0}{a_3} t^3 + \ldots$

$= 1 + t^2 + \frac{1}{2!} t^4 + \frac{1}{3!} t^6 + \ldots$

$= e^{t^2}$ ← is one solution of the diff. eqn

(since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ with $x = t^2$)

→ (2) $\boxed{a_0 = 0,\ a_1 = 1}$

This time all even coefficients are zero & only the odd ones are nonzero.

Recall $a_{n+2} = \frac{2a_n}{n+2}$

$$n=1: \quad a_3 = \frac{2a_1}{3} = \frac{2}{3}$$

$$n=3: \quad a_5 = \frac{2a_3}{5} = \frac{2}{5}\left(\frac{2}{3}\right)$$

$$n=5: \quad a_7 = \frac{2a_5}{7} = \frac{2}{7}\left(\frac{2}{5}\right)\left(\frac{2}{3}\right)$$

$$\vdots$$

Thus $a_{2n+1} = \frac{2^n}{3\ 5\cdot 7\cdot\ \cdot(2n+1)}$ (you can show this by induction)

Therefore, $y_2(t) = \cancelto{0}{a_0} + a_1 t + \cancelto{0}{a_2} t^2 + a_3 t^3 + \cdots$

$$= t + \frac{2}{3}t^3 + \frac{2^2}{3\cdot 5}t^5 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2^n t^{2n+1}}{3\cdot 5\cdots(2n+1)}$$ ← is a second sol^n of the diff. eqn.

Notes:

(A) Infinite Series $y(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n$: power series about $t = t_0$

(B) Radius of convergence of the power series: $\rho \geq 0$ s.t.

$|t-t_0| < \rho$: infinite series converges

$|t-t_0| > \rho$: infinite series diverges

(C) You can differentiate and integrate each term separately, maintaining the same interval of convergence.

(D) Use the ratio test to determine the interval of convergence.

i.e Compute $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lambda$.

$|t-t_0| < \frac{1}{\lambda}$: power series converges

$|t-t_0| > \frac{1}{\lambda}$: power series diverges

(E) The product of $\sum_{n=0}^{\infty} a_n(t-t_0)^n$ and $\sum_{n=0}^{\infty} b_n(t-t_0)^n$ is a power series of the form $\sum_{n=0}^{\infty} c_n(t-t_0)^n$ where $c_n = a_0b_n + a_1b_{n-1} + \dots + a_nb_0$.

The quotient $\dfrac{a_0+a_1t+a_2t^2+\dots}{b_0+b_1t+b_2t^2+\dots}$ is also a power series given that $b_0 \neq 0$

Theorem 6 (from textbook)

Let the variable t assume complex values. Let z_0 be the point closest to t_0 at which f or one of its derivatives fails to exist. Compute the distance $\rho \in \mathbb{C}$ between t_0 and z_0. Then the Taylor series of f about t_0 converges for $|t-t_0| < \rho$ and diverges for $|t-t_0| > \rho$.

Theorem 7 (from textbook)

Consider the diff. eqn. $L[y](t) = P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0$

Let the functions $\frac{P(t)}{Q(t)}$ and $\frac{R(t)}{P(t)}$ have convergent Taylor series expansions about $t=t_0$ for $|t-t_0|<\rho$. Then every solution $y(t)$ of the diff. eqn. is analytic at $t=t_0$ and the radius of convergence about $t=t_0$ is at least ρ.

You can determine the coeff. $a_2, a_3, \dots$ in the Taylor series expansion

$$y(t) = a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + \dots$$

by plugging the series above into the diff. eqn. and setting the sum of the coefficients of like powers of t, equal to zero.

Example: (a) Find two linearly independent solutions of

$$L[y](t) = \frac{d^2y}{dt^2} + \frac{3t}{1+t^2}\frac{dy}{dt} + \frac{1}{1+t^2}y = 0$$

(b) Solve the diff. eqn in (a) with initial conditions $y(0)=2, y'(0)=3$.

(a) It's easier to multiply the diff. eqn. by $(1+t^2)$ to get it in the form

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0$$

$$\Rightarrow (1+t^2)\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + y = 0$$

Now set $y(t) = \sum_{n=0}^{\infty} a_n t^n$. We get

$$(1+t^2)\sum_{n=0}^{\infty} a_n n(n-1)t^{n-2} + 3t\sum_{n=0}^{\infty} a_n n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty} a_n n(n-1)t^{n-2}}_{\text{rewrite this such that the power of } t \text{ is } n \text{ instead of } n-2} + \underbrace{\sum_{n=0}^{\infty} a_n n(n-1)t^n + 3\sum_{n=0}^{\infty} a_n n t^n + \sum_{n=0}^{\infty} a_n t^n}_{\text{We can combine these 3 terms}} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)t^n + \sum_{n=0}^{\infty} a_n \underbrace{\left(n(n-1)+3n+1\right)}_{\substack{\| \\ n^2-n+3n+1 \,=\, n^2+2n+1 \,=\, (n+1)^2}} t^n = 0$$

$$\Rightarrow \sum_{n=0} \left[a_{n+2}(n+2)(n+1) + a_n(n+1)^2\right]t^n = 0$$

$$\Rightarrow a_{n+2}(n+2)(n+1) = -a_n(n+1)^2$$

$$\Rightarrow a_{n+2} = \frac{-a_n(n+1)^{\cancel{2}}}{(n+2)\cancel{(n+1)}} = -\frac{a_n(n+1)}{n+2}$$

Recurrence relationship for the coefficients

As before, to find two linearly independent solutions of the diff. eqn consider the simplest cases (i) $a_0 = 1,\ a_1 = 0$

(ii) $a_0 = 0,\ a_1 = 1$

→ (i) $\boxed{a_0 = 1,\ a_1 = 0}$

All odd coefficients are zero

The even ones are $a_{n+2} = -\frac{a_n(n+1)}{n+2}$.

$n=0 \quad a_2 = \frac{-a_0}{2} = -\frac{1}{2}$

$n=2 \quad a_4 = \frac{-a_2(3)}{4} = -\left(-\frac{1}{2}\right)\left(\frac{3}{4}\right)$

$n=4 \quad a_6 = \frac{-a_4(5)}{6} = -\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)$

$\vdots$

$$a_{2n} = (-1)^n \frac{1\cdot3\cdot5\cdots(2n-1)}{\boxed{2\cdot4\cdot6\cdots(2n)}} = (-1)^n \frac{1\cdot3\cdots(2n-1)}{2^n n!}$$

$= 2(2\cdot2)(2\cdot3)\cdots(2\ n)$

$= 2^n(1\ 2\cdot3\cdots n)$

$= 2^n n'$

Thus, the first solution is
$$y_1(t) = a_0 + \cancelto{0}{a_1 t} + a_2 t^2 + \cancelto{0}{a_3 t^3} + \cdots$$
$$= 1 - \frac{1}{2}t^2 + \frac{1\cdot3}{2\cdot4}t^4 + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot3\cdots(2n-1)}{2^n n!} t^{2n}$$

is one solution.

in the absolute value this won't matter

$2n+2-1 = 2n+1$

<u>Ratio test</u> :
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\boxed{(-1)^{\cancel{n}+1}}\,\frac{1\cdot3\cdots\overbrace{(2(n+1)-1)}}{2^{\cancel{n}+1}\boxed{(n+1)!}}\,t^{2(\cancel{n}+1)}}{\cancel{(-1)^{n}}\,\frac{1\cdot3\cdots(2n-1)}{\cancel{2^{n}}\cancel{n!}}\,\cancel{t^{2n}}}\right|$$

$= (n+1)\cancel{n!}$

$$= \lim_{n\to\infty} \left| \frac{(2n+1)t^2}{2(n+1)} \right|$$

$$= t^2 \lim_{n\to\infty} \left| \frac{2n+1}{2n+2} \right| \rightarrow \left[\frac{2+\frac{1}{n}}{2+\frac{2}{n}} \rightarrow 0 \text{ as } n\to\infty \right]$$

$$= t^2$$

Thus by the ratio test the infinite series converges by $|t|<1$, diverges $|t|>1$

→ (ii) $\boxed{a_0 = 0,\ a_1 = 1}$

ALL even coefficients are zero

$$a_{n+2} = -\frac{a_n(n+1)}{n+2}$$

Odd coefficients: $a_3 = \frac{-2a_1}{3} = -\frac{2}{3}$

$$a_5 = -\frac{4a_3}{5} = \frac{2\cdot 4}{3\cdot 5}$$

$$a_7 = -\frac{6a_5}{7} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}$$

$$\Rightarrow a_{2n+1} = (-1)^n \frac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)} = \frac{(-1)^n 2^n n!}{3\cdot 5\cdots(2n+1)}$$

Therefore $y_2(t) = t - \frac{2}{3}t^3 + \frac{2\cdot 4}{3\cdot 5}t^5 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{3\cdot 5\cdots(2n+1)} t^{2n+1}$

is the second solution.

It can be shown using the ratio test that this solution also converges for $|t|<1$ and diverges for $|t|>1$.

(b) For the IVP we want to satisfy $y(0)=2$, $y'(0)=3$.

We found $y_1(t) = \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2^n n!} t^{2n} = 1 - \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \cdots$

$$y_2(t) = \sum_{n=0}^{\infty} (-1)^n \frac{2^n n!}{3\cdot 5\cdot \cdots (2n+1)} t^{2n+1} = t - \frac{2}{3}t^3 + \frac{2\cdot 4}{3\cdot 5}t^5 + \cdots$$

$$\begin{bmatrix} y_1(0) = 1, & y_1'(0) = 0 \\ y_2(0) = 0, & y_2'(0) = 1 \end{bmatrix}$$

So if we want to satisfy $y(0) = c_1 y_1(0) + c_2 y_2(0) = 2$

$$y'(0) = c_1 y_1'(0) + c_2 y_2'(0) = 3$$

we must have $\begin{bmatrix} c_1 = 2 \\ c_2 = 3 \end{bmatrix}$ which implies that $\boxed{y(t) = 2y_1(t) + 3y_2(t).}$

Section 2.8.1: Singular points, Euler equations

Consider again $L[y] = P(t)\dfrac{d^2y}{dt^2} + Q(t)\dfrac{dy}{dt} + R(t)y = 0$

If $\boxed{P(t_0) = 0 \text{ at } t = t_0}$ then we call this a singular differential equation.

In the neighborhood of the singular point t_0 the solutions of the diff. eqn can become very large or oscillate very rapidly and solutions may not be continuous at t_0. So the method of power series will, in general, fail to work.

Definition EULER'S EQUATION

The diff. eqn. $\boxed{L[y](t) = t^2\dfrac{d^2y}{dt^2} + \alpha t\dfrac{dy}{dt} + \beta y = 0}$, where α and β are constants is known as Euler's equation.

We assume for simplicity that $t > 0$.

Note: $t^2 y''$ and ty' are both multiples of t^r if $y = t^r$

$= t^2 r \cdot (r-1) t^{r-2}$ $\qquad$ $t\,r(t^{r-1})$

$= r(r-1)t^r$ $\qquad$ $= r t^r$

This suggests that we can try $\boxed{y=t^r}$ as the solution of Euler's equation.

$$L[t^r] = r(r-1)t^r + \alpha r t^r + \beta t^r$$

$$= [r(r-1)+\alpha r+\beta]\,t^r$$

$$= F(r)\,t^r$$

where $F(r) = r(r-1)+\alpha r+\beta$

$$= r^2+(\alpha-1)r+\beta$$

This implies that $y=t^r$ is a solution of Euler's equation iff $F(r)=0$, i.e. $\boxed{r^2+(\alpha-1)r+\beta=0}$

Using the quadratic formula the two roots are:

$$r_1 = \frac{-(\alpha-1)+\sqrt{(\alpha-1)^2-4\beta}}{2}, \quad r_2 = \frac{-(\alpha-1)-\sqrt{(\alpha-1)^2-4\beta}}{2}$$

As before, here too, the term under the square root can be +ve, 0, or -ve.

CASE 1: $\boxed{(\alpha-1)^2-4\beta>0}$ → two real, distinct roots of the form: $\left.\begin{array}{l} y_1 = t^{r_1} \\ y_2 = t^{r_2} \end{array}\right\}$ linearly independent if $r_1 \neq r_2$

⇒ General solution: $\boxed{y(t) = c_1 t^{r_1} + c_2 t^{r_2}}$

CASE 2: $\boxed{(\alpha-1)^2-4\beta=0}$ → only one real solution : $y_1 = t^{r_1}$

$$r_1 = r_2 = -\frac{(\alpha-1)}{2}$$

A second solution can be found by the <u>method of reduction of order</u>.

However, there is another way to do it which we show here:

Notice that $F(r) = r^2+(\alpha-1)r+\beta = 0$

$= (r-r_1)^2$ in the case of equal roots

$$\Rightarrow L[t^r] = (r-r_1)^2 t^r$$

We must find another solution that's linearly independent and satisfies $L[y_2]=0$.

$$\frac{\partial}{\partial r} L[t^r] = 2(r-r_1)t^r + (r-r_1)^2 \underbrace{t^r \ln t}_{\text{derivative w.r.t. r of exponential fcn}}$$

$$= t^r(r-r_1)\left[2+(r-r_1)\ln t\right]$$

when $r=r_1 \Rightarrow \frac{\partial}{\partial r} L[t^r] = 0$

Thus $L[t^{r_1}\ln t]=0$ which implies that $y_2(t)=t^{r_1}\ln t$ is a 2nd solution.

Since $\boxed{t^{r_1}}$ and $\boxed{t^{r_1}\ln t}$ are linearly independent, the general solution for the case of equal roots is

$$\boxed{y(t) = (c_1+c_2\ln t)t^r,\ t>0}$$

CASE 3 : $\boxed{(\alpha-1)^2-4\beta<0}$ → complex roots : $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$

with $\lambda = \frac{-(\alpha-1)}{2}$, $\mu = \frac{\sqrt{4\beta-(\alpha-1)^2}}{2}$

Hence $\phi(t) = t^r = t^{\lambda+i\mu} = t^\lambda t^{i\mu}$

$$t^{i\mu} = (e^{\ln t})^{i\mu} = e^{i\mu\ln t} = \cos(\mu\ln t)+i\sin(\mu\ln t)$$

$$= t^\lambda\left[\cos(\mu\ln t)+i\sin(\mu\ln t)\right]$$ ← complex-valued solution

$$\Rightarrow \left.\begin{aligned} y_1(t) &= Re\{\phi(t)\} = t^\lambda\cos(\mu\ln t) \\ y_2(t) &= Im\{\phi(t)\} = t^\lambda\sin(\mu\ln t) \end{aligned}\right\} \text{ real-valued independent solutions}$$

Thus, the general solution in the case of complex roots is

$$\boxed{y(t) = t^{\lambda}\left[c_1\cos(\mu\ln t) + c_2\sin(\mu\ln t)\right]}$$

with $\boxed{\lambda = \frac{-(\alpha-1)}{2}}$ and $\boxed{\mu = \frac{\sqrt{4\beta-(\alpha-1)^2}}{2}}$ as above.

Examples. Case 1

Find the general solution of $L[y] = t^2\frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + 2y = 0,\ t>0$

→ Substituting $y=t^r$ gives $L[t^r] = [r(r-1) + 4r + 2]t^r = 0$

$$\Rightarrow r^2 - r + 4r + 2 = r^2 + 3r + 2 = (r+2)(r+1) = 0$$

$$\Rightarrow r = -2, -1$$

Hence $y(t) = c_1t^{r_1} + c_2t^{r_2}$

$$= c_1t^{-2} + c_2t^{-1}$$

$$= \frac{c_1}{t^2} + \frac{c_2}{t}.$$

Case 2

Find the general solution of $L[y] = t^2\frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = 0,\ t>0$

→ Substituting $y=t^r$ gives $L[t^r] = [r(r-1) - 5r + 9]t^r = 0$

$$\Rightarrow r^2 - r - 5r + 9 = r^2 - 6r + 9 = (r-3)^2 = 0$$

$r = 3$ twice.

$y_1(t) = t^3$ and $y_2(t) = t^3\ln t$

Hence $y(t) = t^3(c_1 + c_2\ln t),\ t>0.$

Case 3

Find the general solution of $L[y] = t^2\frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 25y = 0.\ t>0$

$\rightarrow$ Substituting $y = t^r$ gives $L[t^r] = [r(r-1) - 5r + 25]t^r = 0$

$$\Rightarrow r^2 - r - 5r + 25 = r^2 - 6r + 25 = 0$$

$$\Rightarrow r_{1,2} = \frac{6 \pm \sqrt{36 - 4(25)}}{2} = \frac{6 \pm \sqrt{-64}}{2} = 3 \pm 4i$$

Thus $\phi(t) = t^{3+4i} = t^3 (e^{\ln t})^{4i}$

$$= t^3[\cos(4\ln t) + i\sin(4\ln t)]$$

$$y_1(t) = \mathrm{Re}\{\phi(t)\} = t^3 \cos(4\ln t)$$

$$y_2(t) = \mathrm{Im}\{\phi(t)\} = t^3 \sin(4\ln t)$$

Hence $y(t) = c_1 y_1(t) + c_2 y_2(t) \Rightarrow y(t) = t^3[c_1\cos(4\ln t) + c_2 \sin(4\ln t)]$ for $t > 0$.

Q: What happens if $\boxed{t<0}$?

A: $y = t^r$ may not be defined if $t<0$

$y = t^r \ln t$ is not defined if $t<0$.

$\Big]$ Both of these difficulties are avoided if $\boxed{t = -x, \; x > 0}$

change of variables

Let $\boxed{y = u(x), \; x > 0.}$ From the chain rule:

with $x = -t$

$$\frac{dy}{dt} = \frac{du}{dx} \cdot \underbrace{\frac{dx}{dt}}_{=-1} = -\frac{du}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(-\frac{du}{dx}\right)$$

$$= -\frac{d^2u}{dx^2} \underbrace{\frac{dx}{dt}}_{=-1}$$

$$= \frac{d^2u}{dx^2}$$

Thus, we can write

$$L[y] = t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0$$

$$= (-x)^2 \frac{d^2u}{dx^2} + \alpha(-x)\left(-\frac{du}{dx}\right) + \beta u$$

$$= x^2 \frac{d^2u}{dx^2} + \alpha x \frac{du}{dx} + \beta u = 0, \; x > 0.$$

But after this change of variables this equation is exactly the same as before but with t replaced by x and y replaced by u.

Thus, the solutions are

$$u(x) = \begin{cases} c_1 x^{r_1} + c_2 x^{r_2}, & \text{if } (\alpha-1)^2 - 4\beta > 0 \\ (c_1 + c_2 \ln x) x^{r_1}, & \text{if } (\alpha-1)^2 - 4\beta = 0 \\ x^{\lambda}[c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)], & \text{if } (\alpha-1)^2 - 4\beta < 0 \end{cases}$$

Notice that $\boxed{x = -t = |t|}$ for $\boxed{t < 0}$ which implies that

$$y(t) = \begin{cases} c_1 |t|^{r_1} + c_2 |t|^{r_2} \\ (c_1 + c_2 \ln|t|)|t|^{r_1} \\ |t|^{\lambda}[c_1 \cos(\mu \ln|t|) + c_2 \sin(\mu \ln|t|) \end{cases}$$

Section 2.8.2 : Regular singular points, the method of Frobenius

Can we find a class of singular diff. eqns, more general than the Euler equation $t^2 y'' + \alpha t y' + \beta y = 0$ but still solvable analytically?

Rewrite it as
$$\boxed{\begin{aligned} & y'' + \frac{\alpha}{t} y' + \frac{\beta y}{t^2} = 0 \\ & L[y] = y'' + p(t) y' + q(t) y = 0 \end{aligned}}$$

where $p(t)$ and $q(t)$ can be expanded in series of the form

$$\left\{ \begin{aligned} p(t) &= \frac{p_0}{t} + p_1 + p_2 t + p_3 t^2 + \cdots \\ q(t) &= \frac{q_0}{t^2} + \frac{q_1}{t} + q_2 + q_3 t + q_4 t^2 + \cdots \end{aligned} \right\} \quad (+)$$

Definition: $L[y] = y'' + p(t)y' + q(t)y = 0$ is said to have a regular singular point at $t=0$ if $p(t)$ and $q(t)$ have series expansions of the form (+). Equivalently, $t=0$ is a regular singular point of $L[y] = y'' + p(t)y' + q(t)y = 0$ if the functions $tp(t)$ and $t^2q(t)$ are analytic at $t=0$.

Example ① Classify the singular points of Bessel's equation of order ν

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2-\nu^2)y = 0$$

where ν is a constant.

→ Here $P(t) = t^2$ vanishes at $t=0$. Hence $t=0$ is the only singular point. If we divide by t^2 we get

$$\frac{d^2y}{dt^2} + \underset{\underset{p(t)}{\shortparallel}}{\frac{1}{t}}\frac{dy}{dt} + \underset{\underset{q(t)}{\shortparallel}}{\left(1-\frac{\nu^2}{t^2}\right)}y = 0$$

$tp(t) = 1$ and $t^2q(t) = t^2-\nu^2$ are both analytic at $t=0$.

Thus, Bessel's equation of order ν has a regular singular point at $t=0$.

Example ② Classify the singular points of the Legendre equation

$$(1-t^2)y'' - 2ty' + \alpha(\alpha+1)y = 0$$

where α is a constant.

$(1-t^2)$ vanishes at $t=\pm1$. So the eqn is singular there.

If we divide by $(1-t^2)$ we obtain

$$y'' \underset{\underset{p(t)}{\shortparallel}}{- \frac{2t}{1-t^2}} y' + \underset{\underset{q(t)}{\shortparallel}}{\frac{\alpha(\alpha+1)}{1-t^2}} y = 0$$

since t=1 is a singular pt

So: $(t-1)p(t) = (t-1)\left(\frac{-2t}{1-t^2}\right) = (t-1)\left(\frac{-2t}{(1-t)(1+t)}\right) = \frac{2t}{1+t}$

$$(t-1)^2 q(t) = (t-1)^2 \frac{\alpha(\alpha+1)}{1-t^2} = (t-1)^2 \frac{\alpha(\alpha+1)}{(1-t)(1+t)} = \alpha(\alpha+1)\frac{1-t}{1+t}$$

which are both analytic at $t=1$.

since $t=-1$ is a singular point

Similarly, $(t+1)p(t) = (t+1)\left(\frac{-2t}{1-t^2}\right) = \cancel{(t+1)}\left(\frac{-2t}{(1-t)\cancel{(1+t)}}\right) = -\frac{2t}{1-t}$

$$(t+1)^2 q(t) = (t+1)^2 \frac{\alpha(\alpha+1)}{1-t^2} = (t+1)^{\cancel{2}} \frac{\alpha(\alpha+1)}{(1-t)\cancel{(1+t)}} = \alpha(\alpha+1)\frac{t+1}{1-t}$$

which are also both analytic at $t=-1$

Hence $t=-1$ and $t=1$ are regular singular points.

FROBENIUS METHOD

We consider again $L[y] = y'' + p(t)y' + q(t)y = 0$ where $t=0$ is a regular singular point

If we multiply throughout by t^2 we get

$$t^2 y'' + t(tp(t))y' + t^2 q(t) y = 0 \qquad (*)$$

Recall: Euler's equation $t^2 y'' + \alpha t y' + \beta y = 0$

So $(*)$ is viewed as being obtained from Euler's equation by adding higher powers of t to the coefficients α and β.

Let's try solutions of the form

$$\boxed{y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}} = t^r \sum_{n=0}^{\infty} a_n t^n$$

Example Find two linearly independent solutions of the equation

$$L[y](t) = 2ty'' + y' + ty = 0 \quad , \quad 0 < t < \infty$$

Let $y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}$, $a_0 \neq 0$

$$y'(t) = \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1}$$

$$y''(t) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2}$$

Plugging them into the diff. eqn. we get

$$L[y] = 2t \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2} + \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1} + t \sum_{n=0}^{\infty} a_n t^{n+r}$$

$$= 2\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1} + \sum_{n=0}^{\infty} a_n t^{n+r+1}$$

Pull out t^r

$$= t^r \Big[2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n-1} + \sum_{n=0}^{\infty} a_n (n+r) t^{n-1} + \underbrace{\sum_{n=0}^{\infty} a_n t^{n+1}}_{\sum_{n=2}^{\infty} a_{n-2} t^{n-1}} \Big]$$

Let's make all of them start at $n=2$

$$= t^r \Big[2a_0 r(r-1) t^{-1} + 2a_1 (1+r) r + 2\sum_{n=2}^{\infty} a_n (n+r)(n+r-1) t^{n-1}$$
$$+ a_0 r t^{-1} + a_1 (1+r) + \sum_{n=2}^{\infty} a_n (n+r) t^{n-1}$$
$$+ \sum_{n=2}^{\infty} a_{n-2} t^{n-1} \Big]$$

$$= [2a_0 r(r-1) + a_0 r] t^{r-1} + [2a_1 (1+r) r + a_1 (1+r)] t^r$$
$$+ \sum_{n=2}^{\infty} [2a_n (n+r)(n+r-1) + a_n (n+r) + a_{n-2}] t^{n+r-1}$$

$= 0$

Setting the coefficients of each power of t equal to zero gives

(i) $2a_0 r(r-1)+a_0 r = 0 \rightarrow 2a_0r^2-2a_0r+a_0r = 2a_0r^2-a_0r = a_0r(2r-1)=0$

$\boxed{r=0,\ r=\frac{1}{2}}$

(ii) $2a_1(1+r)r+a_1(1+r)=0 \rightarrow a_1(1+r)[2r+1]=0$

$\downarrow$

but since $r=0,\ r=\frac{1}{2}$ from (i), the (ii) implies that $\boxed{a_1=0}$

(iii) $2a_n(n+r)(n+r-1)+a_n(n+r)+a_{n-2}=0$

$\downarrow$

$$a_n(n+r)\left[2(n+r-1)+1\right] = -a_{n-2}$$

$$\boxed{a_n = \frac{-a_{n-2}}{(n+r)(2(n+r)-1)}} \quad \text{for } n\geq 2$$

Solution 1:

$\boxed{r=0}$ $\quad \boxed{a_n = \frac{-a_{n-2}}{n(2n-1)}}$, $n\geq2$

and since $a_1=0$ from (ii) we have that all the odd coefficients are zero.

