Math UA 009

Section 1

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9/7/22

Lesson 1

Exponents and radicals (1.2)

Integer exponents

If a is my real number and n is a positive integer, then the n^{th} power of a is

$$\underset{\substack{\uparrow\\ \text{base}}}{a}^{\overset{\substack{\text{exponent}\\ \swarrow}}{n}} = \underbrace{a\cdot a\cdot a\cdots a}_{n\text{ factors}}$$

e.g. (a) $4^3 = 4\cdot 4\cdot 4$

(b) $\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)$

(c) $(-3)^4 = (-3)(-3)(-3)(-3) = 81$

(d) $-3^4 = -(3\cdot 3\cdot 3\cdot 3) = -81$

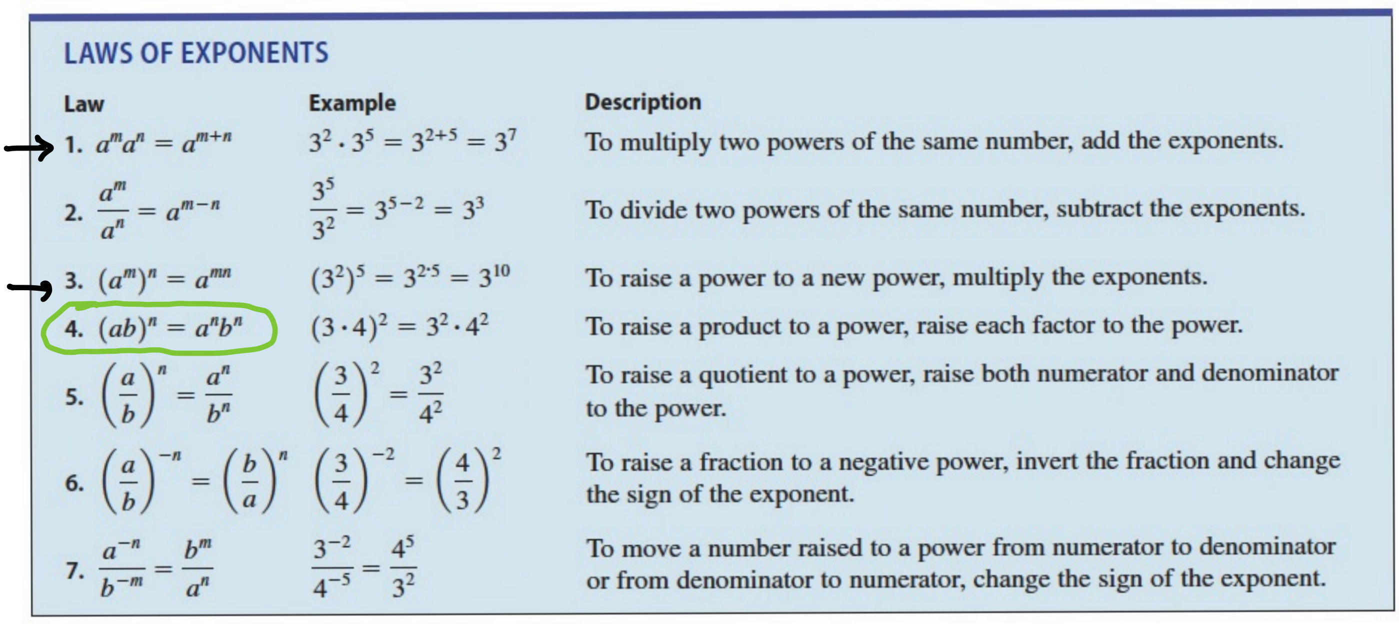
Note $\boxed{a^0 = 1}$ for any base a

$$\boxed{a^{-n} = \frac{1}{a^n}}$$

Example (a) $\left(\frac{2}{3}\right)^0 = 1$

(b) $y^{-2} = \frac{1}{y^2}$

(c) $(-4)^{-2} = \frac{1}{(-4)^2} = \frac{1}{16}$



LAWS OF EXPONENTS

Law	Example	Description
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$	To multiply two powers of the same number, add the exponents.
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{3^5}{3^2} = 3^{5-2} = 3^3$	To divide two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2\cdot5} = 3^{10}$	To raise a power to a new power, multiply the exponents.
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2$	To raise a product to a power, raise each factor to the power.
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{3}{4}\right)^2 = \dfrac{3^2}{4^2}$	To raise a quotient to a power, raise both numerator and denominator to the power.
6. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$	$\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2$	To raise a fraction to a negative power, invert the fraction and change the sign of the exponent.
7. $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$	$\dfrac{3^{-2}}{4^{-5}} = \dfrac{4^5}{3^2}$	To move a number raised to a power from numerator to denominator or from denominator to numerator, change the sign of the exponent.

Examples.

1. $4^3 \cdot 4^5 = 4^8$
2. $\dfrac{a^m}{a^n} = a^{m-n}$. $\quad a^m \cdot \dfrac{1}{a^n} = a^m \cdot a^{-n} = a^{m-n}$
3. $(2^4)^3 = 2^{12}$
4. $(ab)^n = a^n b^n$ $\quad$ e.g. $(2\cdot5)^3 = 2^3 \cdot 5^3$
5. $\left(\dfrac{2}{5}\right)^4 = \dfrac{2^4}{5^4}$
6. $\left(\dfrac{4}{9}\right)^{-3} = \left(\dfrac{9}{4}\right)^3 = \dfrac{9^3}{4^3}$

Simplifying expressions involving exponents

(a) $$(\underbrace{4a^5b^3})^1 \cdot (\underbrace{5ab^2})^{\boxed{4}} = 4 \cdot 5^4 a^5 b^3 a^4 (b^2)^4$$

$$\downarrow$$

$$5^4 \cdot a^4 \cdot (b^2)^4 = 5^4 a^4 b^8$$

$$= 4 \cdot 5^4 a^5 \cdot a^4 \cdot b^3 \cdot b^8$$

$$= 4 \cdot 5^4 a^9 b^{11}$$

Recall $(a^m)^n = a^{mn}$

Example 2 (b) $\underline{\left(\frac{x}{y}\right)^4 \cdot \left(\frac{xy^2}{z^3}\right)^2} = \frac{x^4}{\cancel{y^4}} \cdot \frac{x^2 \cdot \cancel{y^4}}{z^6}$

$$= \frac{x^{4+2}}{z^6} \cdot 1$$

$$= \frac{x^6}{z^6}$$

Note

$y^{-4} \cdot y^4$

$= y^{-4+4}$

$= y^0$

$= 1$

Note: $\frac{y^{\cancel{2}}}{y^{\cancel{4}\,2}} = y^{2-4} = y^{-2} = \frac{1}{y^2}$

$\frac{y^{\cancel{3}}}{y^{\cancel{8}\,5}} = \frac{1}{y^5}$ $\quad y^{3-8} = y^{-5} = \frac{1}{y^5}$

$\frac{x^1}{x^4} = x^{1-4} = x^{-3} = \frac{1}{x^3}$

$\frac{x^2}{x^6} = \frac{1}{x^4}$

$\downarrow$

$x^2 \cdot \frac{1}{x^6} = x^2 \cdot x^{-6} = x^{2-6} = x^{-4} = \frac{1}{x^4}$

e.g. $\frac{x^6}{x^2} = x^4$

Examples involving negative exponents

(a) $$\frac{10a^{1}b^{-3}}{2a^{-2}b^{5}} = 5a^{1-(-2)}b^{-3-5} = \underline{5a^{3}b^{-8}} = \frac{5a^{3}}{b^{8}}$$

↓

$$5\boxed{a^{2}}\cdot a\cdot\frac{1}{b^{5}}\cdot\frac{1}{b^{3}} = 5a^{3}\cdot\frac{1}{b^{8}}$$

Recall

$a^{m}\cdot a^{n} = a^{m+n}$

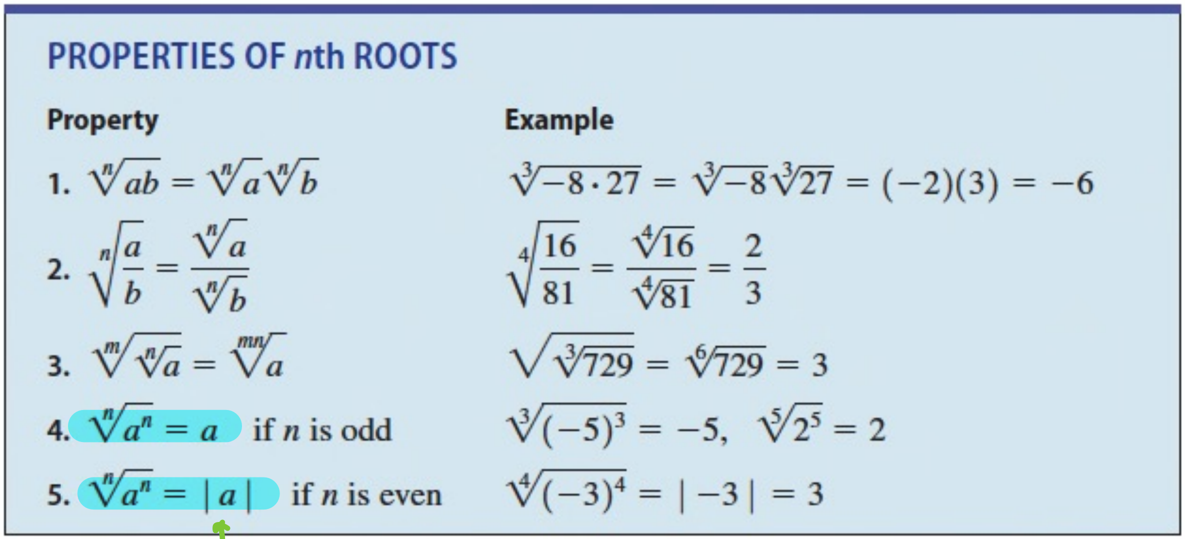
Radicals

If n is any positive integer, then the principal n^{th} root of a is defined as :

$$\sqrt[n]{a} = b$$

This means $b^{n} = a$.

Note. If n is even then both a and b must be greater or equal to 0.



PROPERTIES OF *n*th ROOTS

Property	Example
1. $\sqrt[n]{ab}=\sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[3]{-8\cdot 27}=\sqrt[3]{-8}\sqrt[3]{27}=(-2)(3)=-6$
2. $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\frac{16}{81}}=\frac{\sqrt[4]{16}}{\sqrt[4]{81}}=\frac{2}{3}$
3. $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}}=\sqrt[6]{729}=3$
4. $\sqrt[n]{a^n}=a$ if n is odd	$\sqrt[3]{(-5)^3}=-5,\quad \sqrt[5]{2^5}=2$
5. $\sqrt[n]{a^n}=\lvert a\rvert$ if n is even	$\sqrt[4]{(-3)^4}=\lvert -3\rvert=3$

↑ absolute value of a

Examples.

(a) $\sqrt[4]{81} = 3$

(b) $\sqrt[3]{z^4} \underset{\uparrow}{=} \sqrt[3]{z^3\cdot z} = \sqrt[\cancel{3}]{\cancel{z^3}}\cdot\sqrt[3]{z} \doteq z\cdot\sqrt[3]{z}$

$\downarrow$

$\sqrt[3]{z^2\cdot z^2}$

$= \underbrace{\sqrt[3]{z^2}\cdot\sqrt[3]{z^2}}$

Recall

$$\boxed{\sqrt[n]{a\cdot b} = \sqrt[n]{a}\cdot\sqrt[n]{b}}$$

(c) $\sqrt[4]{81x^{16}y^8} = \sqrt[4]{81}\cdot\sqrt[4]{x^{16}}\cdot\sqrt[4]{y^8}$

$= 3\,\underset{\uparrow\,x^4}{\sqrt[\cancel{4}]{(x^4)^{\cancel{4}}}}\;\underbrace{\sqrt[\cancel{4}]{(y^2)^{\cancel{4}}}}_{\uparrow\,y^2}$

$= 3\cdot x^4\cdot y^2$

Exponent law:

$(a^m)^n = a^{m\cdot n}$

$\sqrt[n]{a^n} = a$

Rational exponents (fractional exponent).

$$\left(a^{\frac{1}{m}}\right)^m = a^{\frac{m}{m}} = a^1 = a \iff \boxed{a^{1/m} = \sqrt[m]{a}} \leftarrow$$

For any exponent m/n where m and n are integers, $n>0$ then we define $\boxed{a^{m/n} = \left(\sqrt[n]{a}\right)^m} \leftarrow$

$\downarrow$

$\nearrow\sqrt[n]{a^m}$

Examples

(a). $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

(b). $125^{-1/3} = \frac{1}{125^{1/3}} = \frac{1}{\sqrt[3]{125}} = \frac{1}{5}$

Simplifying expressions

$\sqrt[5]{(a^4)^2}$

(a). $(2a^4b^6)^{2/5} = 2^{2/5}(a^4)^{2/5}(b^6)^{2/5}$

$= \sqrt[5]{2^2}\ \sqrt[5]{(a^4)^2}\ \sqrt[5]{(b^6)^2}$

$= \sqrt[5]{4}\ \sqrt[5]{a^8}\ \sqrt[5]{b^{12}}$

$\frac{y^4}{y^1} = y^{4-1}$
$= y^3$

(b) $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3 \left(\frac{y^4}{x^{-1/2}}\right) = \left(\frac{2^3\,x^{9/4}}{y}\right)\left(\frac{y^4}{x^{-1/2}}\right) = 8x^{\frac{9}{4}}\cdot x^{\frac{1}{2}}y^3$