The even coeff. are:

$n=2$: $\quad a_2 = \frac{-a_0}{2(3)}$

$n=4$: $\quad a_4 = \frac{-a_2}{4(7)} = -\frac{1}{4\cdot7}\cdot\frac{-a_0}{2\cdot3} = \frac{a_0}{2\cdot3\cdot4\cdot7}$

$n=6$: $\quad a_6 = -\frac{a_4}{6(11)} = -\frac{1}{6\cdot11}\cdot\frac{a_0}{2\cdot3\,4\,7} = -\frac{a_0}{2\cdot3\cdot4\cdot6\,7\cdot11}$

overall, $\quad a_{2n} = \frac{(-1)^n a_0}{2\cdot3\cdot(2n)(2(2n)-1)} = \frac{(-1)^n a_0}{2^n n!\,(4n-1)\cdot3\,7\cdots}$

If we set $a_0 = 1$ then

$$
\begin{aligned}
y_1(t) &= a_0 + a_2 t^2 + a_4 t^4 + \cdots \\
&= 1 - \frac{1}{2\cdot 3} t^2 + \frac{1}{2\cdot 3\cdot 4\cdot 7} t^4 + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{2^n n!(4n-1)\cdot 3\cdot 7\cdots}
\end{aligned}
$$

is one solution of the diff. eqn

Solution 2

$\boxed{r = \frac{1}{2}}$ Recall that we obtained the recurrence relation

$$\boxed{a_n = \frac{-a_{n-2}}{(n+r)(2(n+r)-1)}} \quad \text{for } n \geq 2$$

Subst. $r = \frac{1}{2}$ we get

$$
\begin{aligned}
a_n &= \frac{-a_{n-2}}{(n+\frac{1}{2})(2(n+\frac{1}{2})-1)} \\
&= \frac{-a_{n-2}}{\frac{1}{2}(2n+1)(\cancel{\frac{1}{2}}\cancel{2}(2n+1)-1)} \\
&= \frac{-\cancel{2}a_{n-2}}{(2n+1)(\cancel{2}n\cancel{+1}\cancel{-1})}
\end{aligned}
$$

$$\Rightarrow \boxed{a_n = \frac{-a_{n-2}}{n(2n+1)}}, \quad n \geq 2$$

All the odd coefficients are as before ~~zero~~ since from (ii) we got $a_1 = 0$.

The even coefficients are now given by

$n=2 \qquad a_2 = \frac{-a_0}{2(5)}$

$n=4 \qquad a_4 = \frac{-a_2}{4(9)} = -\frac{1}{4(9)} \cdot \frac{-a_0}{2\cdot 5} = \frac{a_0}{2\cdot 4\cdot 5\cdot 9}$

$n=6 \qquad a_6 = \frac{-a_4}{6(13)} = -\frac{1}{6\cdot 13}\frac{a_0}{2\cdot 4\cdot 5\cdot 9} = \frac{-a_0}{2\cdot 4\cdot 5\cdot 6\ 9\cdot 13}$

Setting $a_0=1$ we get that the 2nd solution is given by

$$y_2(t) = a_0 + a_2 t^2 + a_4 t^4 + \dots$$

$$= 1 - \frac{1}{2\cdot 5}t^2 + \frac{1}{2\cdot 4\cdot 5\cdot 9}t^4 + \dots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{n!\,2^n (4n+1)5\cdot 9\cdot \cdots} \quad , \quad 0<t<\infty.$$

CHAPTER 3: Systems of differential equations

Section 3.1 Algebraic properties of solutions of linear systems

We consider simultaneous 1st-order diff. equations in several variables:

$$\left.\begin{aligned}\frac{dx_1}{dt} &= f_1(t, x_1, \dots, x_n)\\ \frac{dx_2}{dt} &= f_2(t, x_1, \dots, x_n)\\ &\vdots\\ \frac{dx_n}{dt} &= f_n(t, x_1, \dots, x_n)\end{aligned}\right\}\ \text{system of } n \text{ first-order differential equations}$$

The solution is n functions $x_1(t), \dots, x_n(t)$ s.t. $\frac{dx_j(t)}{dt} = f_j(t, x_1(t), \dots, x_n(t))$, $j = 1, 2, \dots, n$. We can also impose initial conditions of the form

$$\begin{aligned}x_1(t_0) &= x_1^0\\ x_2(t_0) &= x_2^0\\ &\vdots\\ x_n(t_0) &= x_n^0\end{aligned}$$

This would then make it an initial-value problem.

<u>Note</u>: Every n^{th}-order differential equation for the single variable y can be converted into a system of n first-order equations for the variables

$$x_1(t) = y \quad , \quad x_2(t) = \frac{dy}{dt} , \ \dots \ , \quad x_n(t) = \frac{d^{n-1}y}{dt^{n-1}}$$

<u>Example</u> Convert the diff. eqn.

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_0 y = 0$$

into a system of n first-order equations.

Let $x_1(t) = y$, $x_2(t) = \frac{dy}{dt}$, ..., $x_n(t) = \frac{d^{n-1}y}{dt^{n-1}}$

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2$$

$$\frac{dx_2}{dt} = \frac{d^2y}{dt^2} = x_3$$

$$\vdots$$

$$\frac{dx_{n-1}}{dt} = x_n$$

and this implies that

$$a_n(t)\frac{dx_n}{dt} + a_{n-1}(t)\,x_n + a_{n-2}(t)\,x_{n-1} + \cdots + a_0 x_1$$

$$\Rightarrow \quad \frac{dx_n}{dt} = \frac{-(a_{n-1}(t)x_n + a_{n-2}(t)\,x_{n-1} + \cdots + a_0 x_1)}{a_n(t)}$$

<u>Example</u> : Convert the IVP $\frac{d^3y}{dt^3} + \left(\frac{dy}{dt}\right)^2 + 3y = e^t$; $y(0)=1, y'(0)=0, y''(0)=0$

into an IVP for $y, \frac{dy}{dt}, \frac{d^2y}{dt^2}$

→ set $x_1 = y$, $\frac{dx_1}{dt} = \frac{dy}{dt} = x_2$, $\frac{dx_2}{dt} = \frac{d^2y}{dt^2} = x_3$, $\frac{dx_3}{dt} = \frac{d^3y}{dt^3}$

$$\frac{dx_3}{dt} + x_2^2 + 3x_1 = e^t$$

Thus the system of 1st order diff. eqns is

$$\left[\begin{array}{l} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = x_3 \\ \frac{dx_3}{dt} = -x_2^2 - 3x_1 + e^t \end{array}\right.$$

We also have to convert the initial conditions

$$y(0)=1 \Rightarrow x_1(0)=1$$

$$y'(0)=0 \Rightarrow x_2(0)=0$$

$$y''(0)=0 \Rightarrow x_3(0)=0$$

If each of the functions $f_1, f_2, \ldots, f_n$ is a linear function of the dependent variables $x_1, \ldots, x_n$ then the system of equations is said to be linear.

Most general system of n first-order linear equations has the form

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + g_1(t)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + g_n(t)$$

if each of $g_1, g_2, \ldots g_n$ is identically zero then the system is homogeneous.

Otherwise, it's called non homogeneous

Actually in this chapter the coefficients will be constant

We'll be using vector and matrix notation to write down the system of diff. eqns.

In particular, we'll use the concise form

$$\dot{\underset{\sim}{x}} = \frac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x} \quad \text{where} \quad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

with initial conditions $\underset{\sim}{x}(t_0) = \underline{x}^\circ = \begin{pmatrix} x_1^\circ \\ x_2^\circ \\ \vdots \\ x_n^\circ \end{pmatrix}$.

Example

$$\frac{dx_1}{dt} = x_1 - x_2 + x_3 , \quad x_1(0) = 1$$

$$\frac{dx_2}{dt} = 3x_2 - x_3 , \quad x_2(0) = 0$$

$$\frac{dx_3}{dt} = x_1 + 7x_3 , \quad x_3(0) = -1$$

$$\dot{\underline{x}} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 1 & 0 & 7 \end{pmatrix} \underline{x} , \quad \underline{x}(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ where } \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

constant c

Definitions ① $c\underline{x} = c\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix}$. The process of multiplying a vector $\underline{x}$ by a number c is called scalar multiplication

② $\underline{x} + \underline{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$. This process of adding two vectors together is called vector addition

Theorem: Let $\underline{x}(t)$ and $\underline{y}(t)$ be two solutions of $\dot{\underline{x}} = \frac{d\underline{x}}{dt} = A\underline{x}$. Then

(i) $c\underline{x}(t)$ is a solution for any constant c

(ii) $\underline{x}(t) + \underline{y}(t)$ is again a solution

Lemma: Let A be an $n \times n$ matrix. For any vectors $\underline{x}$ and $\underline{y}$ and constant c,

(i) $A(c\underline{x}) = cA\underline{x}$

(ii) $A(\underline{x} + \underline{y}) = A\underline{x} + A\underline{y}$

Proof of theorem: (i) If $\underline{x}(t)$ is a solution of $\dot{\underline{x}} = \frac{d\underline{x}}{dt} = A\underline{x}$ then

$$\frac{d(c\underline{x})}{dt} = c\frac{d\underline{x}}{dt} = cA\underline{x} = A(c\underline{x})$$

Hence $c\underline{x}$ is also a solution

(ii) If $\underline{x}(t)$ and $y(t)$ are solutions of $\dot{\underline{x}} = \frac{d\underline{x}}{dt} = A\underline{x}$ then

$$\frac{d}{dt}(\underline{x}+y) = \frac{d\underline{x}}{dt} + \frac{dy}{dt} = A\underline{x} + Ay = A(\underline{x}+y)$$

Hence $\underline{x}(t) + y(t)$ is also a solution □

__Note__: Any linear combination of solutions of $\frac{d\underline{x}}{dt} = A\underline{x}$ is again a solution.
i.e. if $\underline{x}'(t), \dots, \underline{x}^{j}(t)$ are j solutions of $\frac{d\underline{x}}{dt} = A\underline{x}$ then $c_1\underline{x}'(t) + \dots + c_j\underline{x}^{j}(t)$ is again a solution for any choice of the constants $c_1, c_2, \dots, c_j$.

__Example__ Consider $\frac{dx_1}{dt} = x_2$, $\frac{dx_2}{dt} = -4x_1$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This is derived from $\frac{d^2y}{dt^2} + 4y = 0$ (using $x_1 = y$, $\frac{dy}{dt} = x_2$)

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow \left.\begin{array}{l} y_1(t) = \cos(2t) \\ y_2(t) = \sin(2t) \end{array}\right\} \text{two solutions of the scalar equation.}$$

e.g. $x_1 = y_1 = \cos(2t)$, $\tilde{x}_1 = y_2 = \sin(2t)$

$x_2 = \frac{dy_1}{dt} = -2\sin(2t)$, $\tilde{x}_2 = \frac{dy_2}{dt} = 2\cos(2t)$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1\begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} + c_2\begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} = \begin{pmatrix} c_1\cos(2t) + c_2\sin(2t) \\ -2c_1\sin(2t) + 2c_2\cos(2t) \end{pmatrix}$$

is a solution for any choice of constants c_1 and c_2

Section 3.8: The eigenvalue-eigenvector method of finding solutions

$$\dot{\vec{x}} = A\vec{x}, \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

From before: both 1st order and 2nd order linear homogeneous scalar eqns have exponential functions as solutions.

Let's try $\boxed{\vec{x}(t) = e^{\lambda t}\vec{v}}$ where $\vec{v}$ is a constant vector

$$\frac{d\vec{x}}{dt} = \lambda e^{\lambda t}\vec{v} = \lambda \vec{x}(t)$$

and we also have $A(e^{\lambda t}\vec{v}) = e^{\lambda t}A\vec{v}$

Hence $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a solution if and only if $\lambda e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v}$

Divide by $e^{\lambda t}$ $$\boxed{\lambda\vec{v} = A\vec{v}} \quad (\star)$$

Def. A nonzero vector $\vec{v}$ satisfying this condition is called an eigenvector of A with eigenvalue λ.

We can rewrite $(\star)$ as $A\vec{v} - \lambda\vec{v} = \vec{0}$

$$\Rightarrow \boxed{(A-\lambda I)\vec{v} = \vec{0}} \quad (\dagger)$$

$(\dagger)$ has a nonzero solution $\vec{v}$ only if $\boxed{\det(A-\lambda I)=0}$

i.e. $$\det\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} = 0$$

Note for $\vec{v}$ an evector of A with evalue λ:

$$A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v})$$

for any constant c. So any constant multiple $(c \neq 0)$ of an evector of A is again an evector of A with the same evalue.

The general solution of $\dot{\vec{x}} = A\vec{x}$ is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}^{\,1} + c_2 e^{\lambda_2 t}\vec{v}^{\,2} + \dots + c_n e^{\lambda_n t}\vec{v}^{\,n}.$$

Thm When the matrix A has n distinct real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ w/ eigenvectors $\vec{v}^{\,1}, \vec{v}^{\,2}, \dots, \vec{v}^{\,n}$, we are guaranteed that $\vec{v}^{\,1}, \vec{v}^{\,2}, \dots, \vec{v}^{\,n}$ are linearly independent.

Example Find all solutions of the equation

$$\dot{\vec{x}} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \vec{x}$$

→ The characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(-1-\lambda)+1] + 1[3(-1-\lambda)+2] + 4[3-2(2-\lambda)] = 0$$

$$\Rightarrow -(1-\lambda)(1+\lambda)(2-\lambda) + (1-\lambda) + [-3-3\lambda+2] + 4[3-4+2\lambda] = 0$$

$$\Rightarrow -(1-\lambda)(1+\lambda)(2-\lambda) + \underbrace{\cancel{1} - \lambda - 3\lambda - \cancel{1} + [-4+8\lambda]}_{4\lambda - 4 = 4(\lambda-1)} = 0$$

$$\Rightarrow -(1-\lambda)[\underbrace{(1+\lambda)(2-\lambda)+4}] = 0$$

$$-\lambda^2+\lambda+6$$
$$= -(\lambda^2-\lambda-6)$$
$$= -(\lambda-3)(\lambda+2)$$

$$\Rightarrow (1-\lambda)(\lambda-3)(\lambda+2) = 0$$

$$\Rightarrow \boxed{\lambda = -2, 1, 3}$$

Now let's find the eigenvectors:

① $\boxed{\lambda_1 = -2}$

$$(A-\lambda I)\vec{v} = \begin{pmatrix} 1-(-2) & -1 & 4 \\ 3 & 2-(-2) & -1 \\ 2 & 1 & -1-(-2) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3V_1 - V_2 + 4V_3 = 0$$
$$3V_1 + 4V_2 - V_3 = 0 \Rightarrow \boxed{V_3 = 3V_1 + 4V_2}$$
$$2V_1 + V_2 + V_3 = 0$$

$$\rightsquigarrow 3V_1 - V_2 + 4(3V_1 + 4V_2) = 0$$
$$\Rightarrow 15V_1 + 15V_2 = 0$$
$$\boxed{V_1 + V_2 = 0}$$
$$\Rightarrow \boxed{V_2 = -V_1}$$

Thus $V_3 = 3V_1 + 4(-V_1) = -V_1$

$$\boxed{V_2 = V_3}$$

Thus $\vec{v} = c\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue -2. This implies that part of the solution is $\vec{x}(t) = e^{\lambda_1 t}\vec{v}_1 = e^{-2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

Now let's do the same for the other eigenvalues.

② $\boxed{\lambda_2 = 1}$

$$(A-\lambda_2)\vec{v} = \begin{pmatrix}1-1 & -1 & 4\\ 3 & 2-1 & -1\\ 2 & 1 & -1-1\end{pmatrix}\vec{v} = \begin{pmatrix}0 & -1 & 4\\ 3 & 1 & -1\\ 2 & 1 & -2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$-v_2 + 4v_3 = 0 \Rightarrow \boxed{v_2 = 4v_3}$$

$$3v_1 + v_2 - v_3 = 0 \Rightarrow 3v_1 + 4v_3 - v_3 = 0$$

$$\Rightarrow 3v_1 = -3v_3$$

$$\Rightarrow \boxed{v_1 = -v_3}$$

Thus $\vec{v} = c\begin{pmatrix}-1\\ 4\\ 1\end{pmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 1$. This implies that part of the solution is $\vec{x}(t) = e^{\lambda_2 t}\vec{v}_2 = e^{t}\begin{pmatrix}-1\\ 4\\ 1\end{pmatrix}$

③ $\boxed{\lambda_3 = 3}$

$$(A-\lambda_3)\vec{v} = \begin{pmatrix}1-3 & -1 & 4\\ 3 & 2-3 & -1\\ 2 & 1 & -1-3\end{pmatrix}\vec{v} = \begin{pmatrix}-2 & -1 & 4\\ 3 & -1 & -1\\ 2 & 1 & -4\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\Rightarrow -2v_1 - v_2 + 4v_3 = 0$$

$$3v_1 - v_2 - v_3 = 0 \quad \rightsquigarrow v_3 = 3v_1 - v_2$$

$$2v_1 + v_2 - 4v_3 = 0$$

$$\rightsquigarrow -2v_1 - v_2 + 4(3v_1 - v_2) = 10v_1 - 5v_2 = 0$$

$$\boxed{v_2 = 2v_1}$$

$$v_3 = 3v_1 - 2v_1$$

$$\Rightarrow \boxed{v_3 = v_1}$$

Thus $\vec{v} = c\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 3$. This implies that part of the solution is $\vec{x}(t) = e^{\lambda_3 t}\vec{v}_3 = e^{3t}\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}$

Therefore, the general solution is

$$\vec{x}(t) = c_1 e^{-2t}\begin{pmatrix}-1\\1\\1\end{pmatrix} + c_2 e^{t}\begin{pmatrix}-1\\4\\1\end{pmatrix} + c_3 e^{3t}\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix}-c_1e^{-2t} - c_2e^{t} + c_3e^{3t}\\ c_1e^{-2t} + 4c_2e^{t} + 2c_3e^{3t}\\ c_1 e^{-2t} + c_2e^{t} + c_3e^{3t}\end{pmatrix}$$

What do we do in the case of an IVP?

Same as previously...

Example Solve the IVP $\dot{\vec{x}} = \begin{pmatrix}1 & 12\\ 3 & 1\end{pmatrix}\vec{x}$ with $\vec{x}(0) = \begin{pmatrix}0\\1\end{pmatrix}$.

$$\det\left(A - \lambda I\right) = 0 \Rightarrow (1-\lambda)^2 - 36 = 0$$
$$\lambda^2 - 2\lambda + 1 - 36 = 0$$
$$\lambda^2 - 2\lambda - 35 = 0$$
$$(\lambda+5)(\lambda-7) = 0$$
$$\lambda = -5, 7$$

$$\lambda_1 = -5 \Rightarrow \begin{pmatrix}6 & 12\\ 3 & 6\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow 6v_1 + 12v_2 = 0 \Rightarrow v_1 = -2v_2$$
$$\Rightarrow \vec{v} = c\begin{pmatrix}-2\\1\end{pmatrix}$$

$$\lambda_2 = 7 \Rightarrow \begin{pmatrix}-6 & 12\\ 3 & -6\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow -6v_1 + 12v_2 = 0 \Rightarrow v_1 = 2v_2$$
$$\vec{v} = c\begin{pmatrix}2\\1\end{pmatrix}$$

$$\vec{x}(t) = c_1\begin{pmatrix}-2\\1\end{pmatrix}e^{-5t} + c_2\begin{pmatrix}2\\1\end{pmatrix}e^{7t} = \begin{pmatrix}-2c_1e^{-5t} + 2c_2e^{7t}\\ c_1e^{-5t} + c_2e^{7t}\end{pmatrix}$$

Now using $\vec{x}(0) = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow$
$$\begin{matrix}-2c_1 + 2c_2 = 0 \Rightarrow c_1 = c_2\\ c_1 + c_2 = 1 \quad \Rightarrow c_1 = \frac{1}{2}\\ \Rightarrow c_2 = \frac{1}{2}\end{matrix}$$

This implies that the solution to this IVP is

$$\vec{x}(t) = \begin{pmatrix} -e^{-5t}+e^{7t} \\ \frac{1}{2}e^{-5t}+\frac{1}{2}e^{7t} \end{pmatrix}$$

Section 3.9: Complex roots

Lemma If $\boxed{\lambda = \alpha + i\beta}$ is a complex evalue of A with evector $\vec{v} = \vec{y} + i\vec{z}$, then $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a complex-valued solution of the eqn. $\dot{\vec{x}} = A\vec{x}$.

Gives $\boxed{\text{two}}$ real-valued solutions

Pf If $\vec{v}(t) = \vec{y}(t) + i\vec{z}(t)$ is a complex-valued solution of $\dot{\vec{v}} = A\vec{v}$ then

$$\dot{\vec{v}}(t) = \dot{\vec{y}}(t) + i\dot{\vec{z}}(t)$$

$$\begin{aligned} A\vec{v} &= A(\vec{y} + i\vec{z}) \\ &= A\vec{y} + iA\vec{z} \end{aligned}$$

Since $\dot{\vec{v}} = A\vec{v}$ we have $\dot{\vec{y}} + i\dot{\vec{z}} = A\vec{y} + iA\vec{z}$

Equating the real and imaginary parts we have:

Re: $\dot{\vec{y}} = A\vec{y}$

Im: $\dot{\vec{z}} = A\vec{z}$

So both $\vec{y}(t) = \text{Re}\{\vec{v}(t)\}$ and $\vec{z}(t) = \text{Im}\{\vec{v}(t)\}$ are real-valued solutions of $\dot{\vec{v}} = A\vec{v}$.

□

Note. The complex-valued function $\vec{v}(t) = e^{(\alpha+i\beta)t}(\vec{v}^1 + i\vec{v}^2)$ can be written as

identity: $e^{i\beta t} = \cos(\beta t) + i\sin(\beta t)$

$$\vec{v}(t) = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big)(\vec{v}^1 + i\vec{v}^2)$$

$$= e^{\alpha t}\Big[\big(\vec{v}^1\cos(\beta t) - \vec{v}^2\sin(\beta t)\big) + i\big(\vec{v}^1\sin(\beta t) + \vec{v}^2\cos(\beta t)\big)\Big]$$

$$= \vec{y}(t) + i\vec{z}(t)$$

Thus $\vec{y}(t) = e^{\alpha t}[\vec{v}^1\cos(\beta t) - \vec{v}^2\sin(\beta t)]$ are two real-valued solutions of $\dot{\vec{v}} = A\vec{v}$,

$\vec{z}(t) = e^{\alpha t}[\vec{v}^1\sin(\beta t) + \vec{v}^2\cos(\beta t)]$

and they are also linearly independent.

Example. Solve the IVP: $\dot{\vec{x}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}\vec{x}$, $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{pmatrix} = 0 \Rightarrow (1-\lambda)[(1-\lambda)^2 + 1] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 2\lambda + 1 + 1) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 2\lambda + 2) = 0$$

$$\lambda = 1 \qquad \lambda = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm i$$

$$\lambda = 1 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v} = c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \text{ Thus } \vec{x} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^t$$

$$\lambda = 1+i \Rightarrow \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad v_2 - iv_3 = 0$$

$$v_2 = iv_3$$

$$\vec{v} = c\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

Thus $$\vec{x} = c_2\begin{pmatrix}0\\i\\1\end{pmatrix}e^{(1+i)t} = c_2\begin{bmatrix}0\\i\\1\end{bmatrix}e^t(\cos t + i\sin t)$$

$$= c_2e^t\left[\begin{pmatrix}0\\-\sin t\\\cos t\end{pmatrix} + i\begin{pmatrix}0\\\cos t\\\sin t\end{pmatrix}\right]$$

Thus $\vec{x}^2(t) = e^t\begin{pmatrix}0\\-\sin t\\\cos t\end{pmatrix}$, $\vec{x}^3(t) = e^t\begin{pmatrix}0\\\cos t\\\sin t\end{pmatrix}$, are real-valued solutions.

The three solutions $\vec{x}^1(t)$, $\vec{x}^2(t)$, $\vec{x}^3(t)$ are linearly independent since their initial values

$$\vec{x}^1(0) = \begin{pmatrix}1\\0\\0\end{pmatrix},\quad \vec{x}^2(0) = \begin{pmatrix}0\\0\\1\end{pmatrix},\quad \vec{x}^3(0) = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

are linearly independent vectors.

The general solution is $\vec{x}(t) = c_1e^t\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2e^t\begin{pmatrix}0\\-\sin t\\\cos t\end{pmatrix} + c_3e^t\begin{pmatrix}0\\\cos t\\\sin t\end{pmatrix}$

Setting $t=0$ we see that

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\0\\1\end{pmatrix} + c_3\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}c_1\\c_3\\c_2\end{pmatrix}$$

$\Rightarrow \boxed{c_1 = 1 = c_2 = c_3}$

Thus the particular solution is

$$\vec{x}(t) = e^t\begin{pmatrix}1\\0\\0\end{pmatrix} + e^t\begin{pmatrix}0\\-\sin t\\\cos t\end{pmatrix} + e^t\begin{pmatrix}0\\\cos t\\\sin t\end{pmatrix} = e^t\begin{pmatrix}1\\-\sin t+\cos t\\\cos t+\sin t\end{pmatrix}$$

<u>Note</u> If $\vec{v}$ is an eigenvector of A with eigenvalue λ, then $\bar{\vec{v}}$ (the complex conjugate of $\vec{v}$) is an eigenvector of A with eigenvalue $\bar{\lambda}$.

Section 3.10: Equal roots

If $\det(A-\lambda I)=0$ does not have n distinct roots then A may not have n linearly independent eigenvectors.

Suppose that an $n\times n$ matrix A has only $k<n$ linearly independent eigenvectors. Then the diff. eqn. $\dot{\vec{x}}=A\vec{x}$ has only k linearly indep. solutions of the form $e^{\lambda t}\vec{v}$.

Q: How do we find an additional $n-k$ linearly independent solutions?

A: Since for a scalar diff. eqn we used $x(t)=e^{at}c$ as the solution to $\dot{x}=ax$, for a constant c, we use $\vec{x}(t)=e^{At}\vec{v}$ as the solution to the vector diff. eqn $\dot{\vec{x}}=A\vec{x}$ for every constant vector $\vec{v}$.

What is e^{At} for A, a matrix?

$$e^{At}=I+At+\frac{A^2t^2}{2!}+\dots+\frac{A^nt^n}{n!}+\dots$$

We can also differentiate this infinite series term by term:

$$\frac{d}{dt}(e^{At})=A+A^2t+\dots+\frac{A^{n+1}}{n!}t^n+\dots$$

$$=A\left[I+At+\dots+\frac{A^n}{n!}t^n+\dots\right]$$

$$=Ae^{At}$$

Therefore, $e^{At}\vec{v}$ is a solution of $\dot{\vec{x}}=A\vec{x}$ for every constant vector $\vec{v}$ since

$$\frac{d}{dt}(e^{At}\vec{v})=Ae^{At}\vec{v}=A(e^{At}\vec{v}).$$

Properties. $(e^{At})^{-1}=e^{-At}$ and $e^{A(t+s)}=e^{At}e^{As}$

Q. How do we find n linearly independent vectors $\vec{v}$ for which the infinite series $e^{At}\vec{v}$ can be summed exactly?