$= 8x^{\frac{11}{4}}y^3.$

$(a^m)^n$
$= a^{m\cdot n}$

$(x^{3/4})^3 = x^{\frac{3}{4}\cdot 3} = x^{\frac{9}{4}}$

$\frac{9}{4} + \frac{1}{2} = \frac{11}{4}$

$(x^{3/4})^3 \cdot \frac{1}{x^{-1/2}} = x^{\frac{9}{4}}\cdot\frac{1}{x^{-1/2}} = x^{\frac{9}{4}}\cdot x^{\frac{1}{2}}$
$= x^{\frac{9}{4}+\frac{1}{2}}.$
$= x$

$\frac{2^3\,x^{\frac{3}{4}\cdot 3}}{y^{\frac{1}{\cancel{3}}\cdot\cancel{3}}}\cdot\frac{y^4}{x^{-1/2}} = \frac{8x^{\frac{9}{4}}}{y}\cdot\frac{y^4}{x^{-1/2}} = \frac{8x^{\frac{9}{4}}}{x^{-1/2}}\cdot\frac{y^4}{y} = 8x^{\frac{9}{4}}\cdot x^{\frac{1}{2}}\cdot y^3$

$= 8x^{\frac{9}{4}+\frac{1}{2}}y^3$

Recall $x^{-m} = \frac{1}{x^m}$ or $\frac{1}{x^{-m}} = x^m$

Upcoming deadlines

9/12
Monday

- Homework 1 : Sep 18 at 11:59 pm (on Gradescope)
- WebAssign 1.2, 1.3 : Sep 17
- Quiz 1: Sections 2 and 4 $\downarrow$ Sep 20 , Sections 3 and 5 $\downarrow$ Sep 22 : During recitations

Rationalizing the denominator (Sec 1.2)

Sometimes we want to get rid of the radical in the denominator by multiplying both the numerator and the denominator by an appropriate expression

e.g

$$\frac{1}{\sqrt{3}} \cdot \boxed{\frac{\sqrt{3}}{\sqrt{3}}} \quad \Big\Vert \quad 1$$

Recall from last class

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

$$\left(\frac{\sqrt{3}}{\sqrt{3 \cdot 3}} = \frac{\sqrt{3}}{\cancel{\sqrt{3^2}}} \right) = \frac{\sqrt{3}}{3}$$

In general if we have $\frac{1}{\sqrt{a}} \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$ ← standard form

If we are given $\frac{1}{\sqrt[n]{a^m}}$ we have to do the following.

$$\frac{1}{\sqrt[n]{a^{\boxed{m}}}} \cdot \frac{\sqrt[n]{a^{n-m}}}{\sqrt[n]{a^{n\boxed{-m}}}} = \frac{\sqrt[n]{a^{n-m}}}{\sqrt[n]{a^{\cancel{m}+n\cancel{-m}}}} = \frac{\sqrt[n]{a^{n-m}}}{\sqrt[n]{a^n}} = \frac{\sqrt[n]{a^{n-m}}}{a}$$

Examples

(1) Rationalize $\frac{1}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5}}{\sqrt[3]{5}} = \frac{\sqrt[3]{5}}{\sqrt[3]{5\cdot 5}} = \frac{\sqrt[3]{5}}{\sqrt[3]{5^2}}$

$\uparrow$ $\qquad\qquad$ $\uparrow$

$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{a\cdot b}$

$\boxed{\frac{1}{5^{\frac{1}{3}}}} \cdot \frac{5^{\frac{2}{3}}}{5^{\frac{2}{3}}} = \frac{5^{\frac{2}{3}}}{5}$

$\sqrt[3]{5}\cdot\sqrt[3]{5^2}$

$= \sqrt[3]{5^{1+2}}$

$= \sqrt[\cancel{3}]{5^{\cancel{3}}}$

$= 5$

$\frac{1}{5^{\frac{1}{3}}} \cdot \frac{5^x}{5^x} = \frac{5^x}{5}$

$5^{\frac{1}{3}} 5^x = 5^{\frac{1}{3}+x}$ $\qquad$ we want 5^1

$\frac{1}{3} + x = 1$

$x = \frac{2}{3}$

(2) $\frac{2}{\sqrt[2]{5}} \cdot \frac{\sqrt[2]{5}}{\sqrt[2]{5}} = \frac{2\sqrt{5}}{5}$ $\qquad$ $\sqrt[2]{5\cdot 5} = \sqrt[\cancel{2}]{5^{\cancel{2}}} = 5$

(3) $\sqrt[5]{\frac{1}{b^2}} = \frac{1}{\sqrt[5]{b^2}}$ $\qquad$ Recall from last class $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$= \frac{1}{b^{\frac{2}{5}}} \cdot \frac{b^{\frac{3}{5}}}{b^{\frac{3}{5}}}$

$= \frac{b^{\frac{3}{5}}}{b^1}$

$= \frac{\sqrt[5]{b^3}}{b}$

(4) $\frac{1}{\sqrt{5x}} \cdot \frac{\sqrt{5x}}{\sqrt{5x}} = \frac{\sqrt{5x}}{5x}$

Algebraic expressions (1.3)

A polynomial in the variable x is an expression of the form

$$\underbrace{a_n x^n}_{\text{term}} + \underbrace{a_{n-1} x^{n-1}}_{\text{term}} + \dots + a_1 x + a_0$$

where $a_0, a_1, \dots, a_n$ are constants and n is a non-negative integer. If $a_n \neq 0$ then the polynomial has degree n.

The terms $a_k x^k$ are called the terms

Adding and subtracting polynomials

Example: $8x^9 + 2x^9 + 1 \left(= (8+2)x^9 + 1\right) = 10x^9 + 1$

$5x^3 - x^3 = 4x^3$

Note: $-(a+b) = -a-b$

Multiplying algebraic expressions

$$(a+b)\cdot(c+d) = ac+ad+bc+bd$$

Example: (a) $(2x+1)(x-5) = 2x^2 - 10x + x - 5$

$= 2x^2 - 9x - 5$

(b) $(2x+3)(x^2-4x+6) =$ ↓ use distributive law

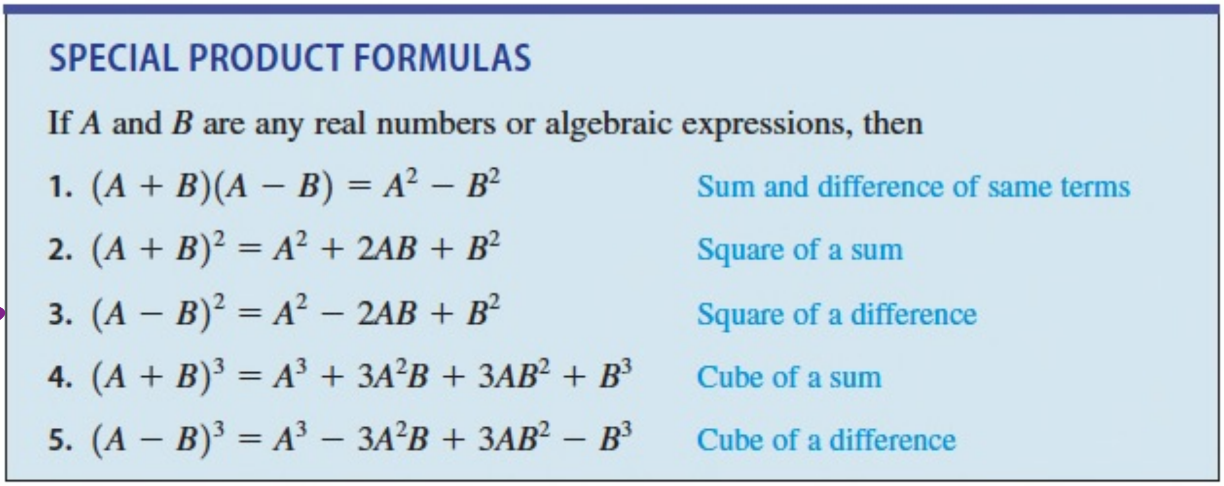
$2x(x^2-4x+6)$

$+3(x^2-4x+6)$

$$= 2x^3 - 8x^2 + \cancel{12x}$$
$$+ 3x^2 - \cancel{12x} + 18$$
$$= 2x^3 - 5x^2 + 18$$

1. $(A+B)(A-B) = A^2 - \cancel{AB} + \cancel{AB} - B^2 = A^2 - B^2$

2. $(A+B)^2 = (A+B)(A+B) = A^2 + AB + AB + B^2 = A^2 + 2AB + B^2$



SPECIAL PRODUCT FORMULAS

If A and B are any real numbers or algebraic expressions, then

1. $(A + B)(A - B) = A^2 - B^2$	Sum and difference of same terms
2. $(A + B)^2 = A^2 + 2AB + B^2$	Square of a sum
3. $(A - B)^2 = A^2 - 2AB + B^2$	Square of a difference
4. $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$	Cube of a sum
5. $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$	Cube of a difference

Principle of substitution:

Example.(1) $(x^2 + y^5)^2 = (x^2+y^5)(x^2+y^5)$

$(A + B)^2$ = FOIL

(formula 2.) $= (x^2)^2 + 2(x^2)(y^5) + (y^5)^2$

$A = x^2$
$B = y^5$

$$= x^4 + 2x^2y^5 + y^{10}$$

(2) $(3x-5)^2 = (3x)^2 - 2(3x)(5) + 5^2 \leftarrow$

$A = 3x$
$B = 5$

$= 9x^2 - 30x + 25$

$(3x-5)(3x-5) = 9x^2 - 15x - 15x + 25$

$= 9x^2 - 30x + 25$

(3) $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})$

Special product formula : $(A+B)\cdot(A-B) = A^2 - B^2$

where $A = \sqrt{x}$ and $B = \sqrt{y}$

$(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) = x - y$

$= (\sqrt{x})^2 - (\sqrt{y})^2$

$(A+B)^2 = A^2 + 2AB + B^2$

(4) $((y+x)-1)\cdot((y+x)+1) = \overbrace{(y+x)^2} - 1$

$= y^2 + 2yx + x^2$
-1

$(A-B)\cdot(A+B) = A^2 - B^2$

where $A = y + x$
$B = 1$

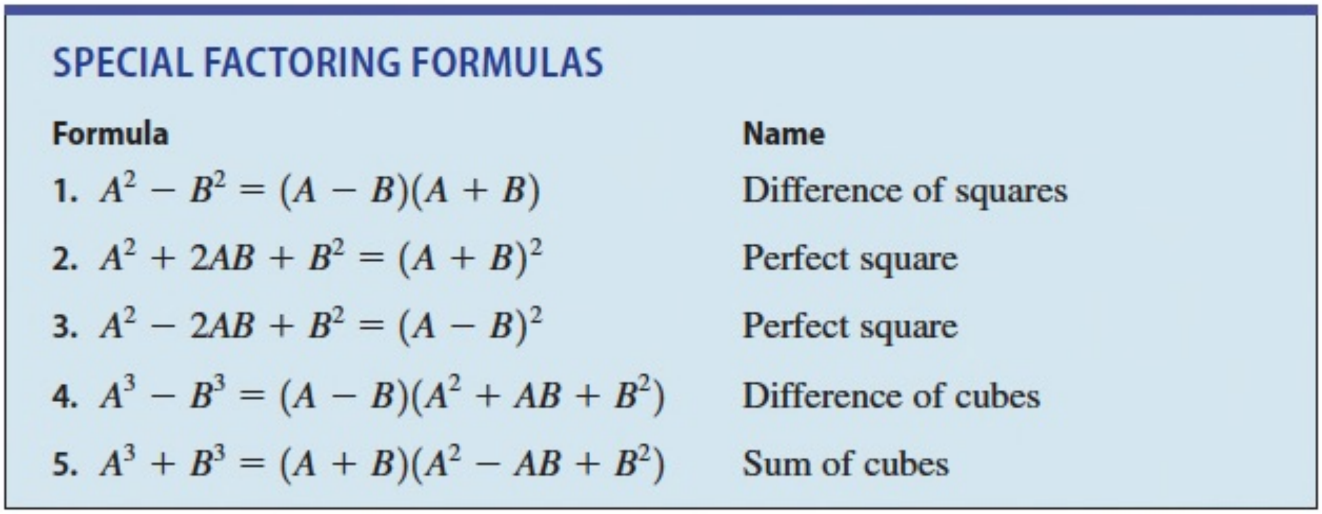
Factoring algebraic expressions

(1) $3x^2 - 9x^7 = 3x^2(1 - 3x^5)$

(2) $8x^4y^2 + 6x^3y^3 - 2xy^4 = 2xy^2(4x^3 + 3x^2y$
$-y^2)$

(3) $\overbrace{(2x^2+5)}\boxed{(x-1)} - \overbrace{4}\boxed{(x-1)} = (x-1)\cdot(2x^2+5-4)$

NB $5(x-1)-4(x-1)=(x-1)$ $\qquad = (x-1)\cdot(2x^2+1)$



SPECIAL FACTORING FORMULAS

Formula	Name
1. $A^2 - B^2 = (A - B)(A + B)$	Difference of squares
2. $A^2 + 2AB + B^2 = (A + B)^2$	Perfect square
3. $A^2 - 2AB + B^2 = (A - B)^2$	Perfect square
4. $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$	Difference of cubes
5. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$	Sum of cubes

$$\underbrace{(2x^2+5)(x-1)} - 4(x-1) = (x-1)\cdot\left[(2x^2+5)-4\right]$$

$$= (x-1)(2x^2+5) - 4(x-1)$$

In general, factoring quadratic expressions:

x^2+bx+c where b, c are real numbers.