A. $e^{At}\vec{v} = e^{(A-\lambda I)t}e^{\lambda I t}\vec{v}$ for any constant λ Note $(A-\lambda I)\lambda I = \lambda I(A-\lambda I)$

$$e^{\lambda I t}\vec{v} = \left[I+\lambda I t+\frac{(\lambda I t)^2}{2!}+\dots\right]\vec{v} = \left[1+\lambda t+\frac{\lambda^2 t^2}{2!}+\dots\right]\vec{v} = e^{\lambda t}\vec{v}$$

Thus, $e^{At}\vec{v} = e^{\lambda t}e^{(A-\lambda I)t}\vec{v}$

Note also that if $(A-\lambda I)^m\vec{v}=\vec{0}$ for some integer m then $(A-\lambda I)^{m+l}\vec{v}$ is also zero for every positive integer l.

$$\boxed{(A-\lambda I)^{m+l}\vec{v} = (A-\lambda I)^{l}\left[(A-\lambda I)^m\vec{v}\right] = \vec{0}}$$

This implies that

$$e^{(A-\lambda I)t}\vec{v} = \left[I+(A-\lambda I)t+\frac{(A-\lambda I)^2t^2}{2!}+\dots+\frac{(A-\lambda I)^{m-1}t^{m-1}}{(m-1)!}\right]\vec{v}$$

$$= \vec{v}+t(A-\lambda I)\vec{v}+t^2\frac{(A-\lambda I)^2}{2!}\vec{v}+\dots\ t^{m-1}\frac{(A-\lambda I)^{m-1}}{(m-1)!}\vec{v}$$

But we also showed that $e^{At}\vec{v} = e^{\lambda t}e^{(A-\lambda I)t}\vec{v}$ which implies that

$$e^{At}\vec{v} = e^{\lambda t}\left[\vec{v}+t(A-\lambda I)\vec{v}+t^2\frac{(A-\lambda I)^2}{2!}\vec{v}+\dots+t^{m-1}\frac{(A-\lambda I)^{m-1}}{(m-1)!}\vec{v}\right]$$

Algorithm for finding n linearly independent solutions of $\dot{\vec{x}}=A\vec{x}$.

① Find all eigenvalues and eigenvectors of A.

If A has n linearly independent eigenvectors, then $\dot{\vec{x}}=A\vec{x}$ has n linearly independent solutions of the form $e^{\lambda t}\vec{v}$

Note. If $\vec{v}$ is an eigenvector of A with eigenvalue λ then the infinite series $e^{(A-\lambda I)t}\vec{v}$ terminates after 1 term.

② Suppose A has only k<n linearly independent eigenvectors

We have only k linearly independent solutions of the form $e^{\lambda t}\vec{v}$.

For additional solutions we pick an eigenvalue λ of A and find $\vec{v}$ s.t.

$\boxed{(A-\lambda I)^2\vec{v}=\vec{0}}$ but $\boxed{(A-\lambda I)\vec{v}\neq\vec{0}}$.

For each $\vec{v}$: $$e^{At}\vec{v} = e^{\lambda t}e^{(A-\lambda I)t}\vec{v} = e^{\lambda t}\left[\vec{v} + t(A-\lambda I)\vec{v} + \cancelto{0}{\frac{t^2(A-\lambda I)^2\vec{v}}{2!}} + \dots\right]$$

is an additional solution of $\dot{\vec{x}}=A\vec{x}$. We repeat this ∀ eigenvalues λ of A.

③ If there are still not enough solutions, then we find all vectors $\vec{v}$ s.t

$\boxed{(A-\lambda I)^3\vec{v}=\vec{0}}$ but $\boxed{(A-\lambda I)\vec{v}\neq\vec{0}}$.

For each $\vec{v}$: $$e^{At}\vec{v} = e^{\lambda t}e^{(A-\lambda I)t}\vec{v} = e^{\lambda t}\left[\vec{v} + t(A-\lambda I)\vec{v} + \frac{t^2(A-\lambda I)^2\vec{v}}{2!} + \cancelto{0}{\frac{t^3(A-\lambda I)^3\vec{v}}{3!}} + \dots\right]$$

is an additional solution of $\dot{\vec{x}}=A\vec{x}$.

④ We repeat this process until we obtain n linearly independent solutions.

Example. Find three linearly independent solutions of the diff. eqn.

$$\dot{\vec{x}} = \begin{pmatrix}1&1&0\\0&1&0\\0&0&2\end{pmatrix}\vec{x}$$

→ Characteristic polynomial: $\det\begin{pmatrix}1-\lambda&1&0\\0&1-\lambda&0\\0&0&2-\lambda\end{pmatrix}=0$

$\Rightarrow (1-\lambda)^2(2-\lambda)=0$

Hence $\lambda=1$ with multiplicity 2
$\lambda=2$ 〃 1.

$\boxed{\lambda=1}$ $(A-\lambda I)\vec{v} = \begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ $\Rightarrow$ $v_2=0$
$v_3=0$
v_1 anything

$\Rightarrow \vec{x}^1(t) = e^t\begin{pmatrix}1\\0\\0\end{pmatrix}$

Since A has only one linearly independent eigenvector with eigenvalue 1, we look for solutions of

$$(A-\overset{\lambda=1}{\lambda} I)^2\vec{v} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\vec{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $v_3=0$ and we can choose anything for v_1 and v_2

The vector $\vec{v}=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ satisfies $(A-\lambda I)^2\vec{v}=\vec{0}$ but $(A-\lambda I)\vec{v}\neq\vec{0}$. (So we can choose any $\vec{v}=\begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$ for which $v_2\neq 0$.) $\rightarrow$ Since the other solution was $\vec{v}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

A solution is $\vec{x}^{\,2}(t) = e^{At}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \underset{\uparrow}{=} e^{t}e^{(A-I)t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$e^{\lambda t}e^{(A-\lambda I)t}\vec{v}$

$$= e^{t}\left[I + t(A-I)\right]\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= e^{t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]$$

$$= e^{t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]$$

$= e^{t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$ is the second linearly independent solution

$\boxed{\lambda=2}$ $(A-\lambda I)\vec{v}=\vec{0}$

Recall that $A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and so $(A-\lambda I)=\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$(A-\lambda I)\vec{v} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $v_2=0 \Rightarrow v_1=0$

and v_3 = anything

Thus $\vec{x}^3(t) = e^{2t}\begin{pmatrix}0\\0\\1\end{pmatrix}$ is the other linearly independent solution.

Example. Solve the IVP $\dot{\vec{x}} = \begin{pmatrix}2&1&3\\0&2&-1\\0&0&2\end{pmatrix}\vec{x}$ with $\vec{x}(0)=\begin{pmatrix}1\\2\\1\end{pmatrix}$

The characteristic polynomial is $\det(A-\lambda I)=0$

$$\Rightarrow \det\begin{pmatrix}2-\lambda&1&3\\0&2-\lambda&-1\\0&0&2-\lambda\end{pmatrix}=0$$

$$\Rightarrow (2-\lambda)\left[(2-\lambda)^2\right]-1(0)+3(0)=0$$

$$\Rightarrow \boxed{\lambda=2} \text{ w/ multiplicity } 3.$$

The eigenvectors satisfy $(A-\lambda I)\vec{v}=\vec{0}$

$$\begin{pmatrix}0&1&3\\0&0&-1\\0&0&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow$$

$v_3=0$

$v_2+3v_3=0 \Rightarrow v_2=0$

v_1 = anything

Thus $\vec{x}^1(t)=e^{2t}\begin{pmatrix}1\\0\\0\end{pmatrix}$ is one of the solutions

We now should look for the other two linearly independent solutions. Let's try to solve for $\vec{v}$ in $(A-\lambda I)^2\vec{v}=\vec{0}$.

$$\begin{pmatrix}0&1&3\\0&0&-1\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&3\\0&0&-1\\0&0&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\Rightarrow$ $v_3=0$ and v_1, v_2 = anything

The vector $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ satisfies $(A-2I)^2\vec{v} = \vec{0}$ but $(A-2I)\vec{v} \neq \vec{0}$

Therefore, a 2nd linearly independent solution is

$$\vec{x}^2(t) = e^{At}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{2t}e^{(A-2I)t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= e^{2t}\left[I + t(A-2I) \right]\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= e^{2t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$$

$$= e^{2t}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$

We now look for the third linearly independent solution by computing $\vec{v}$ that satisfies

$(A-\lambda I)^3\vec{v} = \vec{0}$ and $(A-\lambda I)\vec{v} \neq \vec{0}$.

$$\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\underbrace{\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ from } (A-\lambda I)^2 \text{ above}}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So any $\vec{v}$ satisfies the equation above. For example $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ satisfies $(A-\lambda I)^3\vec{v} = \vec{0}$ and does not satisfy $(A-\lambda I)^2\vec{v} = \vec{0}$.

$$\vec{x}^3(t) = e^{At}\begin{pmatrix}0\\0\\1\end{pmatrix} = e^{2t}e^{(A-2I)t}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= e^{2t}\left[I + t(A-2I) + \frac{t^2}{2!}(A-2I)^2\right]\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= e^{2t}\left[I + t\begin{pmatrix}0&1&3\\0&0&-1\\0&0&0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix}\right]\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= e^{2t}\left[\begin{pmatrix}0\\0\\1\end{pmatrix} + t\begin{pmatrix}3\\-1\\0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}-1\\0\\0\end{pmatrix}\right]$$

$$= e^{2t}\begin{pmatrix}3t - t^2/2\\-t\\1\end{pmatrix}$$

is a 3rd linearly independent solution. The general solution is thus

$$\vec{x}(t) = e^{2t}\left[c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}t\\1\\0\end{pmatrix} + c_3\begin{pmatrix}3t - t^2/2\\-t\\1\end{pmatrix}\right]$$

The constants c_1, c_2, c_3 are found using $\vec{x}(0) = \begin{pmatrix}1\\2\\1\end{pmatrix}$

$$\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$$

Thus the solution to this IVP is

$$\vec{x}(t) = e^{2t}\left[\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}t\\1\\0\end{pmatrix} + \begin{pmatrix}3t - t^2/2\\-t\\1\end{pmatrix}\right]$$

$$= e^{2t}\begin{pmatrix}1+2t+3t-t^2/2\\2-t\\1\end{pmatrix}$$

$$= e^{2t}\begin{pmatrix}1+5t-t^2/2\\2-t\\1\end{pmatrix}.$$

Theorem. CAYLEY-HAMILTON

Let $p(\lambda) = p_0 + p_1\lambda + .. + (-1)^n\lambda^n$ be the characteristic polynomial of A.
Then $p(A) = pI + p_1A + \dots + (-1)^nA^n = \vec{0}$.

Section 3.11: Fundamental matrix solutions; e^{At}

If $\vec{x}^1(t), \dots, \vec{x}^n(t)$ are n linearly independent solutions of $\dot{\vec{x}} = A\vec{x}$ then every solution $\vec{x}(t)$ can be written as

$$\vec{x}(t) = c_1\vec{x}^1(t) + c_2\vec{x}^2(t) + \dots + c_n\vec{x}^n(t). \quad (\star)$$

Let $\vec{X}(t)$ be a matrix whose columns are the solutions $\vec{x}^1(t), \dots, \vec{x}^n(t)$. Then $(\star)$ can be written as $\boxed{\vec{x}(t) = \vec{X}(t)\vec{c}}$ where $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$.

Definition: A matrix $\vec{X}(t)$ is called a fundamental matrix solution of $\dot{\vec{x}} = A\vec{x}$ if its columns form a set of n linearly independent solutions of $\dot{\vec{x}} = A\vec{x}$.

Example. Find the fundamental matrix solution of

$$\dot{\vec{x}} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\vec{x}$$

This is the example we did in section 3 8. There we found that the eigenvalues were $\lambda = -2, 1, 3$ and the associated eigenvectors were $v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and they were linearly independent. Thus

$$X(t) = \begin{pmatrix} e^{-2t} & -e^{t} & e^{3t} \\ e^{-2t} & 4e^{t} & 2e^{3t} \\ e^{-2t} & e^{t} & e^{3t} \end{pmatrix}$$

is a fundamental matrix solution of this $\dot{\vec{x}} = A\vec{x}$.

<u>Theorem</u> Let $X(t)$ be a fundamental matrix solution of the differential eq. $\dot{\vec{x}} = A\vec{x}$. Then

$$\boxed{e^{At} = X(t)\,X^{-1}(0)}$$

→ The product of any fundamental matrix solution of $\dot{\vec{x}} = A\vec{x}$ with its inverse at $t=0$ must yield e^{At}.

<u>Lemma</u> A matrix $X(t)$ is a fundamental matrix solution of $\dot{\vec{x}} = A\vec{x}$ iff $\dot{X}(t) = AX(t)$ and $\det(X(0)) \neq 0$.

<u>Proof</u>. Let $\vec{x}^{(1)}(t), \dots, \vec{x}^{n}(t)$ denote the n columns of $X(t)$. Observe that

$$\dot{X}(t) = \left[\dot{\vec{x}}^{(1)}(t), \dots, \dot{\vec{x}}^{(n)}(t)\right]$$

and $AX(t) = \left[A\vec{x}^{(1)}(t), \dots, A\vec{x}^{(n)}(t)\right]$.

The n vector equations $\dot{\vec{x}}^{(1)}(t) = A\vec{x}^{(1)}(t), \dots, \dot{\vec{x}}^{(n)}(t) = A\vec{x}^{(n)}(t)$ are the same as $\dot{X}(t) = AX(t)$. n solutions $\vec{x}^{(1)}(t), \dots, \vec{x}^{(n)}(t)$ are linearly independent iff $\vec{x}^{(1)}(0), \dots, \vec{x}^{(n)}(0)$ are linearly indep. vectors of $\mathbb{R}^n$, which are linearly independent iff $\det X(0) \neq 0$. □

Lemma. Let $X(t)$ and $Y(t)$ be two fundamental matrix soletions of $\vec{x}(t) = A\vec{x}(t)$. Then, there exists a constant matrix C s.t. $Y(t) = X(t)C$.

Proof. The columns $\vec{x}^{(1)}(t), \dots, \vec{x}^{(n)}(t)$ of $X(t)$ and $\vec{y}^{(1)}(t), \dots, \vec{y}^{(n)}(t)$ of $Y(t)$ are linearly indep. sets of solutions of $\dot{\vec{x}} = A\vec{x}$. Thus, every column of $Y(t)$ can be written as a linear combination of the columns of $X(t)$. $\exists$ constants $c_1^j, c_2^j, \dots, c_n^j$ s.t.

$$\vec{y}^j(t) = c_1^j \vec{x}^1(t) + c_2^j \vec{x}^2(t) + \dots + c_n^j \vec{x}^n(t), \quad j = 1, \dots, n \qquad (*)$$

Let C be the matrix $(\vec{c}^1, \vec{c}^2, \dots, \vec{c}^n)$ where

$$\vec{c}^j = \begin{bmatrix} c_1^j \\ \vdots \\ c_n^j \end{bmatrix}$$

Then the n equations $(*)$ are equivalent to the matrix equation $Y(t) = X(t)C$.

□

Example. Find e^{At} if $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix}$.

We want 3 linearly indep. solutions of $\dot{\vec{x}} = A\vec{x}$. We first compute the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 5-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)[(3-\lambda)(5-\lambda)] = 0$$

$$\boxed{\lambda = 1, 3, 5}$$

$\boxed{\lambda=1}$: $(A-\lambda I)\vec{v}=\vec{0}$

$$\begin{pmatrix}0&1&1\\0&2&2\\0&0&4\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{array}{l}v_3=0\\v_2=0\\v_1=\text{anything}\end{array} \Rightarrow \vec{v}^{(1)}=\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

Hence one solution is $\vec{x}^{(1)}(t)=e^{t}\begin{pmatrix}1\\0\\0\end{pmatrix}$.

$\boxed{\lambda=3}$: $(A-\lambda I)\vec{v}=\vec{0}$

$$\begin{pmatrix}-2&1&1\\0&0&2\\0&0&2\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow v_3=0$$

$$-2v_1+v_2+\underset{0}{\cancel{v_3}}=0$$

$$2v_1=v_2$$

$$\vec{v}^{(2)}=\begin{pmatrix}1\\2\\0\end{pmatrix}.$$

Thus, the other solution is $\vec{x}^{(2)}(t)=e^{3t}\begin{pmatrix}1\\2\\0\end{pmatrix}$.

$\boxed{\lambda=5}$ $(A-\lambda I)\vec{v}=\vec{0}$

$$\begin{pmatrix}-4&1&1\\0&-2&2\\0&0&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$-2v_2+2v_3=0 \Rightarrow v_3=v_2$$

$$-4v_1+v_2+\underset{v_2}{\cancel{v_3}}=0$$

$$4v_1=2v_2$$

$$\boxed{2v_1=v_2=v_3}$$

Thus $\vec{v}^{(3)}=\begin{pmatrix}1\\2\\2\end{pmatrix}$

The third solution is $\vec{x}^{(3)}(t)=e^{5t}\begin{pmatrix}1\\2\\2\end{pmatrix}$.

The fundamental matrix solution is therefore

$$X(t) = \begin{pmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix}$$

We now compute $X^{-1}(0)$. $X(0) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow X^{-1}(0) = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

Thus $e^{At} = X(t)X^{-1}(0)$

$$= \begin{pmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t} & -\frac{1}{2}e^{t}+\frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t}+\frac{1}{2}e^{5t} \\ 0 & e^{3t} & -e^{3t}+e^{5t} \\ 0 & 0 & e^{5t} \end{pmatrix}$$

Section 3.12 The nonhomogeneous equation, VARIATION OF PARAMETERS

Consider $\boxed{\dot{\vec x} = A\vec x + \vec f(t)}$, $\vec x(t_0) = \vec x^{0}$

Let $\vec x^{1}(t), \dots, \vec x^{n}(t)$ be n linearly indep. solutions of $\dot{\vec x}(t) = A\vec x(t)$ — the homog. case. Since the general solution for this is $c_1\vec x^{1}(t) + \dots + c_n \vec x^{n}(t)$, we seek a solution of the form

$$\vec x(t) = u_1(t)\vec x^{1}(t) + u_2(t)\vec x^{2}(t) + \dots + u_n(t)\vec x^{n}(t) \qquad (*)$$

This can be written in the form $\boxed{\vec x(t) = X(t)\vec u(t)}$ where $X(t) = [\vec x^{1}(t), \dots, \vec x^{n}(t)]$ and $\vec u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}$. If we plug this into $\dot{\vec x} = A\vec x + \vec f(t)$ we get

$$\underbrace{\dot X(t)\vec u(t) + X(t)\dot{\vec u}(t)}_{\text{from } \frac{d}{dt}\vec x(t) = \frac{d}{dt}(X(t)\vec u(t)) \text{ product rule}} = AX(t)\vec u(t) + \vec f(t) \qquad (\dagger)$$

The matrix $X(t)$ is a fundamental matrix solution of the homogeneous problem $\dot{\vec{x}} = A\vec{x}$. Thus $\dot{X}(t) = AX(t)$ and (+) reduces to

$$\cancel{\dot{X}(t)}\vec{u}(t) + X(t)\dot{\vec{u}}(t) = \underbrace{AX(t)}_{\dot{X}(t)}\cancel{\vec{u}(t)} + \vec{f}(t)$$

$$\Rightarrow \quad X(t)\dot{\vec{u}}(t) = \vec{f}(t)$$

We already saw that the columns of $X(t)$ are linearly independent vectors of $\mathbb{R}^n$ at every time t. Hence $X^{-1}(t)$ exists, and

$$X(t)\dot{\vec{u}}(t) = \vec{f}(t) \Rightarrow \dot{\vec{u}}(t) = X^{-1}(t)\vec{f}(t).$$

Now we integrate between t_0 and t to get:

$$\vec{u}(t) - \boxed{\vec{u}(t_0)} = \int_{t_0}^{t} X^{-1}(s)\vec{f}(s)\,ds$$

$= X^{-1}(t_0)\vec{x}^0$ (recall that we wrote (*) as $\vec{x}(t) = X(t)\vec{u}(t)$)

$\Rightarrow \vec{u}(t) = X^{-1}(t)\vec{x}(t)$

$$\underbrace{\vec{u}(t)}_{X^{-1}(t)\vec{x}(t)} = X^{-1}(t_0)\vec{x}^0 + \int_{t_0}^{t} X^{-1}(s)\vec{f}(s)\,ds$$

$$\vec{x}(t) = X(t)X^{-1}(t_0)\vec{x}^0 + X(t)\int_{t_0}^{t} X^{-1}(s)\vec{f}(s)\,ds$$

If $X(t)$ is the fundamental matrix solution e^{At} then we can write $X(t) = e^{At}$, $X^{-1}(s) = e^{-As}$

$$\Rightarrow \vec{x}(t) = e^{At}e^{-At_0}\vec{x}^0 + e^{At}\int_{t_0}^{t} e^{-As}\vec{f}(s)\,ds$$

$$= e^{A(t-t_0)}\vec{x}^0 + \int_{t_0}^{t} e^{A(t-s)}\vec{f}(s)\,ds.$$

Example. Solve

$$\dot{\vec{x}} = \underbrace{\begin{pmatrix}1&0&0\\2&1&-2\\3&2&1\end{pmatrix}}_{A}\vec{x} + \begin{pmatrix}0\\0\\e^t\cos 2t\end{pmatrix}, \quad \vec{x}(0)=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\det(A-\lambda I) = \det\begin{pmatrix}1-\lambda&0&0\\2&1-\lambda&-2\\3&2&1-\lambda\end{pmatrix} = (1-\lambda)[(1-\lambda)^2+4] = (1-\lambda)[\lambda^2-2\lambda+1+4]$$

$$= (1-\lambda)(\lambda^2-2\lambda+5)=0$$

$$\lambda=1 \qquad \lambda = \frac{2\pm\sqrt{4-4(5)}}{2} = \frac{2\pm4}{2}i = 1\pm2i$$

$\boxed{\lambda=1}$ $\quad (A-\lambda I)\vec{v}=\vec{0}$

$$\begin{pmatrix}0&0&0\\2&0&-2\\3&2&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \quad 2v_1-2v_3=0$$

$$v_1=v_3$$

$$3v_1=-2v_2 \Rightarrow v_1=-\frac{2}{3}v_2$$

$$\vec{v}^{(1)} = \begin{pmatrix}-2\\3\\-2\end{pmatrix}$$

Thus one of the solutions is $\vec{x}^{1}(t) = e^t\begin{pmatrix}-2\\3\\-2\end{pmatrix}$.

Now we consider $\boxed{\lambda = 1+2i}$

$$\begin{pmatrix}-2i&0&0\\2&-2i&-2\\3&2&-2i\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$-2iv_1=0 \Rightarrow v_1=0$$

$$\underset{0}{\cancel{v_1}}-iv_2-v_3=0 \Rightarrow v_3=-iv_2 \qquad \Rightarrow \vec{v}^{(2)} = \begin{pmatrix}0\\1\\-i\end{pmatrix}$$

Thus $\vec{x}(t) = e^{(1+2i)t}\begin{pmatrix}0\\1\\-1\end{pmatrix} = e^t(\cos 2t+i\sin 2t)\begin{pmatrix}0\\1\\-i\end{pmatrix} = e^t\left[\begin{pmatrix}0\\\cos 2t\\\sin 2t\end{pmatrix}+i\begin{pmatrix}0\\\sin 2t\\-\cos 2t\end{pmatrix}\right]$

which implies that $\vec{x}^2(t) = e^t\begin{pmatrix}0\\\cos 2t\\\sin 2t\end{pmatrix}$, $\vec{x}^3(t) = e^t\begin{pmatrix}0\\\sin 2t\\-\cos 2t\end{pmatrix}$. are the real-valued solutions of $\dot{\vec{x}} = A\vec{x}$.