$= (x+r)\cdot(x+s)$ factored form

$= x^2 + sx + rx + rs$

$= x^2 + (s+r)\cdot x + rs$

$$\boxed{\begin{aligned} b &= s+r \\ c &= rs \end{aligned}}$$

Example

$$x^2+\underset{\overset{\uparrow}{b}}{7}x+\underset{\overset{\uparrow}{c}}{12}$$

$1\cdot 12 = c = 12$

$3\cdot 4 = c = 12$

$2\cdot 6 = c = 12$

$$=(x+3)(x+4)$$

A bit more complicated...

Factoring $ax^2+bx+c = (px+r)(qx+s)$

$$= pqx^2+psx+rqx+rs$$

$$= pqx^2+(ps+rq)x+rs$$

$$\boxed{\begin{aligned}a&=pq\\ b&=ps+rq\\ c&=rs\end{aligned}}$$

Example.

$$\overset{\checkmark}{6x^2}+7x\overset{\downarrow}{-5} = \begin{array}{l}(3x+5)(2x-1)\ \text{attempt 1}\\ (3x\quad 1)(2x\quad 5)\\ (6x\quad 5)(x\quad 1)\\ (6x\quad 1)(x\quad 5)\end{array}$$

$3x\cdot 2x$

$6x\cdot x$

Attempt 1: $6x^2-3x+10x-5$

$$=6x^2+7x-5$$

Note:

F L

$(a+b)(c+d) = ac + ad + bc + bd$

I O

F O I L

first outer inner last

Today we'll finish Section 1.3 and do most of 1.4 too.

Today: Office hours at WWH in Room 1025 at 4:30-5:30pm.

Examples ② Factorize:

$2x^2 + \underline{5x} + 3$

$= (x+1)(2x+3)$

$2x^2 + 6x + x + 3 = 2x^2 + 7x + 3$

$(2x+1)(x+3)$

$(x+1)(2x+3)$

$2x^2 + 3x + 2x + 3$

③ $8x^2 + 10x + 3 = (2x+1)(4x+3)$ $= 2x^2 + 5x + 3$

④ $6y^2 + 11y - 21 = (y+3)(6y-7)$

$\xcancel{(y-3)(6y+7)}$

↑

6 · 1

2 · 3

1 · 8

2 · 4

3 · 7 = 21

1 21 = 21

$6y^2 - 7y + 18y - 21 = 6y^2 + 11y - 21$

Difference of squares:

$A^2 - B^2 = (A-B)(A+B).$

Example :

$$4x^2 - 36 = (2x - 6)(2x + 6)$$

$$9z^2 - 25 = (3z - 5)(3z + 5)$$

$$(a+b)^2 - c^2 = [(a+b) - c][(a+b) + c]$$

$A = a+b \quad B = c$

Perfect square:

$A^2 + 2AB + B^2$ or $A^2 - 2AB + B^2$

To recognize it look if the middle term is plus or minus twice the product of the square root of the two outer terms.

e.g 1. $x^2 + \boxed{6x} + 9 = (x+3)^2 \rightarrow (x+3)(x+3)$

$= x^2 + 3x + 3x + 9$

$= x^2 + 6x + 9$

2 $\underbrace{4x^2}_{(2x)^2} - \underline{4xy} + \underbrace{y^2}_{(y)^2} = (2x - y)^2$

$\sqrt{4x^2}$

$= \sqrt{4}\,\cancel{\sqrt{x^2}}$

$= 2x$

$\Rightarrow (2x)^2 = 4x^2$

Factoring expressions with fractional exponents

Recall

$$\boxed{x^m x^n = x^{m+n}}$$

Factorize: $3x^{3/2} - 9x^{1/2} + 6x^{-1/2} = 3x^{-1/2}(x^2 - 3x^1 + 2)$

factor out the power of x with the smallest exponent

To factor out $x^{-1/2}$ from $x^{3/2}$:

$$x^{3/2} = x^{-1/2}\left(x^{\frac{3}{2}-\left(-\frac{1}{2}\right)}\right)$$

$$= x^{-1/2}(x^2)$$

$$3x^{-1/2}\cdot x^2$$

$$= 3x^{-\frac{1}{2}+2}$$

$$= 3x^{-\frac{1}{2}+\frac{4}{2}}$$

$$= 3x^{\frac{3}{2}}$$

$$\frac{1}{6} + \frac{2}{3} = \frac{1}{6} + \frac{4}{6} = \frac{5}{6}$$

e.g. $(2+x)^{-2/3}x + (2+x)^{1/3} = (2+x)^{-2/3}\cdot\left[x + (2+x)^{\textcircled{1}}\right]$

$$\boxed{x^m x^n = x^{m+n}}$$

NB. $(2+x)^{-\frac{2}{3}}\cdot(2+x)^1$

$= (2+x)^{-\frac{2}{3}+1}$

$= (2+x)^{\frac{1}{3}}$

Factoring by grouping

1. $x^3 + x^2 + 4x + 4 = x^2(x+1) + 4(x+1)$

$x(x^2+x)$

$x(x+1)x$

$= (x+1)(x^2+4)$

★ $(x^2 - 4) = (x-2)(x+2)$

$(x+2)^2 = x^2 + 4x + 4$

2. $3x^3 - x^2 - 12x + 4 = x^2(3x-1) - 4(3x-1)$

$= (3x-1)[x^2-4]$

$= (3x-1)(x-2)(x+2)$

Section 1.4 RATIONAL EXPRESSIONS

Aside:

$$a_n x^{\boxed{n}} + a_{n-1} x^{\boxed{n-1}} + \dots + a_0$$

exponents should be integers for this to be a polynomial.

Definition: A rational expression is a fractional expression where both the numerator and the denominator are polynomials.

e.g. $\frac{3x}{x-2}$, $\frac{z-2}{z^2+4}$, ...

But $\frac{x^3}{\sqrt{x^2+1}}$

this is not a polynomial because of $\sqrt{\dots}$
so it's not a rational expression.

Domain: the set of values of 'x' that the variable is allowed to have

e.g. $\frac{3x}{x-2}$ domain is $\{x \mid x \neq 2\}$

$\uparrow$ such that

$\uparrow$ x is not equal to 2

e.g. $$\frac{x}{x^2-5x+6} = \frac{x}{(x-3)(x-2)}$$

Domain is $\{x \mid x \neq 2 \text{ and } x \neq 3\}$

avoid the x values that give that the denominator $=0$.

e.g. $\frac{3\sqrt{x}}{x-5}$ Domain is $\{x \mid x \neq 5 \text{ and } x \geq 0\}$

$\uparrow$ this comes from $\sqrt{x}$

Simplifying rational expressions

e.g. $$\frac{x^2-1}{x^2+x-2} = \frac{(x+1)\cancel{(x-1)}}{(x+2)\cancel{(x-1)}} = \frac{x+1}{x+2}$$

e.g. $$\left(\frac{x^2+2x-3}{x^2+8x+16}\right)\left(\frac{3x+12}{x-1}\right) = \left(\frac{\cancel{(x-1)}(x+3)}{(x+4)^{\cancel{2}}}\right)\cdot\left(\frac{3\cancel{(x+4)}}{\cancel{x-1}}\right)$$

$\uparrow$

$(x+4)(x+4)$

$$= \frac{3(x+3)}{(x+4)} ✓ \quad \left(= \frac{3x+9}{x+4}\right)$$

e.g. $$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} = \frac{x-4}{x^2-4}\cdot\frac{x^2+5x+6}{x^2-3x-4}$$

$$= \frac{\cancel{(x-4)}}{(x-2)\cancel{(x+2)}}\cdot\frac{\cancel{(x+2)}(x+3)}{\cancel{(x-4)}(x+1)}$$

$$= \frac{x+3}{(x-2)(x+1)}$$

Adding and subtracting rational expressions

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}$$

Note. It's best to use the least common denominator.

Examples

1). $$\frac{3}{x-1} + \frac{x}{x+2}$$

the denominator will be a product of the (x-1) and (x+2).

$$= \frac{3(x+2)+x(x-1)}{(x-1)\cdot(x+2)} \swarrow$$

$$\left(= \frac{3\cancel{(x+2)}}{(x-1)\cancel{(x+2)}} + \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+2)} = \frac{3}{x-1} + \frac{x}{x+2} \right)$$

$$= \frac{3x+6+x^2-x}{(x-1)(x+2)}$$

$$= \frac{x^2+2x+6}{(x-1)(x+2)}$$

$$\begin{array}{c}(x+1)(x+6)\\(x+2)(x+3)\end{array}$$

2). $$\frac{1}{x^2-1} - \frac{2}{(x+1)^2} = \frac{1}{(x-1)(x+1)} - \frac{2}{\boxed{(x+1)^2}}$$

$$= \frac{x+1-2(x-1)}{(x-1)(x+1)^{\boxed{2}}}$$

$$= \frac{x+1-2x+2}{(x-1)(x+1)^2}$$

$$= \frac{-x+3}{(x-1)(x+1)^2}$$

Compound fractions

Note: A compound fraction is a fraction that has a fraction expression in the numerator, denominator, or both.

e.g.

$$\frac{\frac{a}{b}+1}{1-\frac{b}{a}} = \frac{\frac{a}{b}+\frac{b}{b}}{\frac{a}{a}-\frac{b}{a}}$$

$$= \frac{\frac{a+b}{b}}{\frac{a-b}{a}}$$

$$= \frac{a+b}{b}\cdot\frac{a}{a-b}$$

$$= \frac{a(a+b)}{b(a-b)}$$

$$\frac{1}{2}+1$$

$$= \frac{1}{2}+\frac{2}{2}$$

$$= \frac{3}{2}$$

Note.

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\cancel{2}}{3}\cdot\frac{5}{\underset{2}{\cancel{4}}}$$

$$= \frac{5}{6}$$

$$\frac{2\cdot 5}{3\cdot 4}$$

Announcements

① In WebAssign it matters whether you write little x or capital X.

② Submit the homework through Gradescope, not by email.

③ If Gradescope doesn't work and you enrolled in class late, let me know. If it just doesn't work, try a different browser.

④ If you have math questions, use Campuswire.

From the previous section.

Rationalizing denominators or numerators

Use: $\boxed{(A - B\sqrt{C})\cdot(A+B\sqrt{C}) = A^2 - B^2C}$

examples (1)

$$\frac{1}{1+\sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{1-\sqrt{3}}{1-3} = \frac{1-\sqrt{3}}{-2}$$

this is correct

$$\text{or} = \frac{\sqrt{3}-1}{2}$$

$$(A-B)(A+B) = A^2 + \cancel{AB} - \cancel{AB} - B^2$$

$$= A^2 - B^2$$

(2)

$$\frac{\sqrt{4+X} - 2}{X} = \frac{\sqrt{4+x} - 2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$

difference of two squares

$$(\sqrt{4+x})^2 - 2^2$$

$$= 4+x-4$$

$$= \frac{\cancel{4}+x-\cancel{4}}{x(\sqrt{4+x}+2)}$$

$$= \frac{\cancel{x}}{\cancel{x}(\sqrt{4+x}+2)}$$

$$= \frac{1}{\sqrt{4+x}+2}$$

(3) $$\frac{\sqrt{y+5}-10}{2} = \frac{\sqrt{y+5}-10}{2} \cdot \frac{\sqrt{y+5}+10}{\sqrt{y+5}+10}$$

$$= \frac{(y+\underline{5}) \underline{-100}}{2(\sqrt{y+5}+10)}$$

$$= \frac{y-95}{2\sqrt{y+5}+20}$$

Section 1.5: Equations

Example:

unknown variable that I want to solve for

$$2x+4=0$$
$$\quad -4 \qquad -4$$

Linear equation (first degree polynomial).

$$\rightarrow \quad 2x = -4$$

$$x = -2$$

$\uparrow$

the solution is the root of the equation.

Equivalent equations

1. $A = B \iff A+C = B+C$

2. $A = B \iff A \cdot C = B \cdot C$ (where $C \neq 0$).

Linear equation solutions:

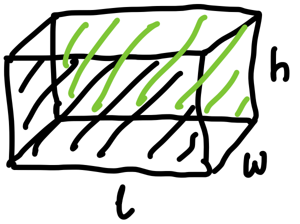
e.g. $7x - 4 = 5x + 9$

$$7x = 5x + 13$$

$$2x = 13$$

$$x = \frac{13}{2}$$

e.g.



$$A = 2lh + \underline{2wh} + \underline{2lw}$$

We want width, w, expressed in terms of all other quantities.

$$A - 2lh = 2wh + 2lw$$

$$A - 2lh = 2w(h + l)$$

$$w = \frac{A - 2lh}{2(h+l)}$$

SOLVING QUADRATIC EQUATIONS

Reminder: A quadratic equation is of the form

$$ax^2 + bx + c = 0 \qquad (*)$$

where a, b, and c are real numbers with $a \neq 0$

Zero-product property

$\boxed{AB = 0}$ if and only if $A=0$ or $B=0$.

Factoring a quadratic to solve it

e.g. $\underline{x^2+5x = 24.}$

$x^2+5x-24 = 0$

$(x-3)(x+8)=0$

$x=3$ or $x=-8$

if $x=3$ LHS $= (3)^2+5(3)$

$= 9+15$

$= 24$

$=$ RHS ✓

if $x=-8$ LHS $= (-8)^2+5(-8$

$= 64-40$

$= 24$

$=$ RHS ✓

DO NOT USE THIS

$x^2+5x=24$

$x(x+5)=\underline{\underline{24}}$

e.g. $x^2 = 24$ ✓

$(-\sqrt{24})^2 = +24$

$x = \pm\sqrt{24}$

For simple quadratic equations

$x^2=c$

$x=\sqrt{c}, -\sqrt{c}$

e.g. $(x-3)^2 = 7$

square root both sides

$x-3 = \pm\sqrt{7}$

$x = 3\pm\sqrt{7}$

COMPLETING THE SQUARE

$(x+b)^2 = \underline{x^2} + 2bx + b^2$

always −

$$\underbrace{x^2 + bx}_{(\text{⌂})} + a = \underbrace{\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2}_{(\bigstar)} + a$$

always half the coefficient of x

$\left(x+\frac{b}{2}\right)\left(x+\frac{b}{2}\right)$

Check that (★) gives (⌂)

$$(\bigstar) = \underbrace{\left(x+\frac{b}{2}\right)^2} - \left(\frac{b}{2}\right)^2 = x^2 + bx + \cancel{\frac{b^2}{4}} - \cancel{\frac{b^2}{4}}$$

$$= x^2 + bx$$

$$= (\text{⌂})$$

Example

① $x^2 - 8x + 13 = 0$ use completing the square.

★ $(x-4)^2 - 4^2 + 13 = 0$

remains unchanged.

always mins

$(x-4)^2 - 16 + 13 = 0$

$(x-4)^2 - 3 = 0$

$(x-4)^2 = 3$

$x - 4 = \pm\sqrt{3}$

$x = 4 \pm \sqrt{3}$

$(x-4)^2 = (x-4)(x-4)$

$= x^2 - 8x + 16$

$(x-4)^2 - 4^2 + 13$

$= x^2 - 8x + \cancel{16} - 16 + 13$

$= x^2 - 8x + 13$

②

$$3x^2 - 12x + 8 = 0$$

$$3\underbrace{(x^2 - 4x)} + 8 = 0$$

★

$$3\left[(x-2)^2 - 2^2\right] + 8 = 0$$

$$3\left[(x-2)^2 - 4\right] + 8 = 0$$

$$3(x-2)^2 - 12 + 8 = 0$$

$$3(x-2)^2 - 4 = 0$$

$$3(x-2)^2 = 4$$

$$(x-2)^2 = \frac{4}{3}$$

$$x - 2 = \pm\sqrt{\frac{4}{3}}$$

$$x = 2 \pm \sqrt{\frac{4}{3}} = 2 \pm \frac{2}{\sqrt{3}}$$

always −

$$\underbrace{x^2 + bx}_{(A)} + a = \underbrace{\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2}_{(*)} + a$$

always half the coefficient of x

Line 2 to line 3

$$3(x^2 - 4x) + 8 = 0$$

$$x^2 - 4x$$

$$b = -4$$

$$\frac{b}{2} = -2$$

$$= (x-2)^2 - 2^2$$

$$= (x-2)^2 - 4$$

③

$$3x^2 - 12x + 9 = 0$$

e.g.

$$12x = 0$$

$$x = 0$$

$$3(x^2 - 4x + 3) = 0$$

(÷3) 3 $x^2 - 4x + 3 = 0$ 3 (÷3) → $\boxed{(x-1)(x-3) = 0}$

$x = 1$ $x = 3$.

$$(x-2)^2 - 2^2 + 3 = 0$$

$$(x-2)^2 - 4 + 3 = 0$$

$$(x-2)^2-1=0$$

$$(x-2)^2=1$$

$$x-2=\pm 1$$

$$x=2\pm 1=3,1$$

$$\underbrace{x^2+bx}_{(⌂)}+a=\underbrace{\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2}_{(*)}+a$$

always −

always half the coefficient of x

④ $$3x^2+\boxed{16}x+5=0$$

$$3\left[x^2+\boxed{\frac{16}{3}}x\right]+5=0$$

b, we want $\frac{b}{2}=\frac{1}{\cancel{2}}\left(\frac{\cancel{16}^{8}}{3}\right)=\frac{8}{3}$

$$\frac{1}{\cancel{2}}\left(\frac{\cancel{16}^{8}}{3}\right)$$

$$3\left[\left(x+\frac{8}{3}\right)^2-\left(\frac{8}{3}\right)^2\right]+5=0$$

$$\Rightarrow 3\left(x+\frac{8}{3}\right)^2-3\left(\frac{8}{3}\right)^2+5=0$$

$$3\left(x+\frac{8}{3}\right)^2-\cancel{3}\left(\frac{64}{\cancel{9}_{3}}\right)+5=0$$

$$3\left(x+\frac{8}{3}\right)^2\underbrace{-\frac{64}{3}+5}=0$$

$$-\frac{64}{3}+\frac{15}{3}$$

$$=-\frac{49}{3}$$

$$3\left(x+\frac{8}{3}\right)^2-\frac{49}{3}=0$$

$$(\div 3)\ 3\left(x+\frac{8}{3}\right)^2=\frac{49}{3}\ (\div 3)$$

$$\left(x+\frac{8}{3}\right)^2 = \frac{49}{9}$$

$$x+\frac{8}{3} = \pm\sqrt{\frac{49}{9}} = \pm\frac{7}{3}$$

$$x = -\frac{8}{3}\pm\frac{7}{3} = -\frac{1}{3}, -5$$

$$\downarrow$$

$$\frac{-8+7}{3} = \frac{1}{3}$$

$$\frac{-8-7}{3} = -\frac{15}{3} = -5$$

Sep 21,22

Question about rationalizing denominator

$$\frac{x}{\sqrt{x^2+1}+1}\cdot\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}-1} = \frac{x(\sqrt{x^2+1}-1)}{x^2\cancel{+1}\cancel{-1}}$$

$$= \frac{\cancel{x}(\sqrt{x^2+1}-1)}{x^{\cancel{2}}}$$

$$= \frac{\sqrt{x^2+1}-1}{x}$$

Solving quadratic equations using the quadratic formula.

$ax^2+bx+c=0$ where a, b, c are real numbers.

Quadratic formula:

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

where $a \neq 0$

Example ①

$$3x^2-5x-1=0$$

$$ax^2+bx+c=0$$

$a=3$
$b=-5$
$c=-1$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2-4(3)(-1)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{25+12}}{6}$$

$$= \frac{5 \pm \sqrt{37}}{6} \quad \leftarrow$$

Completing the square:

$$\boxed{3x^2-5x-1=0}$$

$$x^2-\frac{5}{3}x-\frac{1}{3}=0$$

$$3\left[\underbrace{x^2-\frac{5}{3}x}\right] \underline{-1}=0$$

$$3\left[\left(x-\frac{1}{2}\cdot\frac{5}{3}\right)^2 \bigcirc\!\!\!\!- \left(\frac{1}{2}\cdot\frac{5}{3}\right)^2\right]-1=0$$

$$3\left[\left(x-\frac{5}{6}\right)^2-\left(\frac{5}{6}\right)^2\right]-1=0$$

→ $$3\left(x-\frac{5}{6}\right)^2-3\left(\frac{5}{6}\right)^2-1=0$$

$$3\left(x-\frac{5}{6}\right)^2-\cancel{3}\left(\frac{25}{\underset{12}{\cancel{36}}}\right)-1=0$$

$$3\left(x-\frac{5}{6}\right)^2\boxed{-\frac{25}{12}-1}=0$$

$$-\frac{25}{12}-1=-\frac{25}{12}-\frac{12}{12}$$
$$=-\frac{25-12}{12}$$
$$=-\frac{37}{12}$$

$$3\left(x-\frac{5}{6}\right)^2-\frac{37}{12}=0$$

$$3\left(x-\frac{5}{6}\right)^2=\frac{37}{12}$$

$\div 3$ $\qquad$ $\div 3$

$$\left(x-\frac{5}{6}\right)^2=\frac{37}{36}$$

$$\frac{1}{3}\left(\frac{37}{12}\right)=\frac{37}{36}$$

$$x-\frac{5}{6}=\pm\sqrt{\frac{37}{36}}$$

$$x-\frac{5}{6}=\pm\frac{\sqrt{37}}{6}$$

$$x=\frac{5}{6}\pm\frac{\sqrt{37}}{6}$$

$$x=\frac{5\pm\sqrt{37}}{6}$$

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$$

$$\sqrt{\frac{37}{36}}=\frac{\sqrt{37}}{\sqrt{36}}$$
$$=\frac{\sqrt{37}}{6}$$

Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Example ② $x^2+2x+2=0$

$a = 1$

$b = 2$

$c = 2$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

$$= \frac{-2 \pm \sqrt{4}\sqrt{-1}}{2}$$

$$= \frac{\boxed{-2} \pm 2\sqrt{-1}}{2}$$

$$= \frac{\cancel{2}(-1 \pm \sqrt{-1})}{\cancel{2}}$$

$$\underline{a} + \underline{ib}$$

$$= -1 \pm \underbrace{\sqrt{-1}}_{} = -1 \pm i$$

imaginary numbers, i

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$ ← discriminant

$$\rightarrow \quad D = 0 = b^2 - 4ac$$

$$x = \frac{-b \pm 0}{2a} = \boxed{-\frac{b}{2a}}$$

The discriminant of a quadratic equation $ax^2+bx+c=0$ $(a \neq 0)$ is defined by

$$D = b^2 - 4ac$$

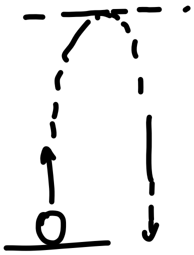
(term under the square root in the quadratic formula)

① If $\boxed{D>0}$ then you have two distinct real solutions

② If $\boxed{D=0}$ then you have only one real solution

③ If $\boxed{D<0}$ then you have no real solution

Example Projectile paths



We throw a ball upwards with an initial speed of v_0 ft/s and it reaches a height h after t seconds. The formula that models this motion is

t is time

$$h = -16t^2 + \underset{\substack{\uparrow \\ \text{constant}}}{v_0} t$$

(a) When does the ball reach the ground?

find t Ground is $h=0$

$$0 = -16t^2 + v_0 t$$

$$0 = \boxed{t}\underbrace{(-16t + v_0)}$$

$t = 0$ ↗ or

$$\begin{array}{rr} & +16t \quad +16t \\ \end{array}$$

$$\boxed{-16t + v_0 = 0}$$

$$v_0 = 16t$$

$t = \frac{V_0}{16}.$

Solve for t : $t = 0$ $\boxed{t = \frac{V_0}{16}}$

(b) When does the ball reach a height of 6400 ft

$V_0 = 800$ ft/s.

$$h = -16t^2 + V_0 t = -16t^2 + 800t$$

$(at^2 + bt + c = 0)$

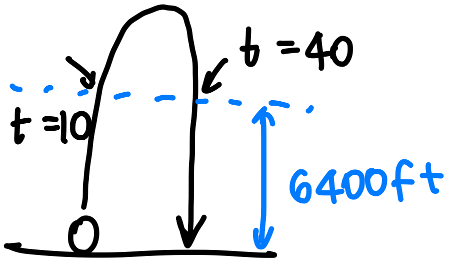
$$6400 = -16t^2 + 800t$$

$$\underline{16}t^2 - \underline{800}t + 6\underline{400} = 0$$

$$t^2 - 50t + 400 = 0$$

$$(t-40)(t-10) = 0$$

$$t = 40 \text{ or } t = 10.$$



(c) When does it reach a height of 12000ft ?

$$h = -16t^2 + 800t$$

$$12000 = -16t^2 + 800t$$

$$16t^2 - 800t + 12000 = 0$$

$$t^2 - 50t \underbrace{+750}=0$$

$D = b^2 - 4ac < 0$
then no real solution

$b^2 - 4ac$ ↙

$(-50)^2 - 4(1)(750)$

$= 250 - 3000$

$= -2750 < 0$

$$t = \frac{50 \pm \sqrt{(-50)^2 - 4(1)(750)}}{2(1)}$$

$$= \frac{50 \pm \sqrt{250 - 3000}}{2}$$

$$= \frac{50 \pm \sqrt{-2750}}{2}$$

no
real solutions

The ball never reaches 12000 ft.

Other types of equations.