We check that they are linearly indep by substituting $t=0$

$\vec{x}^2(0)=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{x}^3(0)=\begin{pmatrix}0\\0\\-1\end{pmatrix}$. These are linearly independent, which implies that

$$X(t)=\begin{pmatrix}-2e^t & 0 & 0\\ 3e^t & e^t\cos 2t & e^t\sin 2t\\ -2e^t & e^t\sin 2t & -e^t\cos 2t\end{pmatrix}$$

is the fundamental matrix solution of $\dot{\vec{x}}=A\vec{x}$.

$$X^{-1}(0)=\begin{pmatrix}-2 & 0 & 0\\ 3 & 1 & 0\\ -2 & 0 & -1\end{pmatrix}^{-1}=\begin{pmatrix}-1/2 & 0 & 0\\ 3/2 & 1 & 0\\ 1 & 0 & -1\end{pmatrix}$$

$\uparrow$ verify

Therefore $e^{At}=X(t)X^{-1}(0)$

$$=\begin{pmatrix}-2e^t & 0 & 0\\ 3e^t & e^t\cos 2t & e^t\sin 2t\\ -2e^t & e^t\sin 2t & -e^t\cos 2t\end{pmatrix}\begin{pmatrix}-1/2 & 0 & 0\\ 3/2 & 1 & 0\\ 1 & 0 & -1\end{pmatrix}$$

$$=\begin{pmatrix}e^t & 0 & 0\\ -\frac{3}{2}e^t+\frac{3}{2}e^t\cos 2t+e^t\sin 2t & e^t\cos 2t & -e^t\sin 2t\\ e^t+\frac{3}{2}e^t\sin 2t-e^t\cos 2t & e^t\sin 2t & e^t\cos 2t\end{pmatrix}$$

can factor out the exponential

Recall that $\boxed{\vec{x}(t)=e^{A(t-t_0)}\vec{x}_0+\int_{t_0}^{t}e^{A(t-s)}\vec{f}(s)ds.}$ and the initial condition is

$\vec{x}(0)=\begin{pmatrix}0\\1\\1\end{pmatrix}$. Thus $t_0=0$

$$\Rightarrow \vec{x}(t)=e^{At}\begin{pmatrix}0\\1\\1\end{pmatrix}+\int_0^t e^{A(t-s)}\begin{pmatrix}0\\0\\e^s\cos 2s\end{pmatrix}ds$$

$$=\begin{pmatrix}0\\ e^t\cos 2t-e^t\sin 2t\\ e^t\sin 2t+e^t\cos 2t\end{pmatrix}+$$

$$+ e^{At}\int_0^t e^{-As}\begin{pmatrix}0\\0\\e^s\cos 2s\end{pmatrix}ds$$

$$= e^t\begin{pmatrix}0\\ \cos 2t-\sin 2t\\ \sin 2t+\cos 2t\end{pmatrix} +$$

$$e^{At}\int_0^t e^{-s}\begin{pmatrix}1 & 0 & 0\\ -\frac{3}{2}+\frac{3}{2}\overset{\cos 2s}{\cancel{\cos(-2s)}}+\overset{-\sin 2s}{\cancel{\sin(-2s)}} & \overset{\cos 2s}{\cancel{\cos(-2s)}} & \overset{-\sin 2s}{\cancel{\sin(-2s)}}\\ 1+\frac{3}{2}\underset{-\sin 2s}{\cancel{\sin(-2s)}}-\underset{\cos 2s}{\cancel{\cos(-2s)}} & \underset{-\sin 2s}{\cancel{\sin(-2s)}} & \underset{\cos 2s}{\cancel{\cos(-2s)}}\end{pmatrix}\begin{pmatrix}0\\0\\e^s\cos 2s\end{pmatrix}ds$$

$$= e^t\begin{pmatrix}0\\ \cos 2t-\sin 2t\\ \sin 2t+\cos 2t\end{pmatrix} + e^{At}\int_0^t e^{-s}\begin{pmatrix}0\\ -e^s\cos(2s)\sin(2s)\\ e^s\cos^2(2s)\end{pmatrix}ds$$

$$= e^t\begin{pmatrix}0\\ \cos 2t-\sin 2t\\ \sin 2t+\cos 2t\end{pmatrix} + e^{At}\int_0^t \begin{pmatrix}0\\ -\frac{1}{2}\sin(4s)\\ \frac{1}{2}+\frac{1}{2}\cos(4s)\end{pmatrix}ds$$

$$= e^t\begin{pmatrix}0\\ \cos 2t-\sin 2t\\ \sin 2t+\cos 2t\end{pmatrix} + e^{At}\left[\begin{pmatrix}0\\ \frac{1}{8}\cos(4s)\\ \frac{1}{2}s+\frac{1}{8}\sin(4s)\end{pmatrix}\right]_0^t$$

$$= e^t\begin{pmatrix}0\\ \cos 2t-\sin 2t\\ \sin 2t+\cos 2t\end{pmatrix} + e^{At}\begin{pmatrix}0\\ \frac{1}{8}\cos(4t)-\frac{1}{8}\\ \frac{1}{2}t+\frac{1}{8}\sin(4t)\end{pmatrix}$$

$$= e^t\begin{pmatrix}0\\ \cos 2t-\sin 2t\\ \sin 2t+\cos 2t\end{pmatrix} +$$

$$\begin{pmatrix}e^t & 0 & 0\\ -\frac{3}{2}e^t+\frac{3}{2}e^t\cos 2t+e^t\sin 2t & e^t\cos 2t & -e^t\sin 2t\\ e^t+\frac{3}{2}e^t\sin 2t-e^t\cos 2t & e^t\sin 2t & e^t\cos 2t\end{pmatrix}\begin{pmatrix}0\\ \frac{1}{8}\cos(4t)-\frac{1}{8}\\ \frac{1}{2}t+\frac{1}{8}\sin(4t)\end{pmatrix}$$

$$= e^t\begin{pmatrix}0\\ \cos 2t-\sin 2t\\ \sin 2t+\cos 2t\end{pmatrix} + e^t\begin{pmatrix}0\\ \cos 2t\left(\frac{1}{8}\cos 4t-\frac{1}{8}\right)-\sin 2t\left(\frac{1}{2}t+\frac{1}{8}\sin 4t\right)\\ \sin 2t\left(\frac{1}{8}\cos 4t-\frac{1}{8}\right)+\cos 2t\left(\frac{1}{2}t+\frac{1}{8}\sin 4t\right)\end{pmatrix}$$

CHAPTER 4: Qualitative theory of differential equations

In cases where $\dot{\vec{x}} = \vec{f}(t,\vec{x})$ where $\vec{f}(t,\vec{x})$ is a nonlinear function of $x_1, \dots, x_n$ we might not have the tools to solve for $\vec{x}$. However, oftentimes it's enough to know the qualitative properties of $\vec{x}$.

Properties of solutions of $\dot{\vec{x}} = \vec{f}(t,\vec{x})$ we're interested in.

① Are there equilibrium values $\vec{x}^0 = \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix}$ for which $\vec{x}(t) = \vec{x}^0$ is a solution of $\dot{\vec{x}} = \vec{f}(t,\vec{x})$?

$\dot{\vec{x}} = \vec{0}$ if $\vec{x}(t) \equiv \vec{x}^0$. Hence $\vec{x}^0$ is an equilibrium value of $\dot{\vec{x}} = \vec{f}(t,\vec{x})$ if and only if $\boxed{\vec{f}(t,\vec{x}^0) \equiv \vec{0}}$

② Let $\vec{\phi}(t)$ be a solution of $\dot{\vec{x}} = \vec{f}(t,\vec{x})$. Suppose that $\vec{\psi}(t)$ is a 2nd solution with $\psi_j(0)$ very close to $\phi_j(0)$, $j = 1, \dots, n$. Will $\vec{\psi}(t)$ remain very close to $\vec{\phi}(t)$ for all time? STABILITY

③ What happens to solutions $\vec{x}(t)$ of $\dot{\vec{x}} = \vec{f}(t,\vec{x})$ as $t \to \infty$?

(a) Do they approach equilibrium values?

(b) If not, do they approach a periodic solution?

—//—

Example. Find all equilibrium values of $\frac{dx_1}{dt} = 1 - x_2$, $\frac{dx_2}{dt} = x_1^3 + x_2$.

→ $\vec{x}^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$ is an equilibrium value iff

$$\left.\begin{array}{l} 1 - x_2^0 = 0 \\ (x_1^0)^3 + x_2^0 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_2^0 = 1 \\ x_1^0 = -1 \end{array}$$

Thus $\vec{x}^0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is the only equilibrium value of this system.

Example. Find all equilibrium solutions of

$$\frac{dx}{dt} = (x-1)(y-1), \quad \frac{dy}{dt} = (x+1)(y+1)$$

→ $\vec{x}^{\,0} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is an equilibrium value iff $\left.\begin{matrix}(x_0-1)(y_0-1)=0\\(x_0+1)(y_0+1)=0\end{matrix}\right\} \Rightarrow \begin{matrix}x_0=\pm1\\ y_0=\pm1\end{matrix}$

Example. Let $y(t)$ denote the position of the particle relative to its equilibrium position Detemine the stability. The relevant equation is

$$\boxed{\frac{d^2y}{dt^2}+y=\cos 2t.}$$

and the initial conditions are $\boxed{y(0)=1\ ,\ y'(0)=0}$.

→ We first convert this 2nd-order diff. eqn into a system of two 1st-order diff. eqn.s by setting $\boxed{\begin{matrix}x_1=y\\ x_2=y'\end{matrix}}$

Thus $y''=x_2' \Rightarrow x_2'+x_1=\cos 2t \Rightarrow x_2' = -x_1+\cos 2t$

$$x_1' = y' = x_2$$

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos 2t \end{pmatrix}$$

If we solve the homogeneous problem $\dot{\vec{x}} = A\vec{x}$ we have

$$\det(A-\lambda I) = \det\begin{pmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{pmatrix} = \lambda^2+1=0 \Rightarrow \lambda = \pm i$$

$$\lambda = i \Rightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} -iv_1+v_2=0 \\ iv_1=v_2 \end{matrix} \qquad \vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\vec{x}(t) = e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

Thus $\vec{x}^1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$, $\vec{x}^2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

For the nonhomogeneous part, i.e. particular solution we have from variation of parameters that

$$X(t) = [\vec{x}^1 \ \ \vec{x}^2] = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \rightarrow X^{-1}(t) = \frac{1}{\cancelto{1}{\cos^2 t + \sin^2 t}} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$\vec{x}(t) = X(t) X^{-1}(t_0) \vec{x}^0 + X(t) \int_{t_0}^{t} X^{-1}(s) \vec{f}(s) ds$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \cancelto{1}{\cos 0} & \cancelto{0}{-\sin 0} \\ \cancelto{0}{\sin 0} & \cancelto{1}{\cos 0} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \int_0^t \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ \cos 2s \end{pmatrix} ds$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \underbrace{\int_0^t \begin{pmatrix} -\sin s \cos 2s \\ \cos s \cos 2s \end{pmatrix} ds}$$

$$\int_0^t \begin{pmatrix} -\sin s (2\cos^2 s - 1) \\ \cos s (1 - 2\sin^2 s) \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} -2\cos^2 s \sin s + \sin s \\ \cos s - 2\sin^2 s \cos s \end{pmatrix} ds$$

$$= \begin{bmatrix} \frac{2}{3}\cos^3 s - \cos s \\ \sin s - \frac{2}{3}\sin^3 s \end{bmatrix}_0^t$$

$$= \begin{pmatrix} \frac{2}{3}\cos^3 t - \cos t - \frac{2}{3} + 1 \\ \sin t - \frac{2}{3}\sin^3 t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3}\cos^3 t - \cos t + \frac{1}{3} \\ \sin t - \frac{2}{3}\sin^3 t \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \frac{2}{3}\cos^3 t - \cos t + \frac{1}{3} \\ \sin t - \frac{2}{3}\sin^3 t \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + \begin{pmatrix} \frac{2}{3}\cos^4 t - \cos^2 t + \frac{1}{3}\cos t + \sin^2 t - \frac{2}{3}\sin^4 t \\ -\frac{2}{3}\cos^3 t \sin t + \sin t \cos t - \frac{1}{3}\sin t \\ + \cos t \sin t - \frac{2}{3}\cos t \sin^3 t \end{pmatrix}$$

First row of 2nd matrix

$$\frac{2}{3}\cos^4 t - \cos^2 t + \frac{1}{3}\cos t + \sin^2 t - \frac{2}{3}\sin^4 t$$

$$= \frac{2}{3}(\cancelto{1}{\cos^2 t + \sin^2 t})(\cos^2 t - \sin^2 t) - (\cos^2 t - \sin^2 t) + \frac{1}{3}\cos t$$

$$= -\frac{1}{3}(\cos^2 t - \overset{= 1-\cos^2 t}{\boxed{\sin^2 t}}) + \frac{1}{3}\cos t$$

$$= -\frac{1}{3}\cos^2 t + \frac{1}{3}(1 - \cos^2 t) + \frac{1}{3}\cos t$$

$$= -\frac{2}{3}\cos^2 t + \frac{1}{3}\cos t + \frac{1}{3}$$

Second row of 2nd matrix

$$-\frac{2}{3}\cos^3 t\sin t+\sin t\cos t-\frac{1}{3}\sin t+\cos t\sin t-\frac{2}{3}\cos t\sin^3 t$$

$$=-\frac{2}{3}\cos t\sin t\underbrace{\left(\cos^2 t+\sin^2 t\right)}_{1}+2\sin t\cos t-\frac{1}{3}\sin t$$

$$=\frac{4}{3}\cos t\sin t-\frac{1}{3}\sin t$$

Thus $\vec{x}(t)=\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}+\begin{pmatrix}-\frac{2}{3}\cos^2 t+\frac{1}{3}\cos t+\frac{1}{3}\\ \frac{4}{3}\underbrace{\cos t\sin t}_{\frac{1}{2}\sin 2t}-\frac{1}{3}\sin t\end{pmatrix}$

$-\frac{2}{3}\left(\frac{1}{2}+\frac{1}{2}\cos 2t\right)$

$=\cancel{-\frac{1}{3}}-\frac{1}{3}\cos 2t$

$$=\begin{pmatrix}\boxed{-\frac{2}{3}\cos^2 t}+\frac{4}{3}\cos t+\cancel{\frac{1}{3}}\\ \frac{2}{3}\sin 2t-\frac{4}{3}\sin t\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{1}{3}\cos 2t+\frac{4}{3}\cos t\\ \frac{2}{3}\sin 2t-\frac{4}{3}\sin t\end{pmatrix}\qquad (*)$$

Section 4.2 Stability of linear systems

Consider the stability of solutions of autonomous differential equations. Let $\vec{x} = \vec{\phi}(t)$ be a solution of $\dot{\vec{x}} = \vec{f}(\vec{x})$. Is $\vec{\phi}(t)$ stable or unstable?

at $t=0$ will it remain close to $\vec{\phi}(t)$ $\forall\ t \geq 0$?

Def^n. The solution $\vec{x} = \vec{\phi}(t)$ of $\dot{\vec{x}} = \vec{f}(\vec{x})$ is stable if every solution $\vec{\psi}(t)$ which starts sufficiently close to $\vec{\phi}(t)$ at $t=0$ must remain close to $\vec{\phi}(t)$ for all future time t. The solution $\vec{\phi}(t)$ is unstable if there exists at least one solution $\vec{\psi}(t)$ of $\dot{\vec{x}} = \vec{f}(\vec{x})$ which starts near $\vec{\phi}(t)$ at $t=0$ but which does not remain close to $\vec{\phi}(t)$ for all future time.

The solution $\vec{\phi}(t)$ is stable if for every $\varepsilon > 0$ $\exists\ \delta = \delta(\varepsilon)$ such that

$$|\psi_j(t) - \phi_j(t)| < \varepsilon \quad \text{if} \quad |\psi_j(0) - \phi_j(0)| < \delta(\varepsilon), \quad j = 1, \ldots, n$$

for every solution $\psi(t)$.

The stability question can be completely resolved

$$\dot{\vec{x}} = A\vec{x}$$

Theorem. (a) Every solution $\vec{x} = \vec{\phi}(t)$ of $\dot{\vec{x}} = A\vec{x}$ is stable if all the eigenvalues of A have negative real part.

(b) Every solution $\vec{x} = \vec{\phi}(t)$ of $\dot{\vec{x}} = A\vec{x}$ is unstable if at least one eigenvalue of A has positive real part.

(c) Suppose that all the eigenvalues of A have real part ≤ 0 and $\lambda_1 = i\sigma_1, \ldots,$ $\lambda_l = i\sigma_l$ have zero real part. Let $\lambda_j = i\sigma_j$ have multiplicity k_j. This means that the characteristic polynomial of A can be factored into the form

$$p(\lambda) = (\lambda - i\sigma_1)^{k_1} \cdots (\lambda - i\sigma_l)^{k_l} \boxed{q(\lambda)}$$

$\uparrow$

all roots of $q(\lambda)$ have negative real part

Then every solution $\vec{x} = \vec{\phi}(t)$ of $\dot{\vec{x}} = A\vec{x}$ is stable if A has k_j linearly independent eigenvectors of each eigenvalue $\lambda_j = i\sigma_j$. Otherwise, every solution $\vec{\phi}(t)$ is unstable.

Def^n. Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a vector with n components, with $x_1, \ldots, x_n$ real or complex. We define the length of $\vec{x}$ as $\boxed{\|\vec{x}\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.}$

So if $\vec{x} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ then $\|\vec{x}\| = 4$ and if $\vec{x} = \begin{pmatrix} 1+2i \\ 2 \\ -1 \end{pmatrix}$ then $\|\vec{x}\| = \sqrt{5}$

Properties. 1. $\|\vec{x}\| \geq 0$ for any vector $\vec{x}$ and $\|\vec{x}\| = 0$ only if $\vec{x} = \vec{0}$.

2. $\|\lambda\vec{x}\| = \max\{|\lambda x_1|, \ldots, |\lambda x_n|\} = |\lambda| \max\{|x_1|, \ldots, |x_n|\} = |\lambda| \cdot \|x\|$.

3.
$$\begin{aligned} \|\vec{x} + \vec{y}\| &= \max\{|x_1 + y_1|, \ldots, |x_n + y_n|\} \\ &\leq \max\{|x_1| + |y_1|, \ldots, |x_n| + |y_n|\} \quad \text{by triangle inequality} \\ &\leq \max\{|x_1|, \ldots, |x_n|\} + \max\{|y_1|, \ldots, |y_n|\} \\ &= \|\vec{x}\| + \|\vec{y}\| \end{aligned}$$

If all eigenvalues of A have $Re(\lambda) < 0$ then every solution $\vec{x}(t)$ of $\dot{\vec{x}} = A\vec{x}$ approaches zero as $t \to \infty$. Therefore, not only is the equilibrium solution $\vec{x}(t) \equiv \vec{0}$ stable but every solution $\vec{\phi}(t)$ approaches it as $t \to \infty$. This is known as asymptotic stability.

Example. Is the solution $\vec{x}(t)$ of $\dot{\vec{x}} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{pmatrix} \vec{x}$ stable, asymptotically stable, or unstable?

$$\det \begin{pmatrix} -1-\lambda & 0 & 0 \\ -2 & -1-\lambda & 2 \\ -3 & -2 & -1-\lambda \end{pmatrix} = (-1-\lambda)[(-1-\lambda)^2+4] = -(1+\lambda)[\lambda^2+2\lambda+1+4]$$

$$= -(1+\lambda)(\lambda^2+2\lambda+5)$$

$$= 0$$

$$\Rightarrow \lambda = -1 \;, \; \lambda = \frac{-2 \pm \sqrt{4-4(5)}}{2} = -1 \pm 2i$$

All 3 eigenvalues have negative real part and so every solution of $\dot{\vec{x}} = A\vec{x}$ is asymptotically stable.

Example Determine the stability of every solution of $\dot{\vec{x}} = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix} \vec{x}$.

$$\det \begin{pmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 25 = \lambda^2 - 2\lambda + 1 - 25 = \lambda^2 - 2\lambda - 24 = (\lambda-6)(\lambda+4) = 0$$

$$\Rightarrow \quad \lambda = -4, 6$$

Since one eigenvalue of $\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$ is positive, every solution $\vec{x} = \vec{\phi}(t)$ of $\dot{\vec{x}} = A\vec{x}$ is unstable.

Example. Show that every solution of $\dot{\vec{x}} = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix} \vec{x}$ is stable but not asymptotically stable.

$$\det \begin{pmatrix} -\lambda & -3 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 + 6 = 0 \quad \Rightarrow \lambda = \pm \sqrt{6}\, i$$

By part (c) of the Theorem, every solution $\vec{x} = \vec{\phi}(t)$ of $\dot{\vec{x}} = A\vec{x}$ is stable. But, no solution is asymptotically stable.

Solving for $\dot{\vec{x}} = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix} \vec{x}$ we see that the eigenvectors are

$$\lambda = \sqrt{6}\, i \Rightarrow \begin{pmatrix} -\sqrt{6}\, i & -3 \\ 2 & -\sqrt{6}\, i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -\sqrt{6}\, i\, v_1 - 3v_2 = 0$$

$$v_2 = \frac{\sqrt{6}\, i}{3} v_1$$

Thus $\vec{v} = \begin{pmatrix} 3 \\ \sqrt{6}\, i \end{pmatrix}$, $\vec{x}_1(t) = e^{\sqrt{6}\, i t}\vec{v} = [\cos(\sqrt{6}\, t) + i\sin(\sqrt{6}\, t)]\begin{pmatrix} 3 \\ \sqrt{6}\, i \end{pmatrix}$

$$= \begin{pmatrix} 3\cos(\sqrt{6}\, t) + 3i\sin(\sqrt{6}\, t) \\ \sqrt{6}\, i\cos(\sqrt{6}\, t) - \sqrt{6}\sin(\sqrt{6}\, t) \end{pmatrix}$$

$$= \begin{pmatrix} 3\cos(\sqrt{6}\, t) \\ -\sqrt{6}\sin(\sqrt{6}\, t) \end{pmatrix} + i\begin{pmatrix} 3\sin(\sqrt{6}\, t) \\ \sqrt{6}\cos(\sqrt{6}\, t) \end{pmatrix}$$

The general solution is thus

$$\vec{x}(t) = c_1\begin{pmatrix} 3\cos(\sqrt{6}\, t) \\ -\sqrt{6}\sin(\sqrt{6}\, t) \end{pmatrix} + c_2\begin{pmatrix} 3\sin(\sqrt{6}\, t) \\ \sqrt{6}\cos(\sqrt{6}\, t) \end{pmatrix}.$$

So every solution $\vec{x}(t)$ is periodic, with period $2\pi/\sqrt{6}$ and no solution $\vec{x}(t)$ (except $\vec{x}(t) \equiv \vec{0}$) approaches zero as $t \to \infty$.

Example. Show that every solution of $\dot{\vec{x}} = \underbrace{\begin{pmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{pmatrix}}_{A}\vec{x}$ is unstable.

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & -3 & 0 \\ 0 & -6-\lambda & -2 \\ -6 & 0 & -3-\lambda \end{pmatrix} = (2-\lambda)[(-6-\lambda)(-3-\lambda)] + 3(-12)$$

$$= (2-\lambda)[(\lambda+6)(\lambda+3)] - 36$$

$$= (2-\lambda)(\lambda^2 + 9\lambda + 18) - 36$$

$$= 2\lambda^2 + \cancel{18\lambda} + \cancel{36} - \lambda^3 - 9\lambda^2 - \cancel{18\lambda} - \cancel{36}$$

$$= -\lambda^3 - 7\lambda^2$$

$$= -\lambda^2(\lambda + 7)$$

$$= 0$$

Thus $\lambda = -7, 0$ (w/ multiplicity 2).

Every eigenvector of A with eigenvalue 0 must satisfy

$$\begin{pmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2v_1 - 3v_2 = 0 \Rightarrow v_1 = \frac{3}{2}v_2$$

$$-6v_2 - 2v_3 = 0 \Rightarrow v_2 = -\frac{1}{3}v_3$$

Thus $\vec{v} = c\begin{pmatrix} -3 \\ -2 \\ 6 \end{pmatrix}$

Since there is only one linearly independent eigenvector, this means that every solution $\vec{x} = \phi(t)$ of $\dot{\vec{x}} = A\vec{x}$ is unstable.

Section 4.3 Stability of equilibrium solutions

Now consider $\dot{\vec{x}} = A\vec{x} + \vec{g}(\vec{x})$ with $\vec{g}(\vec{x}) = \begin{pmatrix} g_1(\vec{x}) \\ g_2(\vec{x}) \\ \vdots \\ g_n(\vec{x}) \end{pmatrix}$ very small compared to $\vec{x}$.

We assume that

$\dfrac{g_1(\vec{x})}{\|\vec{x}\|}, \ldots, \dfrac{g_n(\vec{x})}{\|\vec{x}\|}$ are continuous functions of $x_1, \ldots, x_n$ which vanish for $x_1 = \ldots = x_n = 0$.

e.g. If $\vec{g}(\vec{x}) = \begin{pmatrix} x_1x_2^2 \\ x_1x_2 \end{pmatrix}$ then both $\dfrac{x_1x_2^2}{\|\vec{x}\|} = \dfrac{x_1x_2^2}{\max\{|x_1|, |x_2|\}}$, $\dfrac{x_1x_2}{\|\vec{x}\|}$ are continuous functions of x_1, x_2 which vanish for $x_1 = x_2 = 0$.

If $\vec{g}(\vec{0})=\vec{0}$ then $\vec{x}(t)\equiv\vec{0}$ is an equilibrium solution of $\dot{\vec{x}}=A\vec{x}+g(\vec{x})$

We want to say whether it's stable or unstable.

If $\vec{x}$ is very small then $g(\vec{x})$ is very small compared to $A\vec{x}$. So we will determine the stability of the eqm solution $\vec{x}(t)\equiv\vec{0}$ from the stability of $\dot{\vec{x}}=A\vec{x}$ (w/o $\vec{g}(\vec{x})$)

Theorem Suppose $\dfrac{\vec{g}(\vec{x})}{\|\vec{x}\|}$ is a continuous function of $x_1,\dots,x_n$ which vanishes for $\vec{x}=\vec{0}$. Then

(a) the eqm solution $\vec{x}(t)\equiv\vec{0}$ of $\dot{\vec{x}}=A\vec{x}+\vec{g}(\vec{x})$ is asymptotically stable if the eqm solution $\vec{x}(t)\equiv\vec{0}$ of the linearized equation $\dot{\vec{x}}=A\vec{x}$ is asymptotically stable.

$\Rightarrow$ $\vec{x}(\vec{t})\equiv\vec{0}$ of $\dot{\vec{x}}=A\vec{x}+\vec{g}(\vec{x})$ is asymptotically stable if all eigenvalues of A have negative real part.

(b) The eqm solution $\vec{x}(t)\equiv\vec{0}$ of $\dot{\vec{x}}=A\vec{x}+\vec{g}(\vec{x})$ is unstable if at least one eigenvalue of A has positive real part.

(c) The stability of $\vec{x}(t)\equiv\vec{0}$ cannot be determined from the stability of the eqm solution $\vec{x}(t)\equiv\vec{0}$ of $\dot{\vec{x}}=A\vec{x}$ if all eigenvalues of A have real part ≤ 0 but at least one eigenvalue of A has zero real part.

Example. Consider

$$\left.\begin{aligned}\frac{dx_1}{dt}&=x_2-x_1(x_1^2+x_2^2)\\ \frac{dx_2}{dt}&=-x_1-x_2(x_1^2+x_2^2)\end{aligned}\right\}\ (*)$$

The linearized equation is $\begin{pmatrix}dx_1/dt\\ dx_2/dt\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

and the eigenvalues of the matrix are $\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2+1=0$

$\lambda = \pm i$

To analyze the behavior of (*) we multiply the first eqn by x_1 and the second equation by x_2 and add them

$$x_1\frac{dx_1}{dt} + x_2\frac{dx_2}{dt} = \cancel{x_1} x_2 - x_1^2(x_1^2+x_2^2) - \cancel{x_1} x_2 - x_2^2(x_1^2+x_2^2)$$

$$= -(x_1^2+x_2^2)(x_1^2+x_2^2)$$

$$\frac{d}{dx}\left(\frac{1}{2}x_1^2+\frac{1}{2}x_2^2\right) = -(x_1^2+x_2^2)^2$$

$$\frac{1}{2}\frac{d}{dt}(x_1^2+x_2^2) = -(x_1^2+x_2^2)^2$$

$$\frac{1}{(x_1^2+x_2^2)^2}\frac{d}{dt}(x_1^2+x_2^2) = -2$$

$$-\frac{1}{(x_1^2+x_2^2)} = -2t + c$$

$t=0 \Rightarrow \quad -\frac{1}{(x_1^2(0)+x_2^2(0))} = c$

Thus $\cancel{-}\frac{1}{(x_1^2(t)+x_2^2(t))} = \cancel{-}2t \cancel{-}\frac{1}{(x_1^2(0)+x_2^2(0))}$

$$\frac{1}{x_1^2(t)+x_2^2(t)} = \frac{2t(x_1^2(0)+x_2^2(0))+1}{x_1^2(0)+x_2^2(0)}$$

$$\Rightarrow \boxed{x_1^2(t)+x_2^2(t) = \frac{x_1^2(0)+x_2^2(0)}{2t[x_1^2(0)+x_2^2(0)]+1}}$$

This implies that as $t \to \infty$, $x_1^2(t) + x_2^2(t) \to 0$ for any solution $x_1(t), x_2(t)$.

Thus $x_1(t) \equiv 0$, $x_2(t) \equiv 0$ is asymptotically stable.

Example. Now consider instead

$$\left.\begin{aligned} \frac{dx_1}{dt} &= x_2 + x_1(x_1^2 + x_2^2) \\ \frac{dx_2}{dt} &= -x_1 - x_2(x_1^2 + x_2^2) \end{aligned}\right\} (+)$$

The linearized system is the same $\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}$.

However if we now follow the same process we have that

$$\frac{1}{2}\frac{d}{dt}(x_1^2 + x_2^2) = (x_1^2 + x_2^2)^2$$

which gives $\boxed{x_1^2(t) + x_2^2(t) = \dfrac{x_1^2(0) + x_2^2(0)}{1 - 2t\left[x_1^2(0) + x_2^2(0)\right]}}$

Note that every solution $x_1(t), x_2(t)$ of $(+)$ with $x_1^2(0) + x_2^2(0) \neq 0$ approaches infinity in finite time.

Thus $x_1(t) \equiv 0$, $x_2(t) \equiv 0$ is unstable.

Example. Consider

$$\begin{aligned} \frac{dx_1}{dt} &= -2x_1 + x_2 + 3x_3 + 9x_2^3 \\ \frac{dx_2}{dt} &= -6x_2 - 5x_3 + 7x_3^5 \\ \frac{dx_3}{dt} &= -x_3 + x_1^2 + x_2^2 \end{aligned}$$

Determine whether the equilibrium solution $x_1(t) \equiv 0$, $x_2(t) \equiv 0$, $x_3(t) \equiv 0$ is stable or unstable.

We rewrite this system as $\dot{\vec{x}} = A\vec{x} + \vec{g}(\vec{x})$ where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} -2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \vec{g}(\vec{x}) = \begin{pmatrix} 9x_2^3 \\ 7x_3^5 \\ x_1^2 + x_2^2 \end{pmatrix}$$

The $\vec{g}(\vec{x})$ satisfies the hypothesis of the Theorem I.e .

$$\frac{\vec{g}(\vec{x})}{\|\vec{x}\|} = \frac{\vec{g}(\vec{x})}{\max\{|x_1|, \ldots, |x_n|\}}$$ is a continuous function of $x_1, \ldots, x_n$ which vanishes for $\vec{x} = \vec{0}$

$$\det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & 1 & 3 \\ 0 & -6-\lambda & -5 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-2-\lambda)[(-6-\lambda)(-1-\lambda)] = 0$$

$$\Rightarrow \quad \lambda = -6, -2, -1$$

Since all the eigenvalues of A are negative, the equlibrium solution $\vec{x}(t) \equiv \vec{0}$ is asymptotically stable

Section 4.4 The phase-plane

Consider the system of differential equations

$$\frac{dx}{dt} = f(x,y)$$

$$\frac{dy}{dt} = g(x,y)$$

and observe that every solution $x = x(t)$, $y = y(t)$ defines a curve in the 3D space (t,x,y).

Example. Solve $\begin{array}{l}\dot{x} = -y\\ \dot{y} = x\end{array}$ and describe the curve the solutions trace out

$$\Rightarrow \quad \dot{\vec{x}} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{A} \vec{x} \qquad \det(A - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$\lambda = i \Rightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} -iv_1 - v_2 = 0 \\ v_2 = -iv_1 \\ \vec{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{array}$$

$$\vec{x} = e^{it}\begin{pmatrix} 1 \\ -i \end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \begin{pmatrix} \cos t + i\sin t \\ -i\cos t + \sin t \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

$x(t) = \cos t$, $y(t) = \sin t$ is a solution As t runs from 0 to 2π, the points $(x,y) = (\cos t, \sin t)$ trace out a circle of radius 1 and center $(0,0)$ i.e. $x^2 + y^2 = 1$. As t runs from 0 to ∞, the set of points $(\cos t, \sin t)$ trace out this circle infinitely often

Example It can be shown that a solution of

$$\frac{dx}{dt} = 6\sqrt{\frac{y-7}{5}} \quad , \quad \frac{dy}{dt} = 10\sqrt{\frac{x-2}{3}}$$

is $x = 3t^2+2$, $y = 5t^2+7$
$x \geqslant 2$ $\quad$ $y \geqslant 7$

Solving for t we have $3t^2 = x-2 \Rightarrow t = \sqrt{\frac{x-2}{3}}$, $x \geqslant 2$

$$y = 5\left(\frac{x-2}{3}\right)+7 \Rightarrow \boxed{y = \frac{5}{3}(x-2)+7} \quad \text{so for } 2 \leq x < \infty$$

$\uparrow$

orbit of the solution.

An advantage of using the orbit of a solution rather than the solution itself is that it's often possible to obtain the orbit of a solution w/o prior knowledge of the solution

Let $\begin{bmatrix} x = x(t) \\ y = y(t) \end{bmatrix}$ be a solution of $\begin{bmatrix} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{bmatrix}$. If $x'(t) \neq 0$ at $t = t_1$ then we can solve for $t = t(x)$ in a neighborhood of $x_1 = x(t_1)$. For t near t_1 the orbit of $x(t), y(t)$ is the curve $\boxed{y = y(t(x))}$

Note that $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x,y)}{f(x,y)}$. Thus, the orbits of the solutions $x = x(t), y = y(t)$ are the solution curves of $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$.

$\Rightarrow$ We do not need to find a solution $x(t), y(t)$ in order to compute its orbit. We only need to solve the single 1st-order scalar diff. equ. $\boxed{\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}}$

Example Find the orbits of $\frac{dx}{dt} = y^2$, $\frac{dy}{dt} = x^2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x^2}{y^2}$$

$$\int y^2\,dy = \int x^2\,dx$$

$$\Rightarrow \frac{y^3}{3} = \frac{x^3}{3} + C$$

$$y^3 = x^3 + A$$

$y = (x^3+A)^{1/3}$ where A is a constant.

Orbits of $\frac{dx}{dt} = y^2$, $\frac{dy}{dt} = x^2$ are the set of all curves $y(x) = (x^3+A)^{1/3}$.

Example. Orbits of $\frac{dx}{dt} = y(1+x^2+y^2)$, $\frac{dy}{dt} = -2x(1+x^2+y^2)$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x\cancel{(1+x^2+y^2)}}{y\cancel{(1+x^2+y^2)}} = -\frac{2x}{y}$$

$$\int y\,dy = \int -2x\,dx$$

$$\frac{y^2}{2} = -x^2 + C$$

$\frac{y^2}{2} + x^2 = C$ ← ellipses.

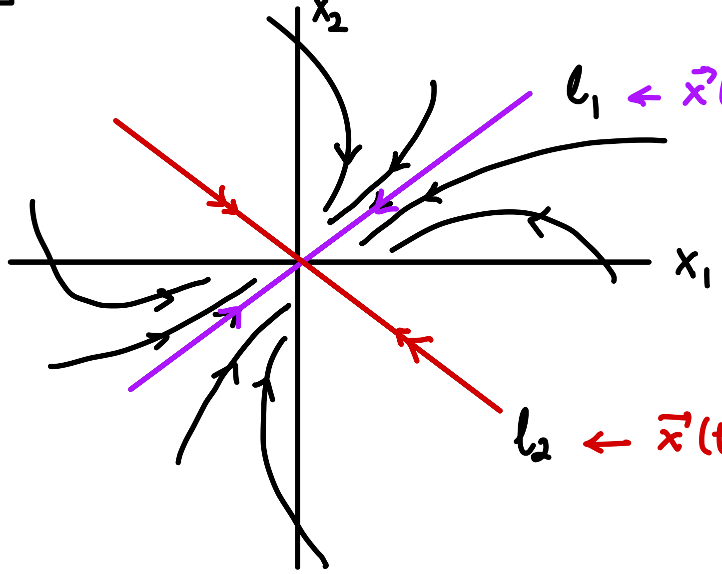
Section 4.7: Phase portraits of linear systems

$$\dot{\vec{x}} = A\vec{x}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A complete picture of all orbits of this linear diff. eqn. is called a phase portrait, and it depends almost completely on the eigenvalues of A. It also changes a lot when the eigenvalues of A change sign or become imaginary.

Cases:

① $\boxed{\lambda_2 < \lambda_1 < 0}$ Let $\vec{v}^1$ and $\vec{v}^2$ be eigenvectors of A with eigenvalues λ_1 and λ_2



eqm sol^n $\vec{x} = \vec{0}$ is a $\boxed{\text{STABLE NODE}}$

The arrows on l_1 and l_2 indicate in what direction $\vec{x}(t)$ moves along its orbit.

$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$ so every solution $\vec{x}(t)$ approaches $\begin{pmatrix}0\\0\end{pmatrix}$ as $t \to \infty$.

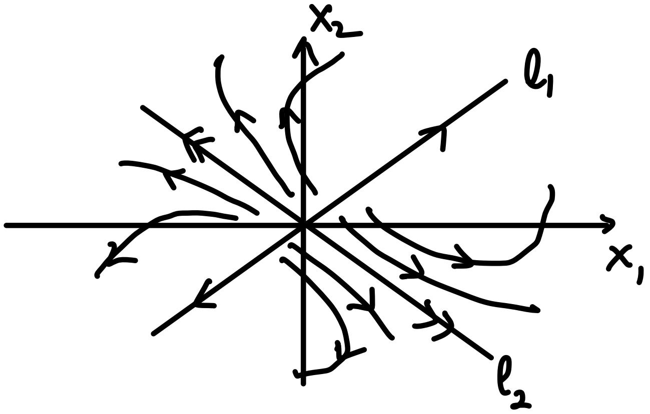
It's helpful to rewrite the general solution as $\vec{x}(t) = e^{\lambda_1 t}\left(c_1\vec{v}_1 + c_2 e^{(\lambda_2-\lambda_1)t}\vec{v}_2\right)$

less negative λ

Observe that $\lambda_2 - \lambda_1 < 0$. Thus, as long as $c_1 \neq 0$ the term $c_2 e^{(\lambda_2-\lambda_1)t}\vec{v}_2$ is negligible compared to $c_1\vec{v}_1$ for t sufficiently large. Therefore, as $t \to \infty$, the trajectory not only approaches the origin but also tends toward the line through $\vec{v}_1$

Tangent to the slow eigenvector

② $\boxed{0 < \lambda_1 < \lambda_2}$ both evalues are positive $\vec{x}(t) \equiv \vec{0}$ is an $\boxed{\text{UNSTABLE NODE}}$



③ $\boxed{\lambda_1 = \lambda_2 < 0}$ Does A have 1 or 2 linearly independent eigenvectors?

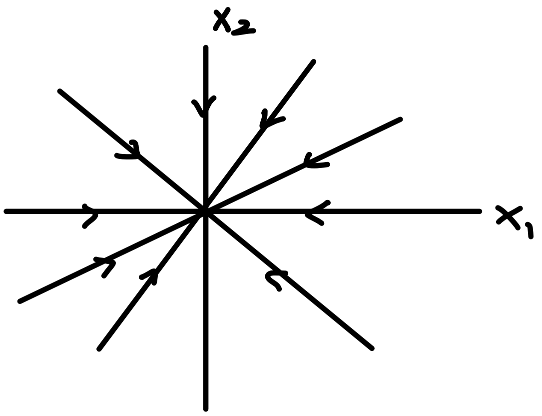
→ If A has 2 linearly indep. evectors $\vec{v_1}$ and $\vec{v_2}$ w/ evalue $\lambda<0$ then every solution can be written as $\vec{x}(t) = e^{\lambda t}(c_1\vec{v_1} + c_2\vec{v_2})$.

Every vector is an eigenvector with this eigenvalue λ.

↳ let's write an arbitrary vector $\vec{x_0}$ as a linear combination of two evectors: $\vec{x_0} = c_1\vec{v_1} + c_2\vec{v_2}$. Then

$$A\vec{x_0} = A(c_1\vec{v_1} + c_2\vec{v_2}) = c_1\lambda\, v_1 + c_2\lambda\, v_2 = \lambda(c_1\vec{v_1} + c_2\vec{v_2} = \lambda\vec{x_0}$$

so $\vec{x_0}$ is also an eigenvector with eigenvalue λ.



STAR NODE

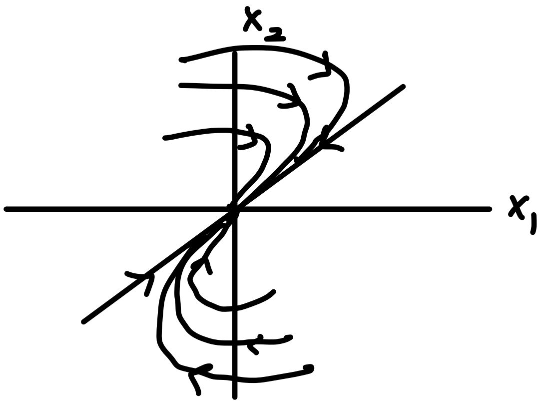
→ If A has 1 linearly indep. evector $\vec{v}$ with λ then

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{v_1} + c_2 e^{\lambda t}(\vec{u} + \boxed{kt\vec{v}})$$

dominant term as $t \to \infty$

Every solution $\vec{x}(t)$ approaches $\binom{0}{0}$ as $t \to \infty$.

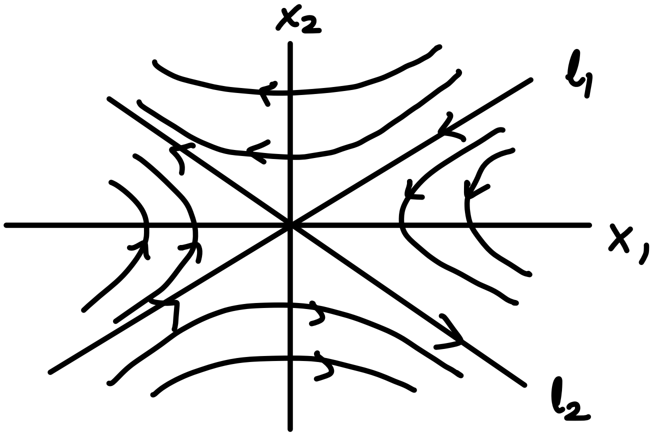
Hence the tangent to the orbit of $\vec{x}(t) \to \pm\vec{v}$ as $t \to \infty$.



DEGENERATE NODE

④ $\boxed{\lambda_1=\lambda_2>0}$ Same as ③ above but w/ direction of arrows reversed.

⑤ $\boxed{\lambda_1<0<\lambda_2}$ $\vec{x}(t) = c_1 e^{\lambda t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2}$



SADDLE POINT

⑥ $\boxed{\begin{matrix}\lambda_1 = \alpha+i\beta \\ \lambda_2 = \alpha-i\beta\end{matrix}}$, $\beta\neq 0$

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda\end{pmatrix} = 0 \Rightarrow (a-\lambda)(d-\lambda)-bc=0$$

$$\lambda^2-(a+d)\lambda+ad-bc=0$$

$$\lambda = \frac{(a+d)\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$$

We get complex λ if

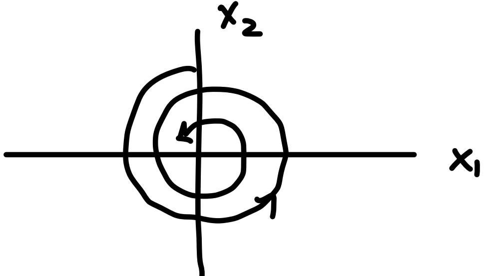
$(a+d)^2-4(ad-bc)<0$

Since $\beta\neq 0$ the eigenvalues are distinct and the general solution is still $\vec{x}(t)=c_1e^{\lambda_1 t}\vec{v_1}+c_2e^{\lambda_2 t}\vec{v_2}$.

The c, $\vec{v}$ are complex since the λ's are. $\vec{x}(t)$ is a linear combination of $e^{(\alpha\pm i\beta)t}$. By Euler's identity $e^{i\beta t}=\cos(\beta t)+i\sin(\beta t)$

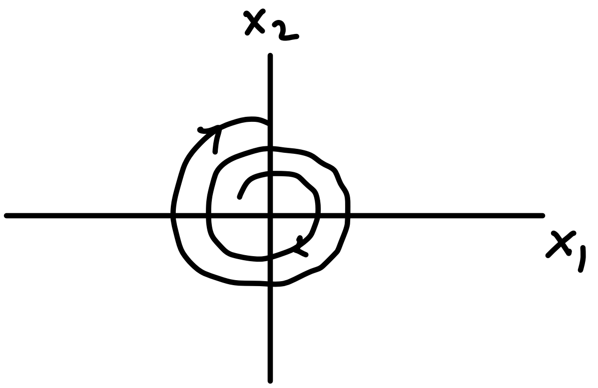
Thus $\vec{x}(t)$ is a combination of terms involving $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$.

- Exponentially decaying oscillations if $\boxed{\alpha=Re(\lambda)<0}$



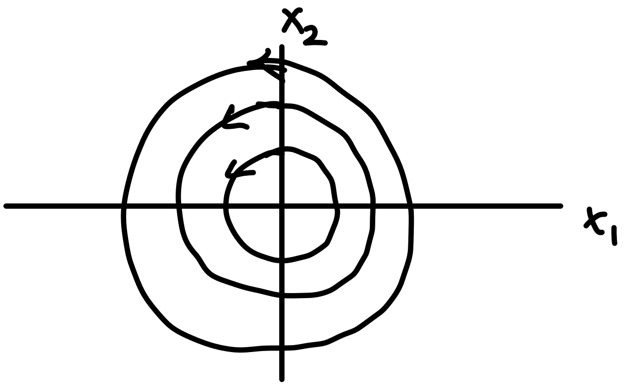
STABLE SPIRAL

- Exponentially growing oscillations if $\boxed{\alpha = Re(\lambda) > 0}$



UNSTABLE SPIRAL

- If the eigenvalues are purely imaginary, i.e. $\boxed{\alpha = 0}$ then the solutions are periodic with period $T = 2\pi/\beta$.



CENTER

<u>Note</u>: The direction of the arrows must be determined from the differential equation $\vec{x} = A\vec{x}$. The simplest way of doing this is to check the sign of $\dot{x}_2$ when $x_2 = 0$

① If $\dot{x}_2 > 0$ for $x_2 = 0$ and $x_1 > 0$ then all the solutions $\vec{x}(t)$ move in the counterclockwise direction

② If $\dot{x}_2 < 0$ for $x_2 = 0$ and $x_1 > 0$ then all solutions $\vec{x}(t)$ move in the clockwise direction.

Example Draw the phase portrait of the linear equation

$$\dot{\vec{x}} = A\vec{x} = \begin{pmatrix} -2 & -1 \\ 4 & -7 \end{pmatrix} \vec{x}$$

$$\det\begin{pmatrix} -2-\lambda & -1 \\ 4 & -7-\lambda \end{pmatrix} = 0 \Rightarrow (-2-\lambda)(-7-\lambda)+4=0$$

$$\lambda^2 + 9\lambda + 14 + 4 = 0$$

$$\lambda^2 + 9\lambda + 18 = 0$$

$$(\lambda + 3)(\lambda + 6) = 0$$

$$\lambda = -3, -6$$

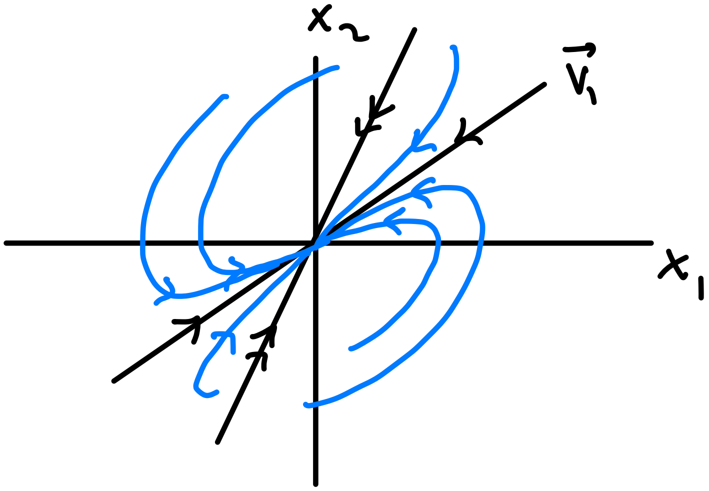
$\lambda_1 = -3$: $\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_1 - v_2 = 0 \Rightarrow v_1 = v_2$. Thus $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -6$: $\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$4v_1 - v_2 = 0$

$v_2 = 4v_1$. Thus $\vec{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$



Stable node

Example Draw the phase portrait of $\dot{\vec{x}} = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \vec{x}$

$$\det\begin{pmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 - 9 = 0$$

$$\lambda^2 - 2\lambda + 1 - 9 = 0$$

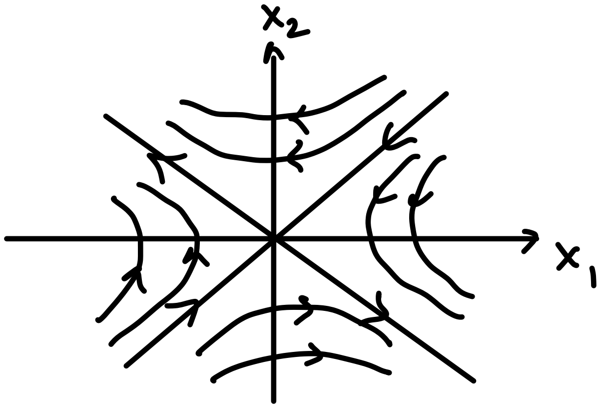
$$\lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda+2)(\lambda-4) = 0$$

$$\lambda_1 = -2, \ \lambda_2 = 4$$

$$\lambda_1 = -2 \Rightarrow \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3v_1 - 3v_2 = 0, \quad v_1 = v_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4 \Rightarrow \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 + v_2 = 0, \quad v_1 = -v_2 \Rightarrow \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

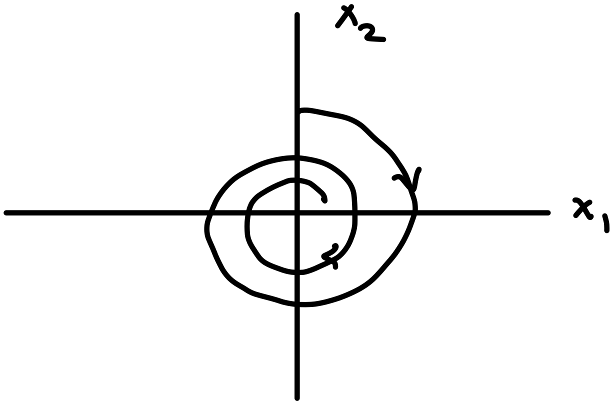


Example Draw the phase portrait of $\dot{\vec{x}} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}$

$$\det\begin{pmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2 + 1 = 0$$

$$-1-\lambda = \pm i \Rightarrow \lambda = -1 \pm i \Rightarrow \text{stable spiral}$$

to decide the direction of the arrows we look at $\dot{x}_2 = -x_1 - x_2$ and see that when $x_2 = 0$ (so along the horizontal axis) $\dot{x}_2 < 0$ when $x_1 > 0$, so the arrows go clockwise



Stability properties of linear systems $\vec{x}' = A\vec{x}$ w/ $\det(A-\lambda I)=0$ and $\det(A)\neq 0$

Eigenvalues	Type of critical point	Stability
$\lambda_1 > \lambda_2 > 0$	node	unstable
$\lambda_1 < \lambda_2 < 0$	node	asympt. stable
$\lambda_2 < 0 < \lambda_1$	saddle point	unstable
$\lambda_1 = \lambda_2 > 0$	proper or improper node	unstable
$\lambda_1 = \lambda_2 < 0$	proper or improper node	asympt. stable
$\lambda_1, \lambda_2 = \alpha + i\beta$	spiral point	
$\alpha > 0$		unstable
$\alpha < 0$		asympt. stable
$\lambda_1 = i\beta, \lambda_2 = -i\beta$	center	stable

Example. Consider $\vec{x}' = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \vec{x}$. Let $p = a_{11} + a_{22} = \text{trace}(A)$

$q = a_{11}a_{22} - a_{12}a_{21} = \det(A)$

Show that the critical point $(0,0)$ is a

(a) node if $q<0$ and $\Delta \geq 0$

(b) saddle point if $q<0$

(c) spiral point if $p \neq 0$ and $\Delta < 0$

(d) center if $p=0$ and $q>0$

Compute: $\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$

$$= a_{11}a_{22} - (a_{11}+a_{22})\lambda + \lambda^2 - a_{12}a_{21}$$

$$= \lambda^2 - \underbrace{(a_{11}+a_{22})}_{p}\lambda + \underbrace{a_{11}a_{22}-a_{12}a_{21}}_{q}$$

$$= \lambda^2 - p\lambda + q$$

$$= 0$$

$$\boxed{\lambda_{1,2} = \frac{p \pm \sqrt{p^2-4q}}{2} = \frac{p \pm \sqrt{\Delta}}{2}}$$

Note $\lambda_1\lambda_2 = \left(\frac{p}{2} + \frac{\sqrt{\Delta}}{2}\right)\left(\frac{p}{2} - \frac{\sqrt{\Delta}}{2}\right) = \frac{p^2}{4} - \frac{\Delta}{4} = \cancel{\frac{p^2}{4}} - \frac{\cancel{p^2}+4q}{\cancel{4}\;\;4} = q$

$\lambda_1 + \lambda_2 = \frac{p}{2} + \cancel{\frac{\sqrt{\Delta}}{2}} + \frac{p}{2} - \cancel{\frac{\sqrt{\Delta}}{2}} = p$

(a) So if $q>0$ this implies that λ_1 and λ_2 have the same sign since $q = \lambda_1\lambda_2 > 0$ and if $\Delta \geq 0$ it means that λ_1 and λ_2 are real. So it has to be a node

(b) if $q<0$ it means that λ_1, λ_2 have opposite signs so it's a saddle point

(c) if $p \neq 0$ and $\Delta < 0$ this implies that λ_1, λ_2 are complex eigenvalues so ~~it~~ must be a spiral

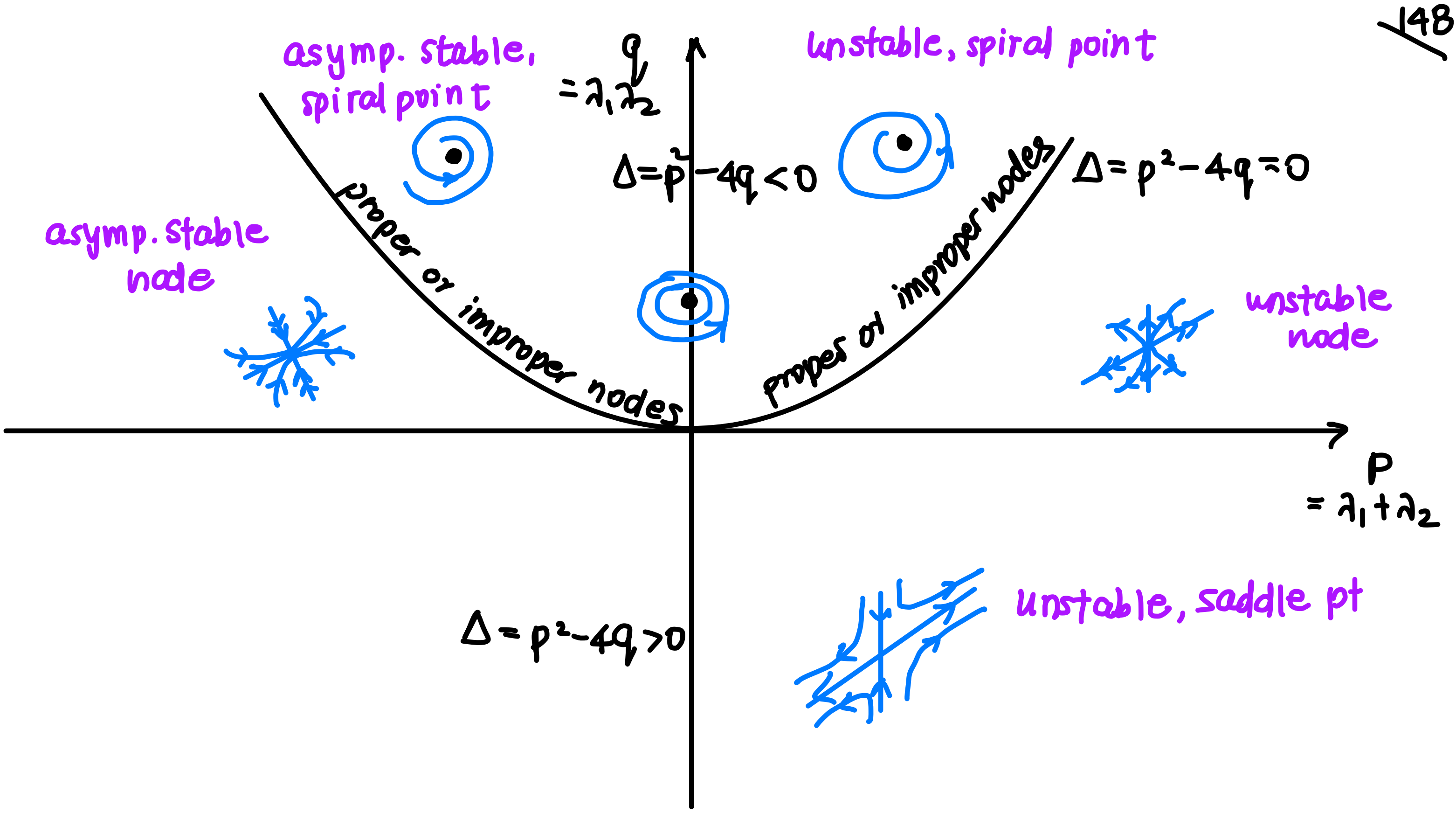
(d) if $p=0$ (real part $=0$) and $q>0 \Rightarrow \lambda_1, \lambda_2$ are the same sign. $\Rightarrow$ center

→ Now show that the equilibrium point $(0,0)$ is

(a) asymptotically stable if $q>0$ and $p<0$

(b) stable if $q>0$ and $p=0$

(c) unstable if $q<0$ or $p>0$



$q = \frac{p^2}{4}$ parabola in p-q axes. ⇒ proper/improper nodes
⇝ repeated, real evalues.