Solve for x: $\frac{3}{x} - \frac{2}{\underset{\sim\sim}{x-3}} = \frac{-12}{\underset{\sim\sim}{x^2-9}}$ $\ast$

$$\frac{3}{x} - \frac{2}{x-3} = \frac{-12}{(x-3)(x+3)}$$

LCD
lowest
common
denominator
is $\boxed{x(x-3)(x+3)}$

↑

↓ multiply by what the denominator is missing

$$\rightarrow \frac{3(x-3)(x+3)}{x(x-3)(x+3)} - \frac{2x(x+3)}{x(x-3)(x+3)} = \frac{-12x}{x(x-3)(x+3)}$$

$$3(x-3)(x+3) - 2x(x+3) = -12x$$

$$3(x^2 - 9) - 2x^2 - 6x = -12x$$

$$\underline{3x^2} - 27 - \underline{2x^2} \underset{\sim\sim}{-6x} \underset{\sim\sim}{+12x} = 0$$

$$x^2 + 6x - 27 = 0$$

$$(x-3)(x+9) = 0$$

$\boxed{\cancel{x = 3}}$ $\quad x = -9$

not a solution.

Example. Solve for x the following

$$2x = 1 - \sqrt{2-x}$$

$$2x - 1 = -\sqrt{2-x} \quad \rightarrow \left(-(2x-1)\right)^2 = \left(\sqrt{2-x}\right)^2$$

square both sides

$$(2x-1)^2 = \left(-\sqrt{2-x}\right)^2$$

$$4x^2 - 4x + 1 = 2 - x$$

1·1 $\quad (4x+1)(x-1)=0$

4·1

$$\rightarrow \underline{4}x^2 \boxed{-3x} - \underline{1} = 0$$

2·2

$$(4x+1)(x-1) = 0.$$

$$\boxed{x = -\frac{1}{4}} \text{ or } \boxed{x = 1}$$

$x = -\frac{1}{4}$ plug it in into original equation $2x = 1 - \sqrt{2-x}$

$$LHS = 2\left(-\frac{1}{4}\right) = -\frac{1}{2}$$

$$RHS = 1 - \sqrt{2 - \left(-\frac{1}{4}\right)} = 1 - \sqrt{2 + \frac{1}{4}} = 1 - \sqrt{\frac{9}{4}} = 1 - \frac{3}{2}$$

$$= -\frac{1}{2}$$

$$= LHS$$

✓ $x = -\frac{1}{4}$ is a solution

$x=1$ plug it in into $2x = 1-\sqrt{2-x}$

$$LHS = 2(1) = 2$$

$$RHS = 1 - \sqrt{2-1} = 1-1=0 \neq 2$$

$$LHS \neq RHS$$

Thus $x=1$ is not a solution.

Inequalities (Section 1.8)

Starting with linear inequalities.

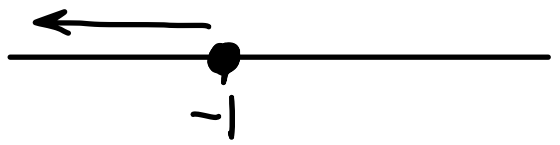
$2x+5=3$ Equality

$$2x=-2$$
$$x=-1$$

Inequality $2x+5 \leq 3$

$$2x \leq -2$$
$$x \leq -1$$

number line



notation

● implies $\geq$ or $\leq$

○ implies $>$ or $<$

Example. $\underline{3x < 9x+8}$

Way 1 $-6x < 8$

$$0 < 6x+8$$
$$-8 < 6x$$

Way 2

$$-8 < 6x$$

(÷6) (÷6)

$$x > \frac{8}{-6} \qquad\qquad -\frac{8}{6} < x$$

$$\boxed{x > \frac{4}{-3}} \longleftrightarrow \boxed{-\frac{4}{3} < x}$$

$$x > -\frac{4}{3}$$

Reminders: ① Homework 2 due tonight at midnight

② Quiz 2 during your recitation this week

③ Office hours on Zoom today at 3:30 - 4:30 pm (link in Brightspace under Course info).

Quiz 2 will include
- Section 1.4 (Rational expressions)
- Section 1.5 (Equations)

Solving a pair of inequalities

Example. $4 < 2x - 3 \leq 8$

Find the values for x.

Step 1 Add 3 everywhere

$$7 < 2x \leq 11$$

Step 2 Divide by 2 throughout

$$\frac{7}{2} < x \leq \frac{11}{2}$$

This in interval notation is $\left(\frac{7}{2}, \frac{11}{2}\right]$

open parenthesis

⇒ strict inequality

square bracket

⇒ less than or equal.

Note: $-2 \le x \le 4$: $[-2, 4]$

$x > 3$: $(3, \infty)$

GUIDELINES FOR SOLVING NONLINEAR INEQUALITIES

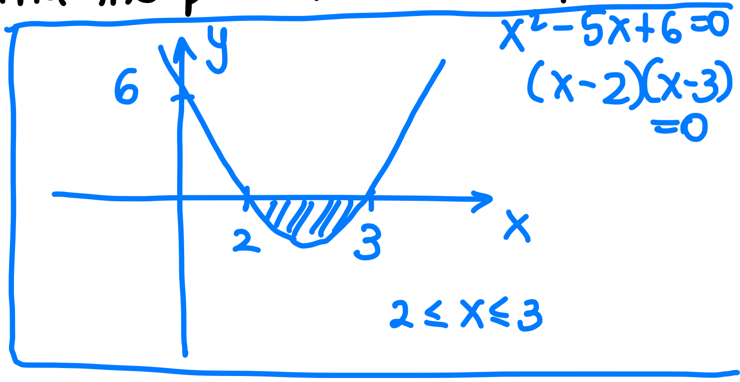
1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals that are determined by these numbers.
4. **Make a Table or Diagram.** Use **test values** to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Use the sign table to find the intervals on which the inequality is satisfied. Check whether the **endpoints** of these intervals satisfy the inequality. (This may happen if the inequality involves $\le$ or $\ge$.)

Example 1 Nonlinear inequalities.

$x^2 \le 5x - 6$ (Find the possible values of x).

$x^2 - 5x + 6 \le 0$

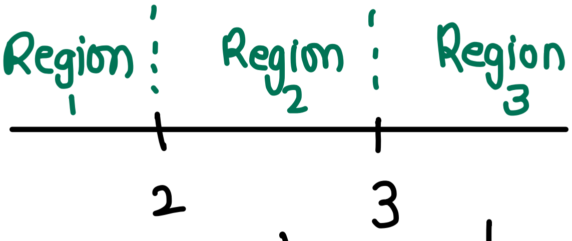
$\Rightarrow \underline{(x-2)(x-3)} \le 0$



$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$2 \le x \le 3$



sign of $(x-2)$	$-$	$+$	$+$
sign of $(x-3)$	$-$	$-$	$+$
sign of $(x-2)(x-3)$	$+$	$-$	$+$

↑ this is the number you think of in each region

$2 \leq x \leq 3$

the inequality we are trying to solve is $(x-2)(x-3) \leq 0$

Thus $(x-2)(x-3) \leq 0$ when $2 \leq x \leq 3$.

or in interval notation $[2,3]$.

This is from HW3

$-2 \leq 2x - 3 < 5$

$1 \leq 2x < 8$

$\frac{1}{2} \leq x < 4$

Interval notation.

$[\frac{1}{2}, 4)$

$-3 < 1 - 4x \leq 17$

$\rightarrow \boxed{-4 < -4x \leq 16}$

Divide by -4 but remember when you divide by a negative number the inequalities reverse.

$\frac{16}{-4} \leq x < \frac{-4}{-4}$

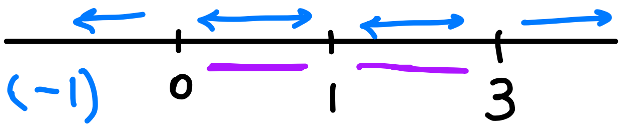
$-4 \leq x < 1$, $[-4, 1)$

Example Nonlinear inequality.

$$\boxed{x(x-1)^2(x-3) < 0}$$

$\uparrow \quad \uparrow \quad \uparrow$

(Repeated factor)



sign of x	✓ −	+	+	+
sign of $(x-1)^2$	✓ +	+	+	+
sign of $(x-3)$	✓ −	−	−	+
sign of $x(x-1)^2(x-3)$	+	−	−	+

$(-)(+)(-)=(+)$ $\quad$ $(+)(+)(-)=(-)$

$0<x<1$

$1<x<3$

The regions that satisfy $x(x-1)^2(x-3)<0$ are

$\boxed{(0,1)\cup(1,3)}$ $\quad$ $\xcancel{(0,3)}$

union

$0<x<1, \quad 1<x<3$

if $x(x-1)^2(x-3)>0$

$x<0, \quad x>3$

$(-\infty,0)\cup(3,\infty)$

Solving inequalities involving quotients

Example Solve for x the inequality $\dfrac{1+x}{1-x} \geq 1.$

STEP 1. Move everything to one side.

$$\frac{1+x}{1-x} - 1 \geq 0$$

$$\left(\frac{1+x}{1-x}\right) - \left(\frac{1-x}{1-x}\right) \geq 0$$

common mistakes

$$\cancel{\frac{1+x - 1-x}{1-x} \geq 0}$$

$$\frac{1+x-(1-x)}{1-x} \geq 0$$

$$\frac{\cancel{1}+x-\cancel{1}+x}{1-x} \geq 0$$

$$\frac{2x}{1-x} \geq 0 \qquad (\div 2)$$

$$\boxed{\frac{x}{\underline{1-x}} \geq 0}$$

$\textcircled{0}$ $\textcircled{1}$

sign of x	$-$	$+$	$+$
sign of $(1-x)$	$+$	$+$	$-$
sign of $\frac{x}{1-x}$	$\frac{(-)}{(+)} = -$	$(+)/(+) = \textcircled{+}$	$(+)/(-) = -$

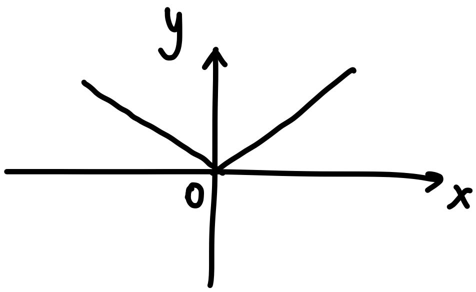
$$\boxed{\underline{0 \leq x < 1}}$$

check endpoints
0, 1

interval notation
$[0,1)$

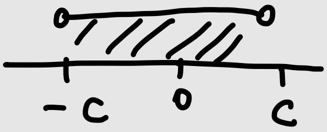
Absolute value inequalities

Aside. $y = |x| = \begin{cases} x, & x>0 \\ -x, & x \leq 0 \end{cases}$

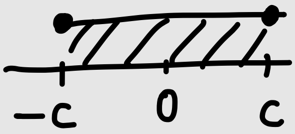


Properties

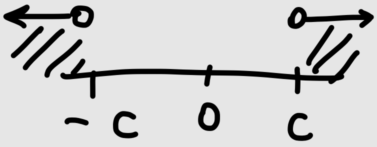
1. $|x| < c \iff -c < x < c$



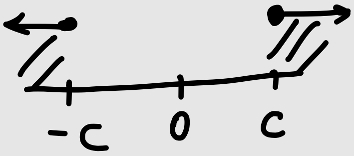
2. $|x| \le c \iff -c \le x \le c$



3. $|x| > c \iff x < -c$ or $c < x$



4. $|x| \geq c \iff x \le -c$ or $c \le x$



Example.

1. Solve $|x-5| < 2$

$$-2 < x-5 < 2$$

(compare to property 1)

Add 5 : $\boxed{3 < x < 7}$ or $(3,7)$

2. Solve $|x-10| < 3$

$$-3 < x-10 < 3$$

Add 10 : $7 < x < 13$

] closed.

3. Solve $\boxed{|3x+2| \geq 4}$

$$3x+2 \le -4$$

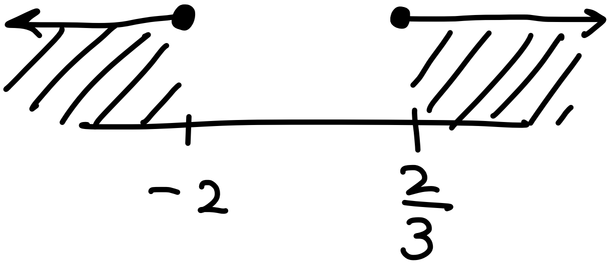
$$3x \le -6$$

$$x \le -2$$

$4 \le 3x+2$

$2 \le 3x$

$\frac{2}{3} \le x$



Interval notation $(-\infty,-2] \cup [\frac{2}{3},\infty)$

↑ it includes -2

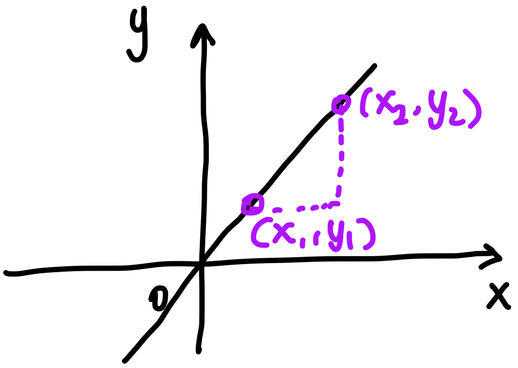
Lines (section 1.10)

Slope of a line

The slope m of a line that is not vertical and passes through the points $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ is given by

$$\boxed{\text{slope} = m = \frac{y_2-y_1}{x_2-x_1}} = \frac{\text{change in output}}{\text{change in input}} = \frac{\text{rise}}{\text{run}}$$

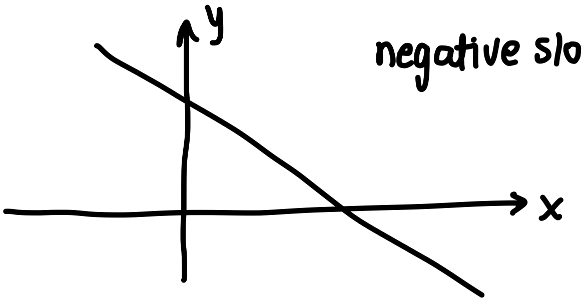
NB The slope of a vertical line is not defined.



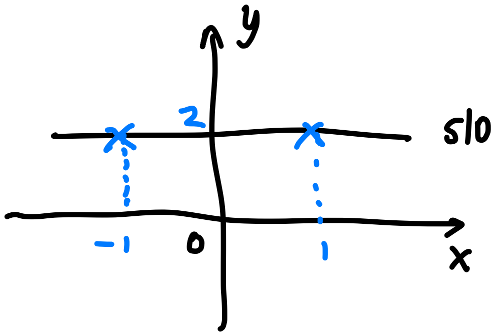
slope is positive

rise = change in y-coordinates

run = change in x-coordinates.



negative slope

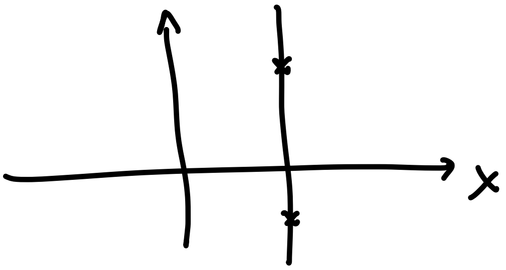


slope is 0

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2-2}{1-(-1)}$$

$$= \frac{0}{2} = 0$$



slope is undefined

Finding the slope of a line from two points.

e.g. $P = (1, 3)$ and $Q = (2, 4)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-3}{2-1} = \frac{1}{1} = 1$$

POINT-SLOPE FORMULA FOR THE EQUATION OF A LINE

slope

$$\boxed{y - y_1 = m(x - x_1)}$$

(x_1, y_1) is the given point on the line

Example. Find the equation of a line that passes through $(1, -3)$ and $(2, 0)$.

(x_1, y_1) (x_2, y_2)

$$m = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-3)}{2-1} = 3$$

Use the point (1, −3).

$$y - y_1 = m(x - x_1)$$

$$y - (-3) = 3(x - 1)$$

$$y + 3 = 3x - 3$$

$$\boxed{y = 3x - 6}$$

to check plug in the other point.

e.g. when $x = 2$

$$y = 3(2) - 6$$

$$= 6 - 6$$

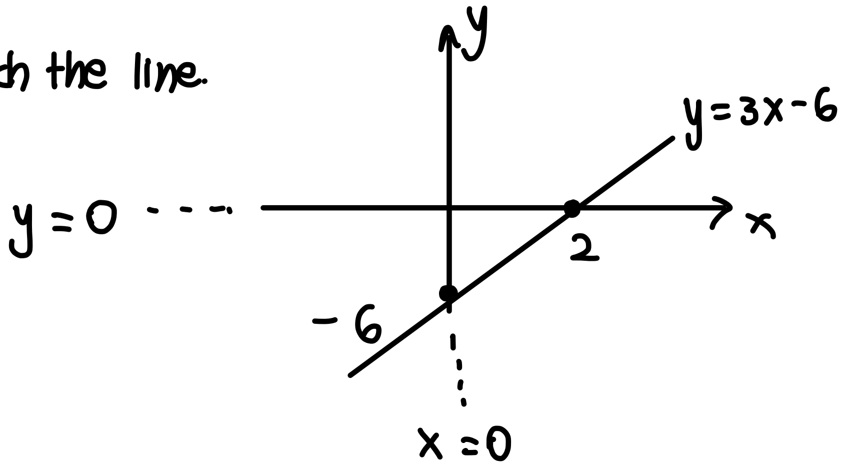
$$= 0 \checkmark$$

Use the point (2, 0)

$$y - 0 = 3(x - 2)$$

$$\boxed{y = 3x - 6}$$

Sketch the line.



x-intercept : $y = 0$

$$0 = 3x - 6$$

$$3x = 6$$

$$x = 2$$

y-intercept : $x = 0$

$$y = 3(0) - 6 = -6$$

<u>Example</u> Find the equation of a line that satisfies the following:

slope $= \frac{2}{5}$ and $\underbrace{\text{y-intercept}}_{\Downarrow\, x = 0}$ is 4. Coordinate is $(\underset{\substack{\uparrow \\ x_1}}{0}, \underset{\substack{\uparrow \\ y_1}}{4})$

Recall $y - \underset{\checkmark}{y_1} = \underset{\checkmark}{m}(x - \underset{\checkmark}{x_1})$

$$y - 4 = \frac{2}{5}(x - 0)$$

$$y - 4 = \frac{2}{5}x$$

$$y = \frac{2}{5}x + 4$$

SLOPE - INTERCEPT FORM OF THE EQUATION OF A LINE

An equation with slope m and y-intercept b is given by

$$\boxed{y = \underbrace{m}_{\text{slope}}x + \underbrace{b}_{\text{y-intercept (where the line crosses the y-axis)}}}$$

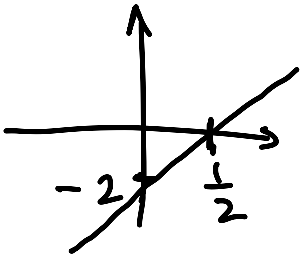
Note

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = mx - mx_1$$

$$y = mx \underbrace{- mx_1 + y_1}_{\text{just a number}} = mx + b$$

Example ① Find the equation of the line with slope 4 and y-intercept -2



$$y = \overset{\text{slope}}{m}x + b \leftarrow \text{y-intercept}$$

$$y = 4x - 2$$

x - Intercept : $y = 0$

$$0 = 4x - 2$$

$$2 = 4x$$

$$x = \frac{1}{2}$$

② Find both the slope and the y-intercept of the line

$$\boxed{3y - 2x = 1}$$

Step 1 Rearrange the equation so that it's of the form $y = mx + b$

$$3y = 2x + 1$$

$$y = \frac{2x+1}{3}$$

$$y = \boxed{m}x + \boxed{b}$$
$$y = \boxed{\frac{2}{3}}x + \boxed{\frac{1}{3}}$$

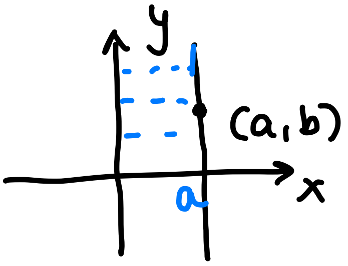
Step 2 Compare with $y = mx + b$ and read off m and b.

$$m = \frac{2}{3}, \quad b = \frac{1}{3}$$

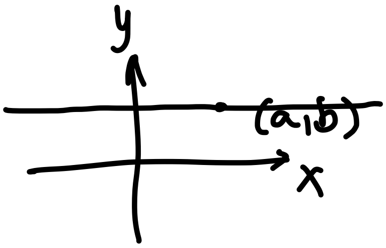
slope y-intercept

Vertical and horizontal lines

- An equation of a vertical line through the point (a,b) is $\boxed{x = a}$



- An equation of a horizontal line through the point (a,b) is $\boxed{y = b}$



Note. If you are given a horizontal line $y = -4$ then the y-intercept is -4.

GENERAL FORM OF THE EQUATION OF A LINE

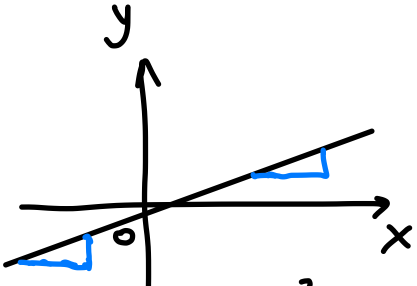
The graph of every linear equation

$$\boxed{Ax + By + C = 0}$$ where A, B are both non-zero

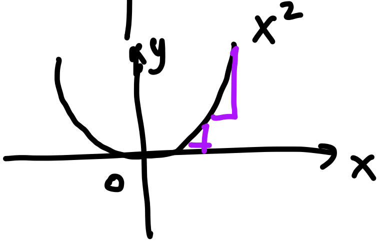
is a line.

$$Ax + By + C = 0 \Rightarrow \quad By = -Ax - C$$

$$y = -\frac{A}{B}x - \frac{C}{B} = mx + b$$



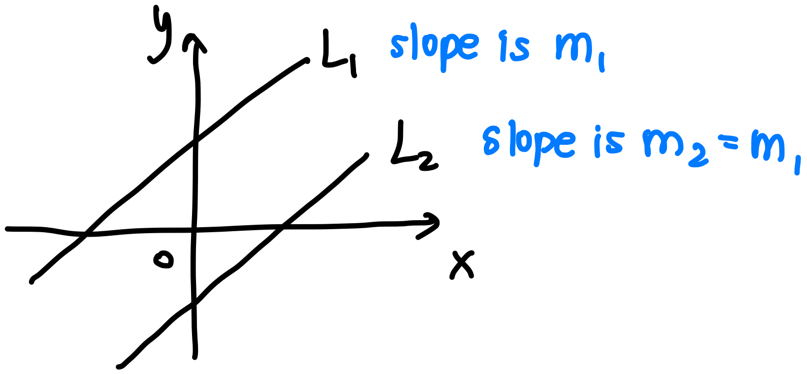
linear ⇒ constant slope



quadratic ⇒ does <u>not</u> have a constant slope, steepness of graph is changing.

PARALLEL AND PERPENDICULAR LINES

<u>Parallel lines</u>. Two nonvertical lines are parallel if and only if their slopes are the same



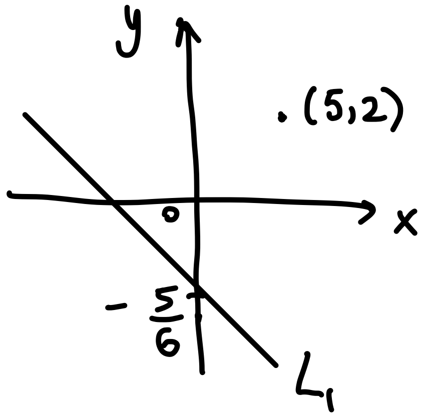
Find an equation of a line that is parallel to $4x + 6y + 5 = 0$ that also passes through $(5,2)$.

Write $4x + 6y + 5 = 0$ in the form of $y = mx + b$

$$6y = -4x - 5$$

$$y = -\frac{4}{6}x - \frac{5}{6} = -\frac{2}{3}x - \frac{5}{6}$$

$$m_1 = -\frac{2}{3}$$



L_2 has the same slope as L_1

$$\Rightarrow \quad m_2 = m_1 = -\frac{2}{3}.$$

Given $\underset{(x_1, y_1)}{(5,2)}$ and $m_2 = -\frac{2}{3}$ we can use $y - y_1 = m(x - x_1)$

$$\Rightarrow \quad y - 2 = -\frac{2}{3}(x - 5)$$

$$y - 2 = -\frac{2}{3}x + \frac{10}{3}$$

$$y = -\frac{2}{3}x + \underbrace{\frac{10}{3} + 2}$$

$$\boxed{y = -\frac{2}{3}x + \frac{16}{3}} \quad \star$$

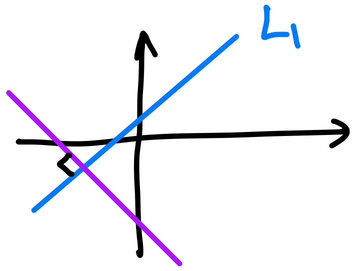
General form of the equation: $Ax + By + C = 0$

$$3y = -2x + 16$$

$$\boxed{3y + 2x - 16 = 0}$$

Perpendicular lines

Two lines with slopes m_1 and m_2 are perpendicular if they satisfy $m_1 m_2 = -1 \quad \Rightarrow \quad \boxed{m_2 = -\frac{1}{m_1}}$



Example Find the equation of the line that is perpendicular to $4x + 6y + 2 \; 0$ that passes through $(1, 2)$.

L_1: $6y = -4x - 2$

$$y = -\frac{4x}{6} - \frac{2}{6} = -\frac{2}{3}x - \frac{1}{3}$$

$m_1 = -\frac{2}{3}$

What is m_2? $\underline{m_2 = -\frac{1}{m_1}} = -\frac{1}{\left(-\frac{2}{3}\right)} = \frac{3}{2}$

$$y - y_1 = m(x - x_1)$$

We have $(x_1, y_1) = (1, 2)$ and $m_2 = \frac{3}{2}$

$$y - 2 = \frac{3}{2}(x - 1)$$

$$y - 2 = \frac{3}{2}x - \frac{3}{2}$$

$$y = \frac{3}{2}x - \frac{3}{2} + 2$$

$$\boxed{y = \frac{3}{2}x + \frac{1}{2}}$$

check: plug in $x=1$ and you should get $y=2$

$$y = \frac{3}{2}(1) + \frac{1}{2}$$

$$= \frac{3}{2} + \frac{1}{2}$$

$$= 2 \quad \checkmark$$

<u>Example 2</u>.