$q > \frac{p^2}{4}$ ⇒ $p^2-4q<0$ ⇒ complex eigenvalues.

above the parabola

Along the q-axis, $p=0$
$(\lambda_1+\lambda_2=0)$
which implies that λ_1, λ_2 are purely imaginary ⇒ center

$q < \frac{p^2}{4}$ ⇒ $p^2-4q>0$ & $q<0$
$(\lambda_1\lambda_2<0)$ ⇒ saddle point

below the parabola

Section 2.9 : The method of Laplace transforms

We want to solve the IVP : $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$; $y(0) = y_0$, $y'(0) = y_0'$

Usually useful when
- $f(t)$ is a discontinuous function of time
- $f(t)$ is zero except for a very short time interval in which it is very large.

Definition Let $f(t)$ be defined for $0 \le t < \infty$. The Laplace transform of $f(t)$, which is denoted by $F(s)$, or $\mathcal{L}\{f(t)\}$ is given by

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt}$$

where $\int_0^\infty e^{-st} f(t)dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)dt$ improper integral

Example Compute the Laplace transform of $f(t) = 1$

$$\mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st} dt$$

$$= \lim_{A\to\infty} \left[-\frac{1}{s} e^{-st} \right]_0^A$$

$$= \lim_{A\to\infty} \left[-\frac{1}{s} e^{-sA} + \frac{1}{s} \right]$$

$$\Rightarrow \boxed{\mathcal{L}\{1\} = \begin{cases} \frac{1}{s} , & s > 0 \\ \infty , & s \le 0 \end{cases}}$$

Example Compute the Laplace transform of e^{at}

$$\mathcal{L}\{e^{at}\} = \lim_{A\to\infty}\int_0^A e^{-st}e^{at}\,dt = \lim_{A\to\infty}\left[\frac{1}{(a-s)}e^{(a-s)t}\right]_0^A$$

$$= \lim_{A\to\infty}\left[\frac{1}{a-s}e^{(a-s)A} - \frac{1}{a-s}\right]$$

$$\Rightarrow \boxed{\mathcal{L}\{e^{at}\} = \begin{cases} \frac{1}{s-a}, & s>a \qquad (a-s)<0 \\ \infty, & s\le a \qquad (a-s)\geq 0 \end{cases}}$$

Example Compute the Laplace transform of cos(wt) and sin(wt)

$$\mathcal{L}\{\cos(\omega t)\} = \lim_{A\to\infty}\int_0^A e^{-st}\cos(\omega t)\,dt$$

$$\mathcal{L}\{\sin(\omega t)\} = \lim_{A\to\infty}\int_0^A e^{-st}\sin(\omega t)\,dt$$

$$\Rightarrow \mathcal{L}\{\cos(\omega t) + i\sin(\omega t)\} = \mathcal{L}\{e^{i\omega t}\} = \int_0^\infty e^{-st}e^{i\omega t}\,dt = \int_0^\infty e^{(i\omega - s)t}\,dt$$

$$= \lim_{A\to\infty}\frac{e^{(i\omega-s)A} - 1}{i\omega - s}$$

$$= \begin{cases} -\frac{1}{i\omega - s} = \frac{1}{s-i\omega}\cdot\frac{s+i\omega}{s+i\omega} = \frac{s+i\omega}{s^2+\omega^2} & s>0 \\ \text{undefined} & s\le 0 \end{cases}$$

$$\mathcal{L}\{\cos(\omega t)\} + i\mathcal{L}\{\sin(\omega t)\} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2} \Rightarrow \boxed{\begin{aligned}\mathcal{L}\{\cos(\omega t)\} &= \frac{s}{s^2+\omega^2} \\ \mathcal{L}\{\sin(\omega t)\} &= \frac{\omega}{s^2+\omega^2}, \; s>0\end{aligned}}$$

Note Here we used the fact that the Laplace transform is a linear operator

$$\mathcal{L}\{c_1f_1(t)+c_2f_2(t)\} = \int_0^\infty e^{-st}[c_1f_1(t)+c_2f_2(t)]\,dt$$

$$= c_1\int_0^\infty e^{-st}f_1(t)\,dt + c_2\int_0^\infty e^{-st}f_2(t)\,dt$$

$$= c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$$

Lemma 1 Let $F(s)=\mathcal{L}\{f(t)\}$. Then $\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)}$

Proof. Use the formula and integrate by parts

$u = e^{-st}$

$\frac{du}{dt} = -se^{-st}$

$\frac{dv}{dt} = f'(t)$

$v = f(t)$

$$\mathcal{L}\{f'(t)\} = \lim_{A\to\infty}\int_0^A e^{-st}f'(t)\,dt$$

$$= \lim_{A\to\infty}\left[e^{-st}f(t)\right]_0^A + \lim_{A\to\infty} s\int_0^A e^{-st}f(t)\,dt$$

$$= -f(0) + s\underbrace{\lim_{A\to\infty}\int_0^A e^{-st}f(t)\,dt}_{F(s)}$$

$$= -f(0) + sF(s)$$

Lemma 2 Let $F(s) = \mathcal{L}\{f(t)\}$. Then $\boxed{\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$

Proof Using Lemma 1 twice:

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left[s\underbrace{\mathcal{L}\{f(t)\}}_{F(s)} - f(0)\right\} - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Now we can reduce the problem of solving the IVP

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_0'$$

to that of solving an algebraic equation. Let $Y(s) = \mathcal{L}\{y(t)\}$ and $F(s) = \mathcal{L}\{f(t)\}$

Taking laplace transforms of both sides of the diff. eqn. gives

$$\mathcal{L}\{ay''(t) + by'(t) + cy(t)\} = F(s)$$

By linearity of the Laplace transform we have:

$$a\mathcal{L}\{y''(t)\} + b\mathcal{L}\{y'(t)\} + c\mathcal{L}\{y(t)\} = F(s)$$

Using Lemmas 1 and 2:

$$a\left[s^2 Y(s) - s\underbrace{y(0)}_{y_0} - \underbrace{y'(0)}_{y_0'}\right] + b\left[sY(s) - \underbrace{y(0)}_{y_0}\right] + cY(s) = F(s)$$

$$\Rightarrow \quad Y(s)\left[as^2 + bs + c\right] - y_0(as + b) - ay_0' = F(s)$$

$$\Rightarrow \quad Y(s) = \frac{(as+b)y_0}{as^2+bs+c} + \frac{ay_0'}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c} \qquad (*)$$

$(*)$ tells us the Laplace transform of the solution $y(t)$ of the IVP

To find $y(t)$ we must consult the inverse Laplace transform tables

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Example Solve $y'' - 3y' + 2y = e^{3t}$, $y(0) = 1$, $y'(0) = 0$

Let $Y(s) = \mathcal{L}\{y(t)\}$ Taking the Laplace transform on both sides gives

$$s^2Y(s) - s\underset{1}{y(\cancel{0})} - \underset{0}{y'(\cancel{0})} - 3(sY(s) - \underset{1}{y(\cancel{0})}) + 2Y(s) = \underbrace{\frac{1}{s-3}}_{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$$

$$Y(s)\left[s^2 - 3s + 2\right] = \frac{1}{s-3} + s - 3$$

$$Y(s) = \frac{1}{(s-3)(s^2-3s+2)} + \frac{s-3}{s^2-3s+2}$$

$$= \frac{1}{(s-3)(s-2)(s-1)} + \frac{s-3}{(s-2)(s-1)}$$

To find $y(t)$ we expand the RHS in partial fractions

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\Rightarrow \quad A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) = 1$$

Let $s = 1 \Rightarrow \quad A(-1)(-2) = 1 \Rightarrow \boxed{A = 1/2}$

$s = 2 \Rightarrow \quad B(1)(-1) = 1 \Rightarrow \boxed{B = -1}$

$s = 3 \Rightarrow \quad C(2)(1) = 1 \Rightarrow \boxed{C = 1/2}$

Thus $\frac{1}{(s-1)(s-2)(s-3)} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}$

Similarly $\frac{s-3}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$

$\Rightarrow \quad s-3 = A(s-2)+B(s-1)$

Let $s=1 \Rightarrow \quad -2=-A \Rightarrow \boxed{A=2}$

$s=2 \Rightarrow \quad -1=B \quad \Rightarrow \boxed{B=-1}$

Thus $\frac{s-3}{(s-1)(s-2)} = \frac{2}{s-1} - \frac{1}{s-2}$

Overall, then we have

$$Y(s) = \frac{1}{(s-3)(s-2)(s-1)} + \frac{s-3}{(s-2)(s-1)}$$

$$= \frac{1}{2}\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3} + \frac{2}{s-1} - \frac{1}{s-2}$$

$$= \underbrace{\frac{5}{2}\frac{1}{s-1}}_{\text{Laplace transform of } \frac{5}{2}e^{t}} - \underbrace{\frac{2}{s-2}}_{\text{Laplace transform of } -2e^{2t}} + \underbrace{\frac{1}{2}\frac{1}{s-3}}_{\text{Laplace transform of } \frac{1}{2}e^{3t}}$$

Thus $Y(s) = \mathcal{L}\{\frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}\} \Rightarrow y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$.

Section 2.10: Some useful properties of Laplace transforms

Property 1: If $\mathcal{L}\{f(t)\} = F(s)$, then $\boxed{\mathcal{L}\{-tf(t)\} = \frac{d}{ds}F(s)}$

Proof $F(s) = \int_0^\infty e^{-st} f(t)\,dt$. Let's differentiate both sides.

$$\begin{aligned} \frac{d}{ds}F(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt \\ &= \int_0^\infty -te^{-st} f(t)\,dt \\ &= \mathcal{L}\{-tf(t)\} \end{aligned}$$

Example: Compute the Laplace transform of te^t

$$\frac{d}{ds}F(s) = -\mathcal{L}\{tf(t)\} \Rightarrow \mathcal{L}\{te^t\} = -\frac{d}{ds}\frac{1}{s-1} = +\frac{1}{(s-1)^2}$$

Example: Compute the Laplace transform of t^{20}

Using Property 1, i.e. $\mathcal{L}\{-tf(t)\} = \frac{d}{ds}F(s)$ 20 times yields

$$\mathcal{L}\{t^{20}\} = (-1)^{20}\frac{d^{20}}{ds^{20}}\mathcal{L}\{1\} = (-1)^{20}\frac{d^{20}}{ds^{20}}\frac{1}{s} = \frac{(20)!}{s^{21}}$$

Example What function has Laplace transform $-\frac{1}{(s-2)^2}$?

$$-\frac{1}{(s-2)^2} = \frac{d}{ds}\frac{1}{s-2} \quad \text{and} \quad \frac{1}{s-2} = \mathcal{L}\{e^{2t}\}$$

So if we use $\mathcal{L}\{-tf(t)\} = \frac{d}{ds}F(s)$ we have

$$\mathcal{L}^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t f(t)$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{-\frac{1}{(s-2)^2}\right\} = -te^{2t}$$

<u>Example</u> What function has Laplace transform $\frac{-4s}{(s^2+4)^2}$?

Use : $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$

$$\frac{-4s}{(s^2+4)^2} = \frac{d}{ds}\left(\frac{2}{s^2+4}\right) \quad \text{and} \quad \mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$$

$\uparrow$

$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$

Thus, using Property 1: $\mathcal{L}\{t\sin(2t)\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{-\frac{4s}{(s^2+4)^2}\right\} = -t\sin 2t$$

<u>Property 2</u>: If $F(s) = \mathcal{L}\{f(t)\}$ then $\boxed{\mathcal{L}\{e^{at}f(t)\} = F(s-a)}$

<u>Proof</u>: $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt$

$$= F(s-a) \qquad \square$$

This states that the Laplace transform of $e^{at}f(t)$ evaluated at the point s equals the Laplace transform of $f(t)$ evaluated at $s-a$.

Example Compute the Laplace transform of $e^{3t}\sin t$

Recall that the Laplace transform of $\sin t$ is $\frac{1}{s^2+1}$.

$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$

So to compute $\mathcal{L}\{e^{3t}\sin t\}$ we need to only replace s by $s-3$:

$$\mathcal{L}\{e^{3t}\sin t\} = \frac{1}{(s-3)^2+1^2} = \frac{1}{(s-3)^2+1}$$

Example What function $g(t)$ has Laplace transform

$$G(s) = \frac{s-7}{25+(s-7)^2}$$

Note that $\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$ and so $\mathcal{L}\{\cos 5t\} = \frac{s}{s^2+25}$

Thus $G(s)$ is obtained from $\mathcal{L}\{\cos 5t\} = \frac{s}{s^2+25}$ by replacing every s by $s-7$. Thus by Property 2, we have

$$\mathcal{L}^{-1}\left\{\frac{s-7}{25+(s-7)^2}\right\} = e^{7t}\cos(5t).$$

Example What function has Laplace transform $\frac{1}{(s^2-4s+9)}$?

$$\frac{1}{(s^2-4s+9)} = \frac{1}{(s-2)^2-4+9} = \frac{1}{(s-2)^2+5} = \frac{1}{(s-2)^2+(\sqrt{5})^2} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{(s-2)^2+(\sqrt{5})^2}$$

$\uparrow$

completing
the square

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{5}}\frac{(\sqrt{5})^2}{(s-2)^2+(\sqrt{5})^2}\right\} = \frac{1}{\sqrt{5}}\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{(s-2)^2+(\sqrt{5})^2}\right\} = \frac{1}{\sqrt{5}}\sin(\sqrt{5}t)e^{2t}$$

Example What function has Laplace transform $\frac{s}{(s^2-4s+9)}$?

trying to relate it to:

$$\mathcal{L}\{\cos wt\} = \frac{s}{w^2+s^2}$$

$$\mathcal{L}\{\sin wt\} = \frac{w}{w^2+s^2}$$

$$\frac{s}{(s^2-4s+9)} = \frac{s}{(s-2)^2-4+9} = \frac{s}{(s-2)^2+5} = \frac{s-2+2}{(s-2)^2+5}$$

$$= \frac{s-2}{(s-2)^2+5} + \frac{2}{(s-2)^2+5} \rightarrow \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{(s-2)^2+5}$$

$$= \frac{s-2}{(s-2)^2+5} + \frac{2}{\sqrt{5}} \frac{\sqrt{5}}{(s-2)^2+5}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{s^2-4s+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+5}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{\sqrt{5}}\frac{\sqrt{5}}{(s-2)^2+5}\right\}$$

$$= \underline{e^{2t}\cos(\sqrt{5}t) + \frac{2}{\sqrt{5}}\sin(\sqrt{5}t)e^{2t}}$$

Lastly, we consider

$$\cosh(at) = \frac{e^{at}+e^{-at}}{2}, \quad \sinh(at) = \frac{e^{at}-e^{-at}}{2}$$

Therefore, by the linearity of the Laplace transform:

$$\mathcal{L}\{\cosh(at)\} = \mathcal{L}\{\tfrac{1}{2}(e^{at}+e^{-at})\} = \tfrac{1}{2}\mathcal{L}\{e^{at}\} + \tfrac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right] = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

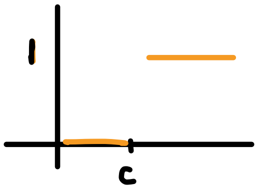
$$\mathcal{L}\{\sinh(at)\} = \mathcal{L}\{\tfrac{1}{2}(e^{at}-e^{-at})\} = \tfrac{1}{2}\mathcal{L}\{e^{at}\} - \tfrac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$$

Section 2.11 Differential equations with discontinuous right-hand sides

Consider again $ay''+by'+cy=f(t)$ where $f(t)$ now has a jump discontinuity at one or more points.

The simplest example is $H_c(t)=\begin{cases}0, & 0\le t<c\\ 1, & t\ge c\end{cases}$ This is called the Heaviside fcn



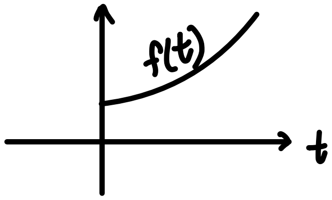
Its Laplace transform is $\mathcal{L}\{H_c(t)\} = \lim\limits_{A\to\infty}\int_0^A e^{-st}H_c(t)dt$

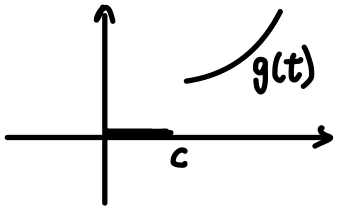
$$= \lim_{A\to\infty}\int_c^A e^{-st}dt = \lim_{A\to\infty}\left[\frac{1}{-s}e^{-st}\right]_c^A$$

$$= \lim_{A\to\infty}\frac{e^{-sc}-e^{-sA}}{s}$$

$$= \frac{e^{-cs}}{s} \text{ for } s>0$$

Next we let f be any function defined on the interval $0\le t<\infty$ and let g be the function obtained from m by shifting the graph of f, c units to the right, i.e.





So we have $g(t)=\begin{cases}0, & 0\le t<c\\ f(t-c), & t\ge c\end{cases}$

An alternative way of writing down this function is $\boxed{g(t)=H_c(t)f(t-c)}$

Property 3: Let $F(s)=\mathcal{L}\{f(t)\}$. Then $\boxed{\mathcal{L}\{H_c(t)f(t-c)\}=e^{-cs}F(s)}$

Proof Using the definition we have

$$\mathcal{L}\{H_c(t)f(t-c)\} = \int_0^\infty e^{-st}H_c(t)f(t-c)\,dt$$

$$= \int_c^\infty e^{-st}f(t-c)dt$$

Using integration by substitution we have $u = t - c \Rightarrow du = dt$

when $t = c \Rightarrow u = 0$

$t = \infty \Rightarrow u = \infty$

Thus $\mathcal{L}\{H_c(t)f(t-c)\} = \int_0^\infty e^{-s(u+c)} f(u)\,du$

$$= e^{-sc}\underbrace{\int_0^\infty e^{-su} f(u)\,du}_{\mathcal{L}\{f(u)\}}$$

$$= e^{-sc}\mathcal{L}\{f(t)\}$$

$\square$

<u>Example.</u> What function has Laplace transform $\frac{e^{-s}}{s^2}$?

Note that $\mathcal{L}\{t\} = \lim_{A\to\infty} \int_0^A e^{-st} t\,dt = \lim_{A\to\infty} \left[-\frac{t}{s}e^{-st}\right]_0^A + \frac{1}{s}\int_0^A e^{-st}dt$

$u = t$ $\quad \frac{dv}{dt} = e^{-st}$

$\frac{du}{dt} = 1$ $\quad v = -\frac{1}{s}e^{-st}$

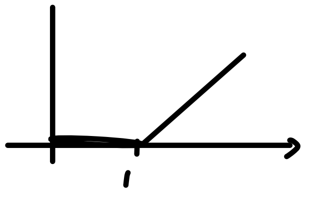
$$= \lim_{A\to\infty} -\cancelto{0}{\frac{A}{s}e^{-sA}} + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_0^A$$

$$= \lim_{A\to\infty} -\cancelto{0}{\frac{1}{s^2}e^{-sA}} + \frac{1}{s^2}$$

$$= \frac{1}{s^2}$$

Thus by Property 3, i.e. $\mathcal{L}\{H_c(t)f(t-c)\} = e^{-sc}\mathcal{L}\{f(t)\}$, we have that

$\frac{e^{-s}}{s^2}$ is the Laplace transform of $\boxed{H_1(t)f(t-1)}$.



Example What function has Laplace transform $\dfrac{e^{-3s}}{s^2-2s-3}$?

Note first that $\dfrac{1}{s^2-2s-3} = \dfrac{1}{(s-1)^2-1-3} = \dfrac{1}{(s-1)^2-4} = \dfrac{1}{(s-1)^2-2^2}$

We know that $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^2-2^2}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{2}\,\dfrac{2}{(s-1)^2-2^2}\right\} = \dfrac{e^t}{2}\sinh(2t)$

Recall $\mathcal{L}\{\sinh(at)\} = \dfrac{a}{s^2-a^2}$ and $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

Thus from property 3 we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2-2s-3}\right\} = H_3(t)\,f(t-3) = H_3(t)\,\frac{e^{t-3}}{2}\sinh(2(t-3)).$$

Example Solve the IVP $y''-3y'+2y = f(t) = \begin{cases} 1, & 0\le t<1\ ; \quad 0,\ 1\le t<2 \\ 1, & 2\le t<3\ ; \quad 0,\ 3\le t<4 \\ 1, & 4\le t<5\ ; \quad 0,\ 5\le t<\infty \end{cases}$

and $y(0)=0$, $y'(0)=0$.

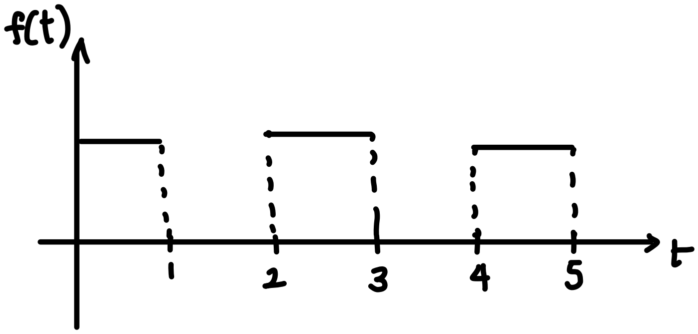
→ Let $Y(s)=\mathcal{L}\{y(t)\}$ and $F(s)=\mathcal{L}\{f(t)\}$. Taking the Laplace transforms of both sides of the diff. eqn. gives

$$s^2Y(s) - s\underset{0}{\cancel{y(0)}} - \underset{0}{\cancel{y'(0)}} - 3sY(s) + 3\underset{0}{\cancel{y(0)}} + 2Y(s) = F(s)$$

$$Y(s)\left[s^2-3s+2\right] = F(s)$$

$$Y(s) = \frac{F(s)}{s^2-3s+2} = \frac{F(s)}{(s-2)(s-1)}$$

How do we compute $F(s)$?



Method 1

$$f(t) = [H_0(t) - H_1(t)] + [H_2(t) - H_3(t)] + [H_4(t) - H_5(t)]$$

where $H_c(t) = \begin{cases} 0, & 0 \leq t < c \\ 1, & t \geq c \end{cases}$

and $\mathcal{L}\{H_c(t)\} = \frac{e^{-cs}}{s}$ for $s > 0$

By the linearity property of Laplace transforms we have

$$F(s) = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s} - \frac{e^{-5s}}{s}$$

Method 2

A second way of computing $F(s)$ is to evaluate

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}dt + \int_2^3 e^{-st}dt + \int_4^5 e^{-st}dt$$

$$= \left[-\frac{1}{s}e^{-st}\right]_0^1 + \left[-\frac{1}{s}e^{-st}\right]_2^3 + \left[-\frac{1}{s}e^{-st}\right]_4^5$$

$$= -\frac{1}{s}e^{-s} + \frac{1}{s} - \frac{1}{s}e^{-3s} + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-5s} + \frac{1}{s}e^{-4s}$$

$$= \frac{1}{s}\left[1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} - e^{-5s}\right]$$

Thus $$Y(s) = \frac{1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} - e^{-5s}}{s(s-1)(s-2)}$$

Use partial fractions

$$\frac{1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \Rightarrow 1 = A(s-1)(s-2) + Bs(s-2) + C\,s(s-1)$$

Let $s = 0 \Rightarrow 1 = A(-1)(-2) \Rightarrow \boxed{A = \frac{1}{2}}$

$s = 1 \Rightarrow 1 = B(-1) \Rightarrow \boxed{B = -1}$

$s = 2 \Rightarrow 1 = 2C \Rightarrow \boxed{C = \frac{1}{2}}$

Thus $\dfrac{1}{s(s-1)(s-2)} = \dfrac{1}{2}\dfrac{1}{s} - \dfrac{1}{s-1} + \dfrac{1}{2}\dfrac{1}{s-2}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2}\frac{1}{s-2}\right\} = \frac{1}{2} - e^{t} + \frac{1}{2}e^{2t}$$

So now that we have to compute

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} - e^{-5s}}{s(s-1)(s-2)}\right\}$$

By property 3,

$$\begin{aligned} y(t) = {} & \frac{1}{2} - e^{t} + \frac{1}{2}e^{2t} - H_1(t)\left[\frac{1}{2} - e^{(t-1)} + \frac{1}{2}e^{2(t-1)}\right] \\ & + H_2(t)\left[\frac{1}{2} - e^{(t-2)} + \frac{1}{2}e^{2(t-2)}\right] - H_3(t)\left[\frac{1}{2} - e^{(t-3)} + \frac{1}{2}e^{2(t-3)}\right] \\ & + H_4(t)\left[\frac{1}{2} - e^{(t-4)} + \frac{1}{2}e^{2(t-4)}\right] - H_5(t)\left[\frac{1}{2} - e^{(t-5)} + \frac{1}{2}e^{2(t-5)}\right] \end{aligned}$$

Section 5.2: Intro to Partial Differential Equations

A partial differential equation is a relation involving one or more functions of several variables, and their partial derivatives.

The order of a PDE is the order of the highest partial derivative that appears in the equation.

Example

$$\frac{\partial^2 u}{\partial t^2} = 2\frac{\partial^2 u}{\partial x \partial t} + u$$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Both are second order PDEs.

Some classic PDEs of order 2

HEAT EQUATION $\quad \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

WAVE EQUATION $\quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

LAPLACE'S EQUATION $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Section 5.3 Heat equation, separation of variables

Consider the boundary-value problem

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad \underbrace{u(x,0)=f(x), \quad 0<x<l}_{\text{initial condition}}; \quad \underbrace{u(0,t)=u(l,t)=0}_{\text{boundary conditions}}$$

We want to find $u(x,t)$.

Recall that when we were considering the IVP $\begin{bmatrix} y''+p(t)y'+q(t)y=0 \\ y(0)=y_0,\ y'(0)=y_0' \end{bmatrix}$

$y(t)$ here is a fcn of a single variable ⇒ ODE

We first showed that $y''+p(t)y'+q(t)y=0$ is linear and so any linear combination of solutions of this would again be a solution. So our solution was $c_1y_1(t)+c_2y_2(t)$ for two linearly indep. solutions $y_1(t)$ & $y_2(t)$.

⇒ Any linear combination of $c_1u_1(x,t)+\dots+c_nu_n(x,t)$ of solutions $u_1(x,t),\dots u_n(x,t)$ of $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2u}{\partial x^2}$ is again a solution, and we also want the boundary conditions to be satisfied.

STRATEGY

Step 1 Find as many solutions $u_1(x,t), u_2(x,t), \dots$ as we can of the BVP

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2u}{\partial x^2}\ ;\quad u(0,t)=u(\ell,t)=0$$

Step 2 Find the solution $u(x,t)$ by taking an appropriate linear combination of the functions $u_n(x,t)$, $n=1,2,\dots$

Regarding $\boxed{\text{step 1}}$. We reduce the problem to solving one or more ODEs.

Set $u(x,t)=X(x)T(t)$ ← this is why the method is called "SEPARATION OF VARIABLES"

Computing $\frac{\partial u}{\partial t}=XT'$ and $\frac{\partial^2u}{\partial x^2}=X''T$ we see that $u(x,t)=X(x)T(t)$ is a solution of $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ if $XT' = \alpha^2 X'' T$

Dividing both sides by $\alpha^2 XT$ we obtain

$$\frac{\cancel{X}T'}{\alpha^2\cancel{X}T} = \frac{\cancel{\alpha^2}X''\cancel{T}}{\cancel{\alpha^2}X\cancel{T}}$$

$$\Rightarrow \underbrace{\frac{T'}{\alpha^2 T}}_{\substack{\text{function of} \\ t \text{ alone}}} = \underbrace{\frac{X''}{X}}_{\substack{\text{function of} \\ x \text{ alone}}}$$

Therefore, this implies that $\frac{X''}{X} = -\lambda$ and $\frac{T'}{\alpha^2 T} = -\lambda$, for some constant λ.

(this is because the only way that a function of x can equal a function of t is if both are constant.)

The boundary conditions $0 = u(0,t) = X(0)T(t)$

$0 = u(\ell,t) = X(\ell)T(t)$

imply that $\boxed{X(0)=0}$ and $\boxed{X(\ell)=0}$ (otherwise, u must be identically zero).

So we have $X'' + \lambda X = 0$ and $X(0)=0$, $X(\ell)=0$

$T' + \alpha^2\lambda T = 0$

Note that $X'' + \lambda X = 0$ is a 2nd order ODE

$$m^2 + \lambda = 0$$
$$m = \pm i\sqrt{\lambda}$$

and $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$, which upon using $X(0) = 0 = X(\ell)$

will determine A, B λ

$X(0)=0 \Rightarrow \boxed{0 = A}$

$X(l)=0 \Rightarrow 0 = B\sin(\sqrt{\lambda}\, l) \Rightarrow$ $B \neq 0$ and $\sqrt{\lambda}\, l = n\pi$

$$\lambda = \left(\frac{n\pi}{l}\right)^2$$

Thus $X(x) = X_n(x) = \sin\left(\frac{n\pi}{l}x\right)$

Similarly, we have $T' + \alpha^2\lambda T = 0$ but we already have $\lambda = \left(\frac{n\pi}{l}\right)^2$

$$\frac{T'}{T} = -\alpha^2\lambda \Rightarrow \frac{T'}{T} = -\frac{\alpha^2 n^2\pi^2}{l^2}$$

$$\ln|T| = -\frac{\alpha^2 n^2\pi^2}{l^2}t$$

$$T(t) = T_n(t) = e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

We would multiply both $X_n(t)$ and $T_n(t)$ by constants but we omit these constants here since we will soon be taking linear combinations of the functions $X_n(x)T_n(t)$

$\Rightarrow$ $u_n(x,t) = \sin\left(\frac{n\pi x}{l}\right)e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$ is a nontrivial solution of the BVP for every positive integer n.

Suppose that $f(x)$ is a finite linear combination of $\sin\left(\frac{n\pi x}{l}\right)$, that is

$$f(x) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{l}\right)$$

Then $u(x,t) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$ is the desired solution as it also satisfies the initial condition $u(x,0) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$, $0<x<l$.

Section 5.4 : Fourier series

An arbitrary function $f(x)$ could be expanded in an infinite series of sines and cosines. Let $f(x)$ be defined on $-l \leq x \leq l$ and compute

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \,, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \,, \quad n=1,2,\dots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \,, \quad n=1,2$$

Then we have

$$f(x) \approx \frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l} + \dots = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{l} + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

Example Let f be $f(x) = \begin{cases} 0, & -1 \leq x < 0 \\ 1, & 0 \leq x \leq 1 \end{cases}$. Compute the Fourier series for f on the interval $-1 \leq x \leq 1$.

In this problem $\boxed{l=1}$ and so $a_0 = \int_{-1}^{1} f(x)\,dx = \int_0^1 1\,dx = 1$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_0^1 \cos(n\pi x)\,dx = \left[\sin(n\pi x)\frac{1}{n\pi}\right]_0^1$$

$$= \underset{0}{\frac{\cancel{\sin(n\pi)}}{n\pi}} - 0 = 0 \text{ for } n \geq 1$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_0^1 \sin(n\pi x)\,dx = \left[\frac{-1}{n\pi}\cos(n\pi x)\right]_0^1$$

$$= -\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi} = \frac{(-1)^{n+1}}{n\pi} + \frac{1}{n\pi} = \frac{1}{n\pi}\left[1-(-1)^n\right]$$

for $n \geq 1$

Note that when $n = \text{even}$, $b_n = 0$

$n = \text{odd}$, $b_n = \frac{2}{n\pi}$

Thus, the Fourier series for f on the interval $-1 \le x \le 1$ is

$$f(x) \approx \overset{1}{\cancel{\frac{a_0}{2}}} + \sum_{n=1}^{\infty}\left[\overset{0}{\cancel{a_n}}\cos\left(\frac{n\pi x}{\underset{1}{\cancel{\ell}}}\right) + \underbrace{b_n}_{\substack{0 \text{ when} \\ n \text{ is even,} \\ \frac{2}{n\pi} \text{ when } n \text{ is odd}}}\sin\left(\frac{n\pi x}{\underset{1}{\cancel{\ell}}}\right)\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \frac{2}{5\pi}\sin(5\pi x) + \cdots$$

<u>Example</u>. Let f be defined as $f = \begin{cases} 1 & \text{for } -2 \le x < 0 \\ x & \text{for } 0 \le x \le 2 \end{cases}$

Compute the Fourier series for f on the interval $-2 \le x \le 2$.

In this problem $\boxed{\ell = 2}$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0} 1\,dx + \frac{1}{2}\int_{0}^{2} x\,dx = \frac{1}{2}\left[x\right]_{-2}^{0} + \frac{1}{2}\left[\frac{x^2}{2}\right]_{0}^{2}$$

$$= \frac{1}{2}(0+2) + \frac{1}{\cancel{2}}\left(\frac{\overset{2}{\cancel{4}}}{2} - 0\right) = 1 + 1 = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^{0}\cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[\frac{2}{n\pi}\underset{0}{\cancel{\sin\left(\frac{n\pi x}{2}\right)}}\right]_{-2}^{0} + \frac{1}{2}\left[x\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_{0}^{2}$$

$$-\frac{1}{\cancel{2}}\int_{0}^{2}\frac{\cancel{2}}{n\pi}\sin\left(\frac{n\pi x}{2}\right)dx$$

<u>by parts</u>

$u = x$

$\frac{du}{dx} = 1$

$\frac{du}{dx} = \cos\left(\frac{n\pi x}{2}\right)$

$v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$

$$= \frac{1}{2}\left(\frac{4}{n\pi}\cancelto{0}{\sin(n\pi)} - 0\right) - \frac{1}{n\pi}\left[-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{2}{(n\pi)^2}\left(\cos(n\pi)-1\right) = \frac{2}{(n\pi)^2}\left((-1)^n-1\right) \quad \text{for} \quad n \geq 1$$

$$= \begin{cases} -\frac{4}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^{0}\sin\frac{n\pi x}{2}dx + \frac{1}{2}\int_0^2 x\sin\left(\frac{n\pi x}{2}\right)dx$$

by parts

$u = x$

$\frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin\left(\frac{n\pi x}{2}\right)$

$v = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$

$$= \frac{1}{\cancel{2}}\left[-\frac{\cancel{2}}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \frac{1}{\cancel{2}}\left[-\frac{\cancel{2}}{n\pi}x\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$+ \frac{1}{\cancel{2}}\int_0^2 \frac{\cancel{2}}{n\pi}\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{1}{n\pi} + \frac{1}{n\pi}\underbrace{\cos(-n\pi)}_{\substack{= \cos(n\pi) \text{ since } \cos \text{ is even} \\ = (-1)^n}} - \frac{1}{n\pi}(2)\cos(n\pi) + \frac{1}{n\pi}(0)\cancelto{0}{\cos(0)}$$

$$+ \left[\frac{2}{(n\pi)^2}\cancelto{0}{\sin\left(\frac{n\pi x}{2}\right)}\right]_0^2$$

$$= \frac{1}{n\pi}\left(-1+(-1)^n\right) - \frac{2}{n\pi}(-1)^n$$

$$= -\frac{1}{n\pi}\left(1+(-1)^n\right) \quad \text{for} \quad n \geq 1$$

$$= \begin{cases} 0 & n \text{ odd} \\ -\frac{2}{n\pi} & n \text{ even} \end{cases}$$

Hence the Fourier series for f on $-2 \leq x \leq 2$ is

$$f(x) \simeq 1 - \frac{4}{\pi^2}\cos\left(\frac{\pi x}{2}\right) - \frac{1}{\pi}\sin(\pi x) - \frac{4}{9\pi^2}\cos\left(\frac{3\pi x}{2}\right) - \frac{1}{2\pi}\sin(2\pi x) + \ldots$$

$$= 1 - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x/2}{(2n+1)^2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}.$$

Note Orthogonality of the Sine and Cosine functions

The standard INNER PRODUCT (u,v) of two-real-valued functions u and v on the interval $\alpha \le x \le \beta$ is defined by

$$(u,v) = \int_{\alpha}^{\beta} u(x)v(x)\,dx.$$

The functions u and v are orthogonal on $\alpha \le x \le \beta$ if their inner product is zero, that is:

$$\int_{\alpha}^{\beta} u(x)v(x)\,dx = 0$$

A set of functions is mutually orthogonal if each distinct pair of functions in the set is orthogonal.

The functions $\sin\left(\frac{n\pi x}{l}\right)$ and $\cos\left(\frac{n\pi x}{l}\right)$, $n=1,2,\ldots$ form a mutually orthogonal set of functions on the interval $-l \le x \le l$. They satisfy the following orthogonality relations:

$$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\cos\left(\frac{(n+m)\pi x}{l}\right) + \cos\left(\frac{(n-m)\pi x}{l}\right)dx$$

$$= \frac{1}{2}\left[\frac{l}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{l}\right) + \frac{l}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{1}{2}\left[\frac{l}{(n+m)\pi}\sin\left((n+m)\pi\right) + \frac{l}{(n-m)\pi}\sin\left((n-m)\pi\right) - \frac{l}{(n+m)\pi}\overbrace{\sin\left(-(n+m)\pi\right)}^{=-\sin((n+m)\pi)\text{ since sine is odd}} - \frac{l}{(n-m)\pi}\sin\left(-(n-m)\pi\right)\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}l}{(n+m)\pi}\sin\left((n+m)\pi\right) + \frac{\cancel{2}l}{(n-m)\pi}\sin\left((n-m)\pi\right)\right]$$

$$= 0$$

as long as m+n and n-m are not zero (otherwise we are dividing by 0)

Since m and n are positive, $n+m \neq 0$. On the other hand if $n-m=0 \Rightarrow n=m$ and the integral must be evaluated in a different way.

using

$$\cos\left(\frac{n\pi x}{l}+\frac{m\pi x}{l}\right)=\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)-\cancel{\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)}$$

$$+\quad \cos\left(\frac{n\pi x}{l}-\frac{m\pi x}{l}\right)=\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)+\cancel{\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)}$$

$$\cos\left(\frac{(n+m)\pi x}{l}\right)+\cos\left(\frac{(n-m)\pi x}{l}\right)=2\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)$$

$$\Rightarrow \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)=\frac{1}{2}\left[\cos\left(\frac{(n+m)\pi x}{l}\right)+\cos\left(\frac{(n-m)\pi x}{l}\right)\right]$$

If $\underline{n=m}$ then

$$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx=\int_{-l}^{l}\left(\cos\left(\frac{n\pi x}{l}\right)\right)^2dx$$

$$=\int_{-l}^{l}\left[\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2n\pi x}{l}\right)\right]dx$$

$$=\left[\frac{1}{2}x+\frac{1}{2}\frac{l}{2n\pi}\sin\left(\frac{2n\pi x}{l}\right)\right]_{-l}^{l}$$

$$=\frac{1}{2}l+\frac{l}{4n\pi}\cancelto{0}{\sin(2n\pi)}+\frac{l}{2}-\frac{l}{4n\pi}\cancelto{0}{\sin(-2n\pi)}$$

$$=l$$

So $\int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{m\pi x}{\ell}\right) dx = \begin{cases} 0, & n \neq m \\ \ell, & n = m \end{cases}$

and similarly, we have that

$$\int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right) dx = 0 \quad \text{for all } n, m$$

$$\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right) dx = \begin{cases} 0, & n \neq m \\ \ell, & n = m \end{cases}$$

Theorem (the Fourier convergence theorem)

Suppose that f and f' are piecewise continuous on the interval $-\ell \leq x < \ell$. Further, suppose that f is defined outside $-\ell \leq x < \ell$ so that it's periodic with period 2ℓ. Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

whose coefficients are given by $a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx$, $n = 0, 1, 2, \ldots$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad n = 1, 2, \ldots$$

The Fourier series converges to $f(x)$ at all points where f is continuous, and to $\frac{[f(x_+) + f(x_-)]}{2}$ at all points where f is discontinuous

Note: $\frac{[f(x_+) + f(x_-)]}{2}$ is the mean value of the right- and left-hand limits at the point x.

Section 5.1   Boundary value problems

Q What values of λ give nontrivial functions $y(x)$ that satisfy

$$y''+\lambda y=0\;;\qquad \begin{matrix} ay(0)+by'(0)=0 \\ cy(\ell)+dy'(\ell)=0 \end{matrix}\quad ?\quad \Big\}\text{ boundary-value problem}$$

$\downarrow$

because we need info about $y(x)$ and $y'(x)$ at two distinct points $x=0$ and $x=\ell$.

Example. What values of λ give nontrivial solutions for

$$y''+\lambda y=0\,,\quad y(0)=0,\ y(\ell)=0\,?$$

$\boxed{\lambda=0}$ $y''=0 \Rightarrow\ y=ax+b$ for some constants a and b.

$$y(0)=0\ \Rightarrow\ b=0$$
$$y(\ell)=0\ \Rightarrow\ a\ell=0 \Rightarrow\ a=0$$

This implies that $y(x)=0$ is the only solution of the BVP for $\lambda=0$.

$\boxed{\lambda<0}$ $y''+\lambda y=0\ \Rightarrow\ y(x)=c_1e^{\sqrt{-\lambda}\,x}+c_2e^{-\sqrt{-\lambda}\,x}$

characteristic eqn: $r^2+\lambda=0$

$r=\pm\sqrt{-\lambda}$

Now using the B.C.s we get

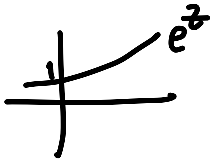
$$y(0)=0\ \Rightarrow\ 0=c_1+c_2$$
$$y(\ell)=0\ \Rightarrow\ 0=c_1e^{\sqrt{-\lambda}\,\ell}+c_2e^{-\sqrt{-\lambda}\,\ell}$$

These two equations have a nonzero solution c_1, c_2 iff

from $\begin{pmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\,l} & e^{-\sqrt{-\lambda}\,l} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we have

$$\det \begin{pmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\,l} & e^{-\sqrt{-\lambda}\,l} \end{pmatrix} = e^{-\sqrt{-\lambda}\,l} - e^{\sqrt{-\lambda}\,l} = 0$$

Thus, $e^{-\sqrt{-\lambda}\,l} = e^{\sqrt{-\lambda}\,l} \Rightarrow e^{2\sqrt{-\lambda}\,l} = 1$ but



we know that $e^z > 1$ for $z > 0$

Thus $c_1 = c_2 = 0$ and the boundary-value problem has no nontrivial solutions $y(x)$ when λ is negative

$\boxed{\lambda > 0}$ From the characteristic equation $r^2 + \lambda = 0 \Rightarrow r = \pm i\sqrt{\lambda}$ we have that the solution of $y'' + \lambda y = 0$ is of the form

$$y(x) = c_1 \cos(\sqrt{\lambda}\,x) + c_2 \sin(\sqrt{\lambda}\,x).$$

B.Cs

$y(0) = 0 \Rightarrow c_1 = 0$

$y(l) = 0 \Rightarrow c_2 \sin(\sqrt{\lambda}\,l) = 0$ but $c_2 \neq 0 \Rightarrow \sqrt{\lambda}\,l = n\pi$

$$\sqrt{\lambda} = \frac{n\pi}{l}$$

$$\lambda = \left(\frac{n\pi}{l}\right)^2 \text{ for some } n \in \mathbb{Z}$$

Thus the BVP has nontrivial solutions

$$y(x) = c_2 \sin\left(\frac{n\pi x}{l}\right) \text{ for } n = 1, 2, \ldots$$

Theorem The BVP has nontrivial solutions $y(\lambda)$ only for a denumerable' set of values $\lambda_1, \lambda_2, \dots$ where $\lambda_1 \le \lambda_2 \le \dots$ and $\lambda_n \to \infty$ as $n \to \infty$. These special values of λ are called eigenvalues and the nontrivial solutions $y(x)$ are called eigenfunctions

Note. In the previous example the eigenvalues are $\lambda = \frac{\pi^2}{\ell^2}, \frac{4\pi^2}{\ell^2}, \frac{9\pi^2}{\ell^2}, \dots$ and the eigenfunctions are all constant multiples of $\sin\left(\frac{\pi x}{\ell}\right), \sin\left(\frac{2\pi x}{\ell}\right), \dots$

Q Why do we use this terminology?

A Let $\vec{V}$ be the set of all functions $y(x)$ which have two continuous derivatives and satisfy $ay(0) + by'(0) = 0$, $cy(\ell) + dy'(\ell) = 0$. $\vec{V}$ is a vector space of infinite dimension.

Consider now the linear operator or transformation L, given by

$$[Ly](x) = -\frac{d^2y}{dx^2}(x)$$

The two solutions $y(x)$ of the BVP are those functions y in $\vec{V}$ for which $Ly = \lambda y$. (since $Ly = -y''$ and the eqn is $y'' + \lambda y = 0$)

Example Find the eigenvalues and eigenfunctions of the BVP

$$y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(1) = 0$$

$\boxed{\lambda = 0}$

$$y'' = 0 \Rightarrow \quad y = c_1 x + c_2$$

$$y'(x) = c_1$$

$$\left.\begin{aligned} c_2 + c_1 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}\right\} \text{ from both B.Cs } \quad \boxed{c_1 = -c_2}$$

' able to be counted by a one-to-one correspondence with the infinite set of integers

$$y(x) = c_1x + c_2 = c_1(x-1) \text{ for } c_1 \neq 0$$

So the eigenfunction is $y(x) = c_1(x-1)$ and the eigenvalue is zero

$\boxed{\lambda<0}$ Every solution $y(x)$ of $y'' + \lambda y = 0$ is given by

$$y(x) = c_1\cosh(\sqrt{-\lambda}\,x) + c_2\sinh(\sqrt{-\lambda}\,x)$$

⌈ Why this and not $y(x) = c_1e^{\sqrt{-\lambda}x} + c_2e^{-\sqrt{-\lambda}x}$?

$$\cosh(\sqrt{-\lambda}\,x) = \frac{e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}}{2}, \quad \sinh(\sqrt{-\lambda}\,x) = \frac{e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}}{2}$$ ⌋

So if we use the B.Cs $y(0) + y'(0) = 0$, $y(1) = 0$ we have

$$c_1\cosh(\sqrt{-\lambda}) + c_2\sinh(\sqrt{-\lambda}) = 0$$

$$y(x) = c_1\sqrt{-\lambda}\sinh(\sqrt{-\lambda}\,x) + c_2\sqrt{-\lambda}\cosh(\sqrt{-\lambda}\,x)$$

$$c_1\underset{1}{\cancel{\cosh(0)}} + c_2\underset{0}{\cancel{\sinh(0)}} + c_1\sqrt{-\lambda}\,\underset{0}{\cancel{\sinh(0)}} + c_2\sqrt{-\lambda}\,\underset{1}{\cancel{\cosh(0)}} = 0$$

Thus

$$c_1\cosh(\sqrt{-\lambda}) + c_2\sinh(\sqrt{-\lambda}) = 0$$
$$c_1 + c_2\sqrt{-\lambda} = 0$$

This implies that the system of equations has a nontrivial solution c_1, c_2 iff

$$\det\begin{pmatrix} \cosh(\sqrt{-\lambda}) & \sinh(\sqrt{-\lambda}) \\ 1 & \sqrt{-\lambda} \end{pmatrix} = \cosh(\sqrt{-\lambda})\sqrt{-\lambda} - \sinh(\sqrt{-\lambda}) = 0$$

$$\Rightarrow \boxed{\sinh(\sqrt{-\lambda}) = \sqrt{-\lambda}\cosh(\sqrt{-\lambda})}$$

But this equation has no solution for $\lambda<0$. To see this we let $\boxed{z=\sqrt{-\lambda}}$ and then consider $h(z)=z\cosh z-\sinh z$.

Note $h(0)=0$ and $h(z)>0$ for $z>0$ since

$$h'(z)=\cancel{\cosh z}+z\sinh z-\cancel{\cosh z}=z\sinh z>0$$

for $z>0$. Thus no $\lambda<0$ can satisfy $\sinh(\sqrt{-\lambda})=\sqrt{-\lambda}\cosh(\sqrt{-\lambda})$

$\boxed{\lambda>0}$ Every solution $y(x)$ of $y''+\lambda y=0$ is of the form

$$y(x)=c_1\cos(\sqrt{\lambda}x)+c_2\sin(\sqrt{\lambda}x)$$

for some c_1, c_2 constants.