Determine whether the lines are parallel or perpendicular.

L_1: $2x - 3y = 10$ and $3y - 2x - 7 = 0$: L_2

Write both in the form $y = mx + b$

L_1: $-3y = 10 - 2x$

$$y = -\frac{10}{3} + \frac{2}{3}x$$

L_2: $3y = 2x + 7$

$$y = \frac{2}{3}x + \frac{7}{3}$$

$$y = \frac{2}{3}x - \frac{10}{3}$$

$$m_1 = \frac{2}{3}$$

$$m_2 = \frac{2}{3}$$

Since $m_1 = m_2$, the lines are parallel.

$$y = \boxed{-\frac{3}{2}}x + 8$$

$$m_2 = -\frac{3}{2}$$

$$m_1 \cdot m_2 = \frac{2}{3}\left(-\frac{3}{2}\right) = -1$$

HW 3

① c) Solve for x.

$$x^{\boxed{\frac{1}{2}}} + 3x^{\boxed{-\frac{1}{2}}} = 10x^{\boxed{-3/2}} \quad \leftarrow$$

Hint : Let $\boxed{t = x^{1/2}}$ and rewrite the entire equation in terms of t.

$$(a^m)^n = a^{mn}$$

$$x^{1/2} = t$$

$$x^{-3/2} = (x^{1/2})^{-3} = t^{-3}$$

$$\boxed{\begin{aligned} x^{1/2} &= t \\ x^{-\frac{1}{2}} &= t^{-1} \\ x^{-\frac{3}{2}} &= t^{-3} \end{aligned}}$$

↓

substitute this in the original equation $x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} = 10x^{-\frac{3}{2}}$

$$t + 3t^{-1} = 10t^{-3}$$

$$t + \frac{3}{t} = \frac{10}{t^3}$$

Multiply throughout by t^3:

$$t^4 + 3t^2 = 10 \quad \leftarrow$$

$$t^3\left(t + \frac{3}{t}\right) = \cancel{t^3}\left(\frac{10}{\cancel{t^3}}\right)$$

$$t^4 + \frac{3\cancel{t^3}^{\,2}}{\cancel{t}} = 10$$

$$t^4 + 3t^2 = 10$$

$$\underset{u^2}{t^4} + 3\underset{u}{\boxed{t^2}} - 10 = 0 \qquad au^2 + bu + c = 0$$

Let $\boxed{u = t^2}$ ⇒ $u^2 + 3u - 10 = 0$

$$(u+5)(u-2) = 0$$

$$\cancel{u = -5} \qquad u = 2$$

$u = -5$ ⇒ $-5 = t^2$ not possible

$u = 2$ ⇒ $2 = t^2$ ⇒ $t = \pm\sqrt{2}$

We also had $t = x^{1/2} = \sqrt{x}$

$$\sqrt{2} = \sqrt{x} \Rightarrow \boxed{x = 2}$$

$$\sqrt{2} = \sqrt{x}$$

$$(\cancel{\sqrt{\ }}2)^{\cancel{2}} = (\cancel{\sqrt{\ }}x)^{\cancel{2}}$$

$$2 = x$$

Section 2.1: Functions

Definition: A function f is a rule that assigns to each element x in a set A exactly one element, which we call $f(x)$, into set B.

e.g. $f(x) = x^2$

Domain: It is set of all possible input values for the function

Range: It is the set of all possible output values $f(x)$

$$\text{range} = \{ f(x) \text{ such that } \underset{\uparrow \text{ it is an element of}}{x \in A} \}$$

$$\overset{\text{dependent variable.}}{y} = f(\underset{\text{independent variable}}{x})$$

Evaluating functions

Example. $f(\overset{\swarrow}{\underset{\uparrow}{x}}) = 2\overset{\swarrow}{\underset{\uparrow}{x}}^2 + 5\overset{\swarrow}{\underset{\uparrow}{x}} - 1$

(a) $f(\underline{\underline{a}}) = 2a^2 + 5a - 1$

(b) $f(-a) = 2(-a)^2 + 5(-a) - 1 = 2a^2 - 5a - 1$

(c) $\dfrac{\overset{\checkmark}{\boxed{f(a+h)}} - \boxed{f(a)}}{h}$

$$\begin{aligned} f(\underline{a+h}) &= 2(a+h)^2 + 5(a+h) - 1 \\ &= 2(a^2 + 2ah + h^2) + 5a + 5h - 1 \\ &= 2a^2 + 4ah + 2h^2 + 5a + 5h - 1 \end{aligned}$$

Thus $\dfrac{f(a+h) - f(a)}{h} = \dfrac{\cancel{2a^2} + 4ah + 2h^2 + \cancel{5a} + 5h - \cancel{1} - (\cancel{2a^2} + \cancel{5a} - \cancel{1})}{h}$

$$= \frac{4ah + 2h^2 + 5h}{h}$$

$$= \frac{\cancel{h}(4a + 2h + 5)}{\cancel{h}}$$

$$= 4a + 2h + 5$$

$f(a+h) \overset{?}{=} f(a) + f(h)$

Let $f(x) = x^2$

$f(a+h) = (a+h)^2$ $\neq$

$f(a) + f(h) = a^2 + h^2$

$$\begin{aligned} (a+h)^2 &= (a+h)(a+h) \\ &= a^2 + \underline{2ah} + h^2 \end{aligned}$$

$(2+3)^2 = 5^2 = 25$

$2^2 + 3^2 = 4 + 9 = 13$

$(a+b)^2 \neq a^2 + b^2$

$(a+b)(a-b) = a^2 - b^2$

Domain and range

Examples.

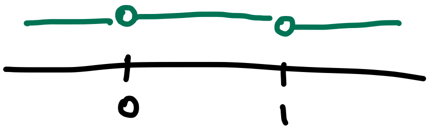
① Find the domain of each function

(a) $f(x) = \frac{1}{x^2 - x} = \frac{1}{x(x-1)}$ So when $x=0$ or $x=1$ the denominator equals to 0

The domain of f is $\{x \mid x \neq 0, x \neq 1\}$

the x-values such that $x \neq 0$ or $x \neq 1$

or in interval notation: $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$



(b) $g(x) = \sqrt{25 - x^2}$

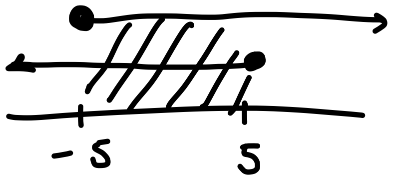
This is a function when $25 - x^2 \geq 0 \rightarrow \underbrace{(5-x)(5+x) \geq 0}$

$25 \geq x^2$

$x^2 \leq 25$

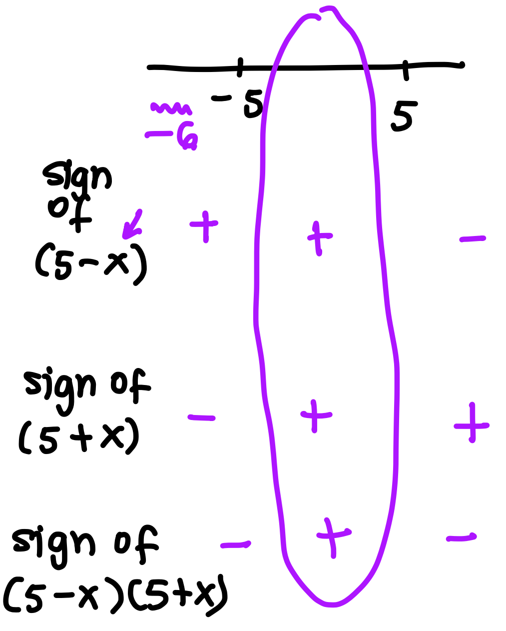
⇒ $x \leq 5$

or $x \geq -5$



The domain is $-5 \leq x \leq 5$

Interval notation: $[-5, 5]$



⇒ $-5 \leq x \leq 5$

(c) $h(w) = \frac{w}{\sqrt{w+1}}$

What should w+1 satisfy so that $\sqrt{w+1}$ is valid?

$\rightarrow$ ~~$w+1 \geq 0$~~

when $w+1 = 0$ then $w = -1$ and then you are dividing by 0

$\Rightarrow w+1 > 0$

$w+1 \geq 0$

$\downarrow$

it also includes $\boxed{w+1 = 0}$

$\Rightarrow w = -1$

cannot be included

$\frac{-1}{\sqrt{-1+1}} = \frac{-1}{\sqrt{0}}$

(d) $z(x) = \frac{1}{\sqrt{x}}$

Domain: $x \geq 0$

$x > 0$ ✓

$z(0) = \frac{1}{\sqrt{0}}$ not OK

(e) $d(x) = \sqrt{x}$

Domain: $x \geq 0$

$d(0) = \sqrt{0} = 0$

From before: $h(w) = \frac{w}{\sqrt{w+1}}$

Domain: ~~$w+1 \geq 0$~~ $w+1 > 0$

~~$w \geq -1$~~ $w > -1$

if $w = -1$ $h(-1) = \frac{-1}{\sqrt{-1+1}} = \frac{-1}{\sqrt{0}}$ ×

$g(w) = \sqrt{w+1}$

Domain: $w+1 \geq 0$

$\Rightarrow w \geq -1$

$f(x) = \sqrt[3]{x}$

$f(x) = \frac{1}{\sqrt[3]{x}}$ $(-\infty, 0) \cup (0, \infty)$

Piecewise - defined functions

Example. ①
$$\underline{\underline{f(x)}} = \begin{cases} x^2 & \text{if } \\ x+1 & \text{if } \end{cases} \boxed{\begin{array}{l} x<0 \\ x \geq 0 \end{array}} \begin{array}{l} \text{quadratic} \\ \text{linear} \end{array}$$

Evaluate

(a) $f(\overset{x}{\underline{-2}}) = (-2)^2 = 4$

(b) $f(-1) = (-1)^2 = 1$

(c) $f(0) = 0+1 = 1$

(d) $f(1) = 1+1 = 2$

(e) $f(2) = 2+1 = 3$

②
$$f(x) = \begin{cases} x^2+2x & \text{if } \underline{x \leq -1} \\ x & \text{if } \underline{-1 < x \leq 1} \\ -1 & \text{if } \underline{x > 1} \end{cases}$$

(a) $f(-4) = (-4)^2 + 2(-4) = 16 - 8 = 8$

(b) $f(-3/2) = (-\frac{3}{2})^2 + \cancel{2}(-\frac{3}{\cancel{2}}) = \frac{9}{4} - 3 = -\frac{3}{4}$

(c) $f(-1) = (-1)^2 + 2(-1) = 1-2 = -1$

(d) $f(0) = 0$

(e) $f(25) = -1$

Example. Evaluate $\dfrac{f(a+h) - f(a)}{h}$ where $h \neq 0$

$f(x) = 3x^2+1$

$f(a+h) = 3(a+h)^2+1 = 3(a^2+2ah+h^2)+1 = 3a^2+6ah+3h^2+1$

$f(a) = 3a^2+1$

Altogether $\dfrac{f(a+h) - f(a)}{h} = \dfrac{3a^2+6ah+3h^2+1-(3a^2+1)}{h}$

$$= \frac{\cancel{3a^2} + 6ah + 3h^2 \cancel{+1} - \cancel{3a^2} \cancel{-1}}{h}$$

$$= \frac{6ah + 3h^2}{h}$$

$$= \frac{3\cancel{h}(2a+h)}{\cancel{h}}$$

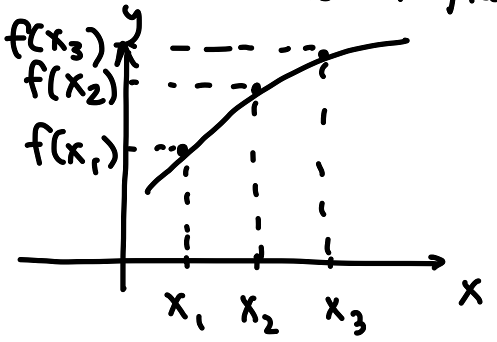
$$= 3(2a+h).$$

2.2. Graphs of functions

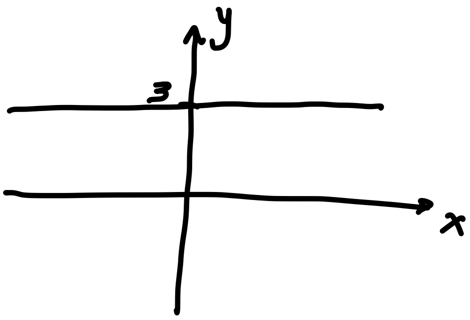
If f is a function with a domain A, then the graph of the function is the set of all ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

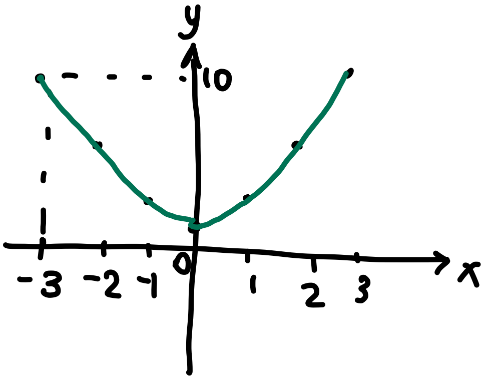
plotted in the coordinate plane.