From the B.Cs $y(0)+y'(0)=0$ and $y(1)=0$ we have

$$\boxed{c_1\cos(\sqrt{\lambda})+c_2\sin(\sqrt{\lambda})=0}$$

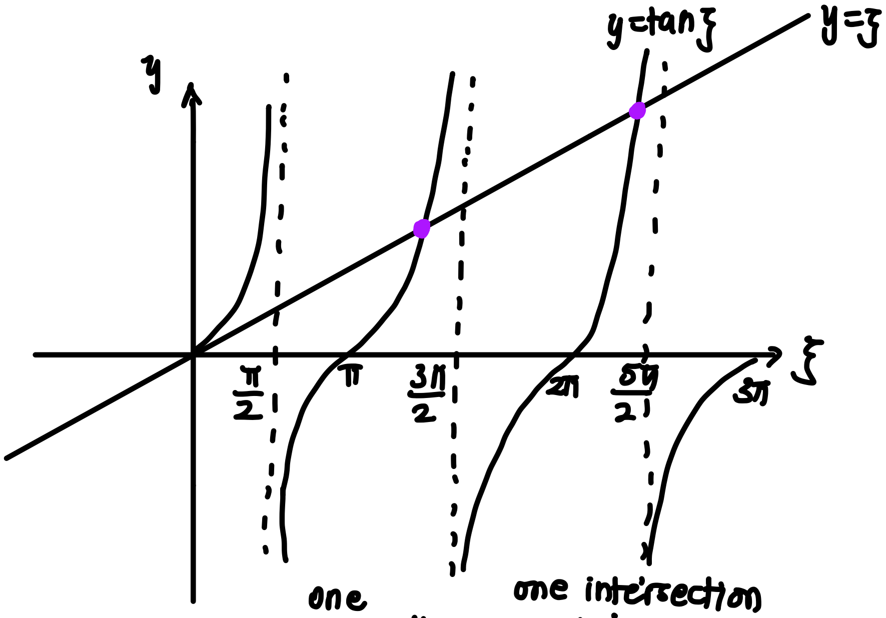
$$y'(x)=-c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x)+c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$\boxed{c_1+c_2\sqrt{\lambda}=0}$$

Thus $$\det\begin{pmatrix}\cos(\sqrt{\lambda}) & \sin(\sqrt{\lambda})\\ 1 & \sqrt{\lambda}\end{pmatrix}=\sqrt{\lambda}\cos(\sqrt{\lambda})-\sin(\sqrt{\lambda})=0$$

$$\Rightarrow\boxed{\tan(\sqrt{\lambda})=\sqrt{\lambda}}$$

So how do we solve this? We set $\xi=\sqrt{\lambda}$, and try to find the intersection points between the graph of $y=\tan(\xi)$ and $y=\xi$, for $\xi>0$



one intersection point in $\frac{\pi}{2}<\xi<\frac{3\pi}{2}$

at $\xi>\pi$

one intersection point in $\frac{3\pi}{2}<\xi<\frac{5\pi}{2}$

at $\xi>2\pi$

More generally, the curves $y=\xi$ and $y=\tan\xi$ intersect exactly once in the interval $\frac{(2n-1)\pi}{2}<\xi<\frac{(2n+1)\pi}{2}$ and this occurs at a point $\xi_n>n\pi$.

Note also that they don't intersect in $0<\xi<\frac{\pi}{2}$. To show this set $h(\xi)=\tan\xi-\xi$

$h'(\xi)=\sec^2\xi-1=\tan^2\xi>0$ for $0<\xi<\frac{\pi}{2} \Rightarrow h(\xi)>0$ for $\xi\in(0,\frac{\pi}{2})$

Thus the eigenvalues are $\lambda_1=\xi_1^2$, $\lambda_2=\xi_2^2$, … and the eigenfunctions are

$\downarrow$

from $\xi=\sqrt{\lambda}$

all constant multiples of the functions $-\sqrt{\lambda_1}\cos(\sqrt{\lambda_1}\,x)+\sin(\sqrt{\lambda_1}\,x)$,

$-\sqrt{\lambda_2}\cos(\sqrt{\lambda_2}\,x)+\sin(\sqrt{\lambda_2}\,x)$, …

We cannot compute λ_n exactly (analytically), but we know that

$$n^2\pi^2<\lambda_n<\frac{(2n+1)^2\pi^2}{4}$$ (look at blue highlight above)

Section 63 : Hermitian operators (orthogonal bases)

Def A set of vectors is orthogonal if the inner product of any two distinct vectors in the set is zero.

Lemma 1: Let $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N$ be mutually orthogonal, that is

$$\underset{\substack{\uparrow \\ \text{inner product notation}}}{\langle \vec{x}_i , \vec{x}_j \rangle} = 0 \qquad i \neq j$$

Then $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N$ are linearly independent.

Proof Suppose that $c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_N \vec{x}_n = \vec{0}$

Taking inner products of both sides with $\vec{x}_j$ gives

$$c_1 \langle \vec{x}_1, \vec{x}_j \rangle + c_2 \langle \vec{x}_2, \vec{x}_j \rangle + \dots + c_N \langle \vec{x}_N, \vec{x}_j \rangle = 0$$

$\Rightarrow \quad c_j \langle \vec{x}_j, \vec{x}_j \rangle = 0$ from the condition that $\langle \vec{x}_i, \vec{x}_j \rangle 0$ for $i \neq j$

$\Rightarrow \quad c_j = 0$ for $j = 1, 2, \dots, N$ since $\langle \vec{x}_j, \vec{x}_j \rangle > 0$.

□

Another advantage of working with orthogonal bases is that it's easy to find the coordinates of a vector wrt a given orthogonal basis.

Let $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ be a mutually orthogonal set of vectors in a real n-dimensional vector space V. By lemma 1, this set of vectors is also a basis for V and every vector $\vec{z} \in V$ can be expanded in the form

$$\vec{x} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_n \vec{u}_n$$

Taking inner products of both sides of the eqn with $\vec{u}_j$ gives $<\vec{x}, \vec{u}_j> = c_j <\vec{u}_j, \vec{u}_j>$

so that $c_j = \dfrac{<\vec{x}, \vec{u}_j>}{<\vec{u}_j, \vec{u}_j>}$, $j = 1, 2, \dots, n.$

Example. Let $V = \mathbb{R}^2$, and define $<\vec{x}, \vec{y}> = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2$

The vector $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are orthogonal and thus form a basis for $\mathbb{R}^2$.

So from $\left[\begin{array}{l} \vec{x} = c_1 \vec{u}_1 + c_2 \vec{u}_2 \\ c_j = \dfrac{<\vec{x}, \vec{u}_j>}{<\vec{u}_j, \vec{u}_j>} \quad j = 1, 2 \end{array} \right]$, any vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ can be written as

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{<\vec{x}, \vec{u}_1>}{<\vec{u}_1, \vec{u}_1>} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{<\vec{x}, \vec{u}_2>}{<\vec{u}_2, \vec{u}_2>} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{x_1 + x_2}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{x_1 - x_2}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Theorem (Gram-Schmidt) Every n-dimensional Euclidean space V has an orthogonal basis

Proof Choose a basis $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ for V. We will inductively construct an orthogonal basis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ by taking suitable combinations of the vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$. Let $\boxed{\vec{v}_1 = \vec{u}_1}$ and set $\boxed{\vec{v}_2 = \vec{u}_2 + \lambda \vec{v}_1}$

Taking the inner product of $\vec{v}_2$ with $\vec{v}_1$ gives

$$<\vec{v}_2, \vec{v}_1> = <\vec{u}_2 + \lambda \vec{v}_1, \vec{v}_1>$$

$$= <\vec{u_2}, \vec{v_1}> + \lambda <\vec{v_1}, \vec{v_1}>$$

So that $\vec{v_2}$ will be orthogonal to $\vec{v_1}$ if $\lambda = -\frac{<\vec{u_2}, \vec{v_1}>}{<\vec{v_1}, \vec{v_1}>}$.

Note that $\vec{v_2} \neq \vec{0}$ since $\vec{v_2} = \vec{u_2} + \lambda\vec{v_1} = \vec{u_2} + \lambda\vec{u_1}$ and $\vec{u_1}$ & $\vec{u_2}$ are linearly independent.

Proceeding inductively, let's assume that $\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}$ are mutually orthogonal and set

$$\vec{v_{k+1}} = \vec{u_{k+1}} + \lambda_1\vec{v_1} + \dots + \lambda_k\vec{v_k}.$$

The requirement that $\vec{v_{k+1}}$ is orthogonal to $\vec{v_1}, \dots, \vec{v_k}$ gives

$$\boxed{\lambda_j = -\frac{<\vec{u_{k+1}}, \vec{v_j}>}{<\vec{v_j}, \vec{v_j}>} \quad \text{for } j = 1, \dots, k.}$$

For this case of $\lambda_1, \dots, \lambda_k$ the vectors $\vec{v_1}, \dots, \vec{v_k}, \vec{v_{k+1}}$ are mutually orthogonal. Also $\vec{v_{k+1}} \neq \vec{0}$ because of the linear independence of $\vec{u_1}, \dots, \vec{u_{k+1}}$.

Proceeding inductively until $k=n$, we obtain n mutually orthogonal nonzero vectors $\vec{v_1}, \dots, \vec{v_n}$.

The above outline is known as the GRAM-SCHMIDT ORTHOGONALIZATION PROCEDURE

Example Let V be the space of all polynomials of degree $n-1$ and define

$$<f, g> = \int_{-1}^{1} f(x) g(x)\, dx$$

$\forall$ fcns f and $g \in V$. It's easy to verify that
$f_0(x) = 1$
$f_1(x) = x$
$\vdots$
$f_{n-1}(x) = x^{n-1}$

form a basis for V. Applying the Gram-Schmidt orthogonalization procedure to $f_0(x), f_1(x), \dots, f_{n-1}(x)$ gives

$$P_0(x) = 1$$

$$P_1(x) = f_1(x) + \lambda_0 P_0(x) = x + \lambda = x + \left(-\frac{\langle f_1, P_0\rangle}{\langle P_0, P_0\rangle}\right)$$

$$= x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} 1\,dx} = x$$

$$P_2(x) = \overset{=x^2}{f_2(x)} + \lambda_0 \overset{=1}{P_0(x)} + \lambda_1 \overset{=x}{P_1(x)}$$

$$= x^2 + \left(-\frac{\langle f_2, P_0\rangle}{\langle P_0, P_0\rangle}\right) 1 + \left(-\frac{\langle f_2, P_1\rangle}{\langle P_1, P_1\rangle}\right) x$$

$$-\frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx} = \frac{-\left[\frac{x^3}{3}\right]_{-1}^{1}}{[x]_{-1}^{1}} = \frac{-\left(\frac{1}{3} - \frac{(-1)^3}{3}\right)}{2} = -\frac{1}{3}$$

$$\frac{-\int_{-1}^{1} x^2 x\,dx}{\int_{-1}^{1} x^2\,dx} = \frac{-\cancelto{0}{\left[\frac{x^4}{4}\right]_{-1}^{1}}}{\left[\frac{x^3}{3}\right]_{-1}^{1}} = 0$$

$$\Rightarrow P_2(x) = x^2 - \frac{1}{3}$$

$$P_3(x) = f_3(x) + \lambda_0 P_0(x) + \lambda_1 P_1(x) + \lambda_2 P_2(x)$$

$$\lambda_0 = = -\frac{\langle f_3, P_0\rangle}{\langle P_0, P_0\rangle} = -\frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} 1\,dx} = -\frac{\left[\frac{x^4}{4}\right]_{-1}^{1}}{2} = 0$$

$$\lambda_1 = -\frac{\langle f_3, P_1\rangle}{\langle P_1, P_1\rangle} = -\frac{\int_{-1}^{1} x^3 \cdot x\,dx}{\int_{-1}^{1} x^2\,dx} = \frac{-\left[\frac{x^5}{5}\right]_{-1}^{1}}{\left[\frac{x^3}{3}\right]_{-1}^{1}} = \frac{-\frac{2}{5}}{\frac{2}{3}} = -\frac{3}{5}$$

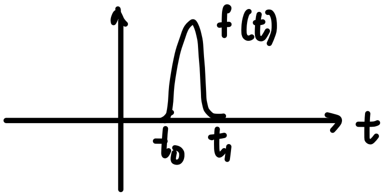
$$\lambda_2 = -\frac{\langle f_3 , p_2 \rangle}{\langle p_2 , p_2 \rangle} = -\frac{\int_{-1}^{1} x^3\left(x^2-\frac{1}{3}\right)dx}{\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2 dx} = \frac{-\left[\frac{x^6}{6}-\frac{x^4}{12}\right]_{-1}^{1}}{\int_{-1}^{1}\left(x^4-\frac{2}{3}x^2+\frac{1}{9}\right)dx} = 0$$

Thus $p_3(x) = x^3 - \frac{3}{5}x$

Section 2.12 The Dirac delta function

Consider the IVP $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$, $y(0) = y_0$, $y'(0) = y_0'$

where $f(t)$ is not known explicitly and $f(t)$ is identically zero except a very short time interval $t_0 \le t \le t_1$



impulsive function $f(t)$

and its integral over this time interval is $I_0 \neq 0$.

Method proposed by Dirac:

Let $t_1 \to t_0$ Then the function $\frac{f(t)}{I_0} \to$ a function equal to $\begin{cases} 0 & \text{for } t \neq t_0 \\ \infty & \text{for } t = t_0 \end{cases}$

and whose integral is equal to 1 over any interval containing t_0.

We denote this function by $\delta(t-t_0)$ and call it the Dirac delta function.

If we set $f(t)$ in $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$ as $I_0\,\delta(t-t_0)$ and impose the condition

$$\boxed{\int_a^b g(t)\delta(t-t_0) = \begin{cases} g(t_0) & \text{if } a \le t_0 \le b \\ 0 & \text{otherwise} \end{cases}}$$

for any continuous function $g(t)$, we'll always obtain the correct solution $y(t)$.

Note Suppose that $f(t)$ is an impulsive function that is positive for $t_0 < t < t_1$ and zero otherwise, and whose integral over $t_0 \le t \le t_1$ is 1. For any continuous function $g(t)$

$$\left[\min_{t_0 \le t \le t_1} g(t)\right] f(t) \le g(t)f(t) \le \left[\max_{t_0 \le t \le t_1} g(t)\right] f(t)$$

$$\Rightarrow \int_{t_0}^{t_1} \underbrace{\left[\min_{t_0 \le t \le t_1} g(t)\right]} f(t)\,dt \le \int_{t_0}^{t_1} g(t)f(t)\,dt \le \int_{t_0}^{t_1} \underbrace{\left[\max_{t_0 \le t \le t_1} g(t)\right]} f(t)\,dt$$

can pull out of the integral

and we know that the integral of $f(t)$ over $t_0 \le t \le t_1$ is 1

$$\Rightarrow \min_{t_0 \le t \le t_1} g(t) \le \int_{t_0}^{t_1} g(t)f(t)\,dt \le \max_{t_0 \le t \le t_1} g(t)$$

So as $t_1 \to t_0 \Rightarrow \int_{t_0}^{t_1} g(t)f(t)dt \to g(t_0)$.

SOLUTION OF $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$ BY THE METHOD OF LAPLACE TRANSFORMS

Apply the definition of the Laplace transform and the property

$$\int_a^b g(t)\delta(t-t_0)\,dt = \begin{cases} g(t_0) & \text{if } a \le t_0 \le b \\ 0, & \text{otherwise} \end{cases}$$

to obtain

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-st_0} \quad (\text{for } t_0 \geq 0)$$

<u>Example</u> Find the solution of the IVP: $y'' - 4y' + 4y = 3\delta(t-1) + \delta(t-2)$ with $y(0)=1$ and $y'(0)=1$

→ Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking Laplace transforms on both sides of the ODE gives

$$s^2Y(s) - s\underset{1}{y(0)} - \underset{1}{y'(0)} - 4(sY(s) - \underset{1}{y(0)}) + 4Y(s) = 3e^{-s(1)} + e^{-s(2)}$$

$$Y(s)\left[s^2 - 4s + 4\right] - s - 1 + 4 = 3e^{-s} + e^{-2s}$$

$$Y(s) = \frac{3e^{-s} + e^{-2s} + s - 3}{s^2 - 4s + 4} = \frac{3e^{-s} + e^{-2s} + s - 3}{(s-2)^2}$$

$$\Rightarrow \quad Y(s) = \frac{s-3}{(s-2)^2} + \frac{3e^{-s}}{(s-2)^2} + \frac{e^{-2s}}{(s-2)^2}$$

$$\frac{s-2-1}{(s-2)^2} = \frac{\cancel{s-2}}{(s-2)^{\cancel{2}}} - \frac{1}{(s-2)^2}$$

$$= \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

So if we want to invert $Y(s)$ we have

$$y(t) = e^{2t} - te^{2t} - 3H_1(t)(t-1)e^{2(t-1)} + H_2(t)(t-2)e^{2(t-2)}$$

$$= (1-t)e^{2t} - 3H_1(t)(t-1)e^{2(t-1)} + H_2(t)(t-2)e^{2(t-2)}.$$

Recall that

$\mathcal{L}\{-tf(t)\} = \frac{d}{ds}F(s)$ and so we show that $\mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$

Additionally $\mathcal{L}\{H_c(t)f(t-c)\} = e^{-cs}F(s)$

Example Solve the IVP $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t} + 3\delta(t-1)$, $y(0)=0$
$y'(0)=0$

Using Laplace transforms:

$$s^2Y(s) - s\underset{0}{\cancel{y(0)}} - \underset{0}{\cancel{y'(0)}} + 2(sY(s) - \underset{0}{\cancel{y(0)}}) + Y(s) = \mathcal{L}\{e^{-t}\} + 3\mathcal{L}\{\delta(t-1)\}$$

$$Y(s)[\underbrace{s^2+2s+1}_{(s+1)^2}] = \frac{1}{s+1} + 3e^{-s}$$

$$Y(s) = \frac{1}{(s+1)^3} + \frac{3e^{-s}}{(s+1)^2}$$

Inverting this we get

$$y(t) = \frac{t^2e^{-t}}{2} + 3H_1(t)(t-1)e^{-(t-1)}$$

For $\mathcal{L}^{-1}\{\frac{1}{(s+1)^3}\}$ we will use $\mathcal{L}\{-tf(t)\} = \frac{d}{ds}F(s)$.

let's integrate $\frac{1}{(s+1)^3}$ to get $-\frac{1}{2(s+1)^2}$ which means that $F(s) = -\frac{1}{2}\frac{1}{(s+1)^2}$

The function whose Laplace transform is $F(s)$ is $-\frac{1}{2}te^{-t}$

Thus $\mathcal{L}\{-t(-\frac{1}{2}te^{-t})\} = \frac{d}{ds}\left(-\frac{1}{2(s+1)^2}\right) = \frac{1}{(s+1)^3}$

$\Rightarrow$ $\mathcal{L}\{\frac{1}{2}t^2e^{-t}\} = \frac{1}{(s+1)^3}$

Example Find the solution of the IVP

$$2y'' + y' + 2y = \delta(t-5)$$

$$y(0) = 0 \,,\ y'(0) = 0$$

Apply Laplace transform

$$2s^2Y(s) - 2s\underset{0}{\cancel{y(0)}} - \underset{0}{\cancel{2y'(0)}} + sY(s) - \underset{0}{\cancel{y(0)}} + 2Y(s) = e^{-5s}$$

$$\left[2s^2 + s + 2\right]Y(s) = e^{-5s}$$

$$Y(s) = \frac{e^{-5s}}{2(s^2+\frac{1}{2}s)+2} \underset{\uparrow}{=} \frac{e^{-5s}}{2\left[(s+\frac{1}{4})^2 - \frac{1}{16}\right]+2}$$

Complete the square

$$= \frac{e^{-5s}}{2(s+\frac{1}{4})^2 - \frac{1}{8} + \frac{16}{8}} = \frac{e^{-5s}}{2(s+\frac{1}{4})^2 + \frac{15}{8}}$$

$$= \frac{1}{2} \ \frac{e^{-5s}}{(s+\frac{1}{4})^2 + \frac{15}{16}}$$

Thus $$\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{\sqrt{15}} \ \frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right\}$$

$$= \frac{4}{\sqrt{15}} \sin\left(\frac{\sqrt{15}}{4} t\right) e^{-\frac{1}{4}t}$$

Thus, by the theorem

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{\sqrt{15}} \ H_5(t)\, e^{-(t-5)/4} \sin\left(\frac{\sqrt{15}}{4}(t-5)\right).$$

The convolution integral

Theorem If $F(s)=\mathcal{L}\{f(t)\}$ and $G(s)=\mathcal{L}\{g(t)\}$ both exist for $s>0$ then

$$H(s)=F(s)G(s)=\mathcal{L}\{h(t)\}\quad , \; s>0,$$

where

$$h(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau=\int_0^t f(\tau)g(t-\tau)\,d\tau$$

↑ this follows from the change of variables

$t-\tau=\xi$

$\boxed{\tau=t-\xi}$

$d\tau=-d\xi$

if $\tau=0\Rightarrow\xi=t$

$\tau=t\Rightarrow\xi=0$

The function h is known as the convolution of f and g.

—//—

The convolution integral can be thought of as a "generalized product" by writing

$$h(t)=(f*g)(t)$$

↑ meaning the integral in the theorem above.

The convolution $f*g$ has many of the properties of ordinary multiplication

It can be shown that

$$f*g=g*f \quad \text{(commutative law)}$$

$$f*(g_1+g_2)=f*g_1+f*g_2 \quad \text{(distributive law)}$$

$$(f*g)*h=f*(g*h) \quad \text{(associative law)}$$

$$f*0=0*f=0.$$

← this is not the number 0 but the function that has the value 0 for each value of t

But there are also properties of ordinary multiplication that the convolution integral does not have. For example it is not in general true that $f*1$ is equal to f.

Note: $(f*1)(t) = \int_0^t f(t-\tau)\,1\,d\tau = \int_0^t f(t-\tau)\,d\tau$

If for example $f(t) = \cos t$:

$$(f*1)(t) = \int_0^t \cos(t-\tau)\,d\tau = \Big[\sin(t-\tau)\Big]_0^t = -\sin(-t)$$

$$= \sin t$$

clearly $(f*1)(t) \neq f(t)$ in this case.

Proof of theorem First we note that $F(s) = \int_0^\infty e^{-s\xi} f(\xi)\,d\xi$

$G(s) = \int_0^\infty e^{-s\tau} g(\tau)\,d\tau$

$$F(s)\,G(s) = \int_0^\infty e^{-s\xi} f(\xi)\,d\xi \int_0^\infty e^{-s\tau} g(\tau)\,d\tau$$

Since the integrand of the first integral does not depend on the integration variable of the second we can write $F(s)G(s)$ as an iterated integral

$$F(s)G(s) = \int_0^\infty \bar{e}^{s\tau}\, g(\tau)\left[\int_0^\infty e^{-s\xi} f(\xi)\,d\xi\right] d\tau$$

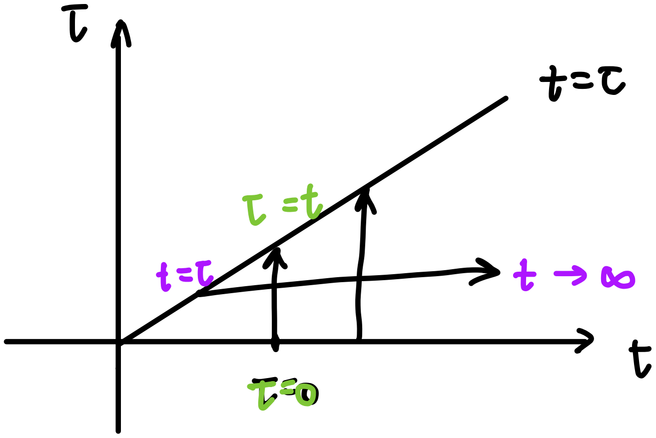
let $\xi = t-\tau$

$d\xi = dt$

$$= \int_0^\infty \cancel{e^{-s\tau}}\, g(\tau)\left[\int_\tau^\infty e^{-s(t-\cancel{\tau})} f(t-\tau)\,dt\right] d\tau$$

$$= \int_0^\infty g(\tau)\left[\int_\tau^\infty e^{-st} f(t-\tau)\,dt\right] d\tau$$

Region of integration in F(s)G(s):



$t = \tau \rightarrow \infty$

$\tau = 0 \rightarrow \infty$

$$= \int_0^\infty e^{-st} \left[\int_0^t f(t-\tau) g(\tau) d\tau \right] dt$$

$$= \int_0^\infty e^{-st} h(t) dt$$

$$= \mathcal{L}\{h(t)\}$$

$\square$

<u>**Example**</u> Find the inverse laplace transform of

$$H(s) = \frac{a}{s^2(s^2+a^2)}$$

It's convenient to think of H(s) as the product of $\frac{1}{s^2}$ and $\frac{a}{s^2+a^2}$ which have inverse Laplace transforms of t and $\sin(at)$, respectively By the theorem. the inverse transform of H(s) is

$$h(t) = \int_0^t (t-\tau) \sin(a\tau) d\tau$$

$$= t\int_0^t \sin(a\tau) d\tau - \int_0^t \tau \sin(a\tau) d\tau$$

$u = \tau$ $\quad \frac{dv}{d\tau} = \sin(a\tau)$

$\frac{du}{d\tau} = 1$ $\quad v = -\frac{1}{a}\cos(a\tau)$

$$= t\left[-\frac{1}{a}\cos(a\tau)\right]_0^t + \left[\frac{1}{a}\tau\cos(a\tau)\right]_0^t - \frac{1}{a}\int_0^t \cos(a\tau)\,d\tau$$

$$= t\left[-\frac{1}{a}\cancel{\cos(at)} + \frac{1}{a}\right] + \frac{t}{a}\cancel{\cos(at)} - \frac{1}{a^2}\left[\sin(a\tau)\right]_0^t$$

$$= \frac{t}{a} - \frac{1}{a^2}\sin(at)$$

Note that we can also find h(t) using partial fractions

Alternative:

$$H(s) = \frac{a}{s^2(s^2+a^2)} = \frac{A}{s^2} + \frac{B}{s^2+a^2}$$

$$A(s^2+a^2) + Bs^2 = a$$

Let $s=0 \Rightarrow Aa^2 = a \Rightarrow A = \frac{1}{a}$

$s = a \Rightarrow A(2a^2) + Ba^2 = a$

$$\frac{1}{\cancel{a}}(2a^{\cancel{2}}) + Ba^2 = a$$

$$2\cancel{a} + Ba^{\cancel{2}} = \cancel{a}$$

$$Ba = -1$$

$$B = -\frac{1}{a}$$

Thus $H(s) = \frac{1}{a}\frac{1}{s^2} - \frac{1}{a}\frac{1}{s^2+a^2} \cdot \frac{a}{a}$

$$h(t) = \frac{t}{a} - \frac{1}{a^2}\sin(at)$$

Which is the same answer as above.

Example Find the solution to the IVP

$$y'' + 4y = g(t), \quad y(0) = 3, \quad y'(0) = 1$$

$$s^2 Y(s) - s\underset{3}{\cancel{y(0)}} - \underset{-1}{\cancel{y'(0)}} + 4Y(s) = G(s)$$

$$Y(s)\left[s^2 + 4\right] = G(s) + 3s - 1$$

$$Y(s) = \frac{G(s)}{s^2+4} + 3\frac{s}{s^2+4} - \frac{1}{s^2+4}\frac{2}{2}$$

$$= \frac{1}{2}\frac{G(s)2}{s^2+4} + 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4}$$

$$y(s) = \frac{1}{2}\int_0^t \sin(2(t-\tau))\, g(\tau)\, d\tau + 3\cos(2t) - \frac{1}{2}\sin(2t)$$

If a specific forcing function g is given then the integral can be evaluated.

THE END