Plot $f(x) = 3$



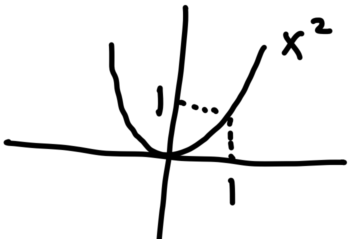
Plot $f(x) = x^2+1$

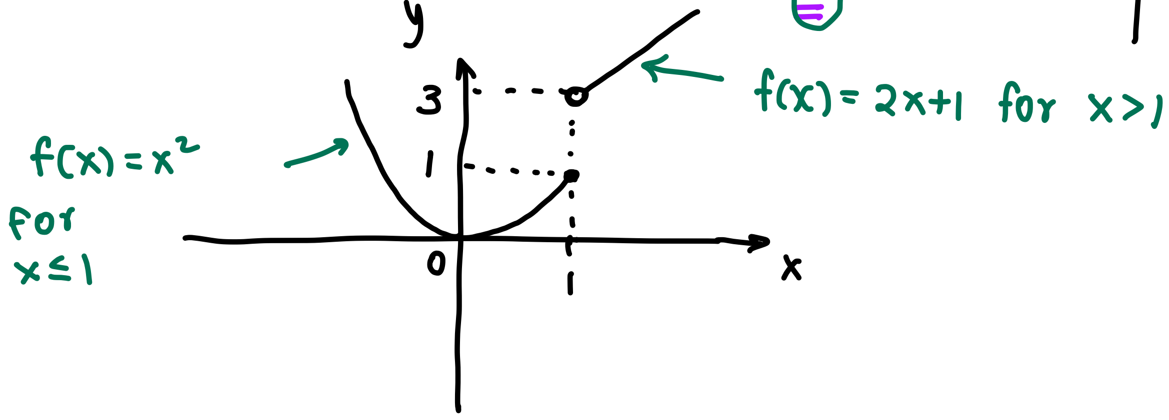


x	y
-3	$(-3)^2+1 = 9+1$ $=10$
-2	5
-1	2
0	1
1	2
2	5
3	10

Graphing piecewise-defined functions

Example ① $f(x) = \begin{cases} x^2 & \text{if } x \leq \boxed{1} \\ 2x+1 & \text{if } x > \boxed{1} \end{cases}$





② $f(x) = \begin{cases} 2x+3 & \text{if } \underline{x < -1} \ ✓ \\ \boxed{3-x} & \text{if } x \geq -1 \end{cases}$

y-intercept $-x+3$

$mx+b$

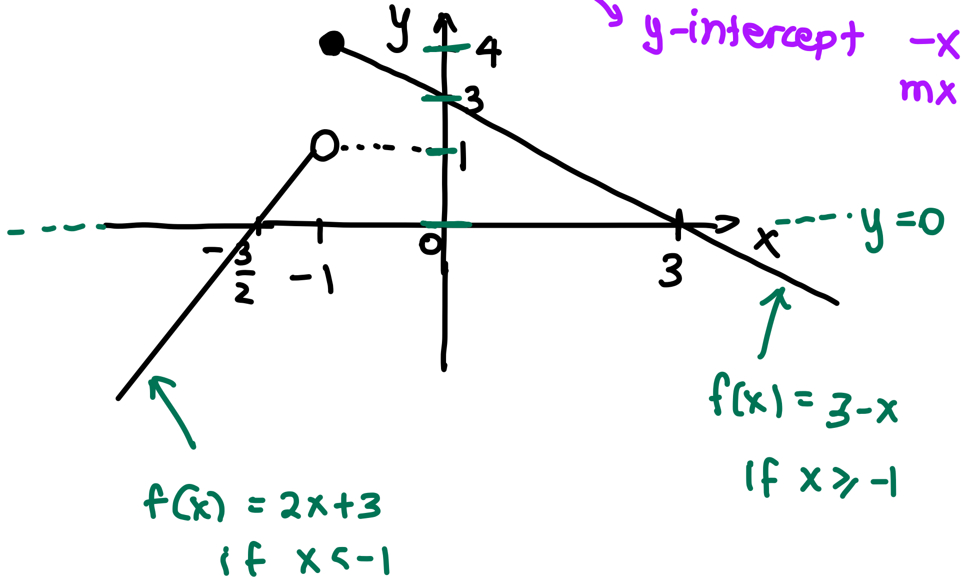
$b=3$

x-intercept

⇒ x-value where $y=0$

$0 = 2x+3 \Rightarrow 2x=-3$

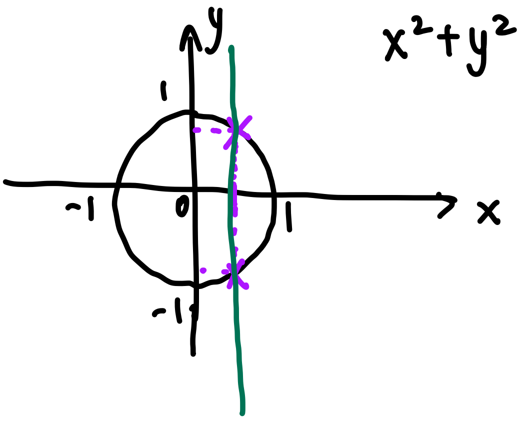
$x = -\frac{3}{2}.$



x-intercept of $f(x)=3-x$

$0 = 3-x$

$x = 3$

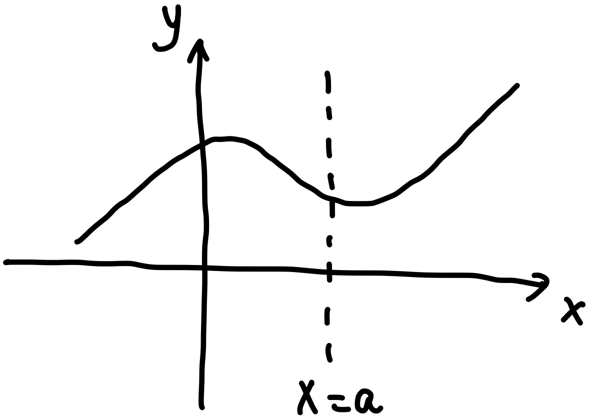


$x^2+y^2=1 \Rightarrow y^2=1-x^2$

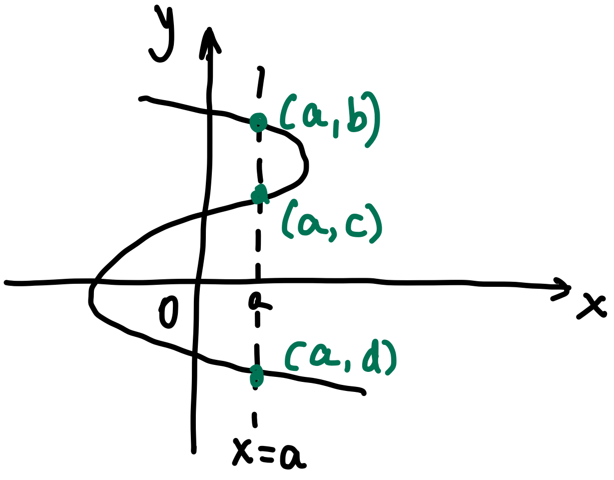
$y = \pm\sqrt{1-x^2}$

Vertical line test

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.



Graph is a function



Not a graph of a function.

Which equations represent functions?

An equation $y = f(x)$ defines a function that gives one value of y for each value of x.

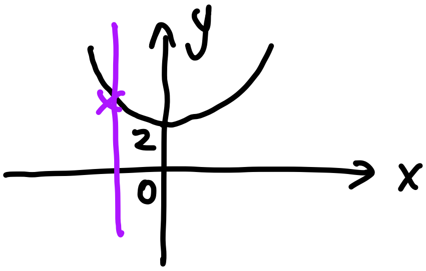
Example

① Does the equation define y as a function of x?

(a) $y - x^2 = 2$

$y = x^2 + 2$

Since this equation gives one value of y for each value of x this defines y as a function of x.



(b) $x^2+y^2=9$

$y^2=9-x^2$

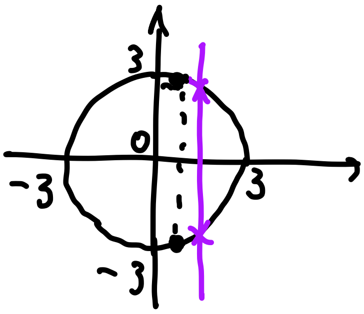
$y=\pm\sqrt{9-x^2}$

check if $x=1$

$y=\pm\sqrt{9-1^2}=\pm\sqrt{8}$

$-\sqrt{8}\ ,\ \sqrt{8}$

Since for each value of x we get more than one value of y this is <u>not</u> a function.



(c) $\boxed{\sqrt{y}-x=5}$

$\sqrt{y}=5+x$

e.g check $x=-6$

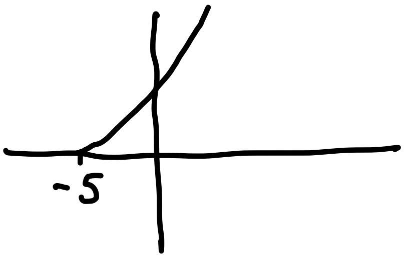
$\sqrt{y}-(-6)=5$

$\sqrt{y}+6=5$

$\sqrt{y}=-1$

$y=(5+x)^2$
$=(x+5)^2$

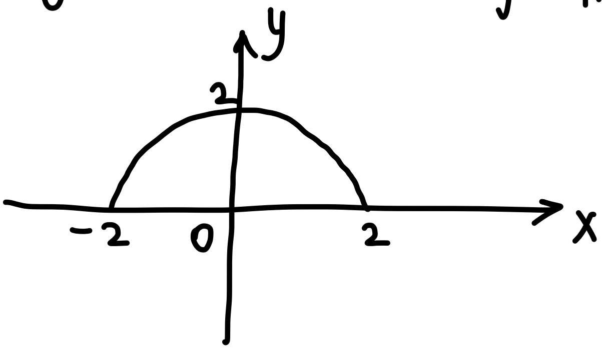
$\pm\sqrt{y}=x+5$



This is a function.

Section 2.3: Getting information from the graph

Finding the domain and range from graphs.



Domain: $[-2,2]$

Range: $[0,2]$

Values of a Function The graph of a function h is given.

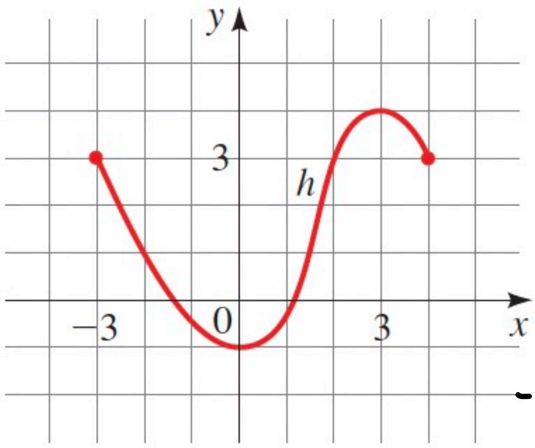
(a) Find $h(-2)$, $h(0)$, $h(2)$, and $h(3)$.

(b) Find the domain and range of h.

(c) Find the values of x for which $h(x) = 3$.

(d) Find the values of x for which $h(x) \leq 3$.

~~**(e)**~~ Find the net change in h between $x = -3$ and $x = 3$.



$h(x) \textcircled{\leq} 3$

$h(x) = 3$

(a) $h(-2) = 1$

$h(0) = -1$

$h(2) = 3$

$h(3) = 4$

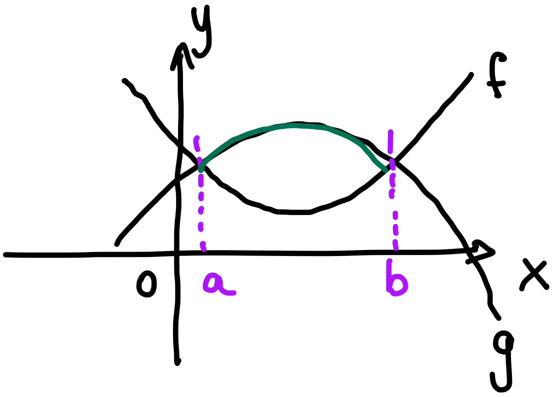
(b) Domain: $[-3, 4]$, Range: $[-1, 4]$

(c) For $h(x) = 3$ we have $x = -3, 2, 4$

(d) For $h(x) \leq 3$ the interval is $[-3, 2]$

SOLVING EQUATIONS AND INEQUALITIES GRAPHICALLY

- The solution(s) of the equation $f(x) = g(x)$ are the values of x where the graphs of f and g intersect.
- The solution(s) of the inequality $f(x) < g(x)$ are the values of x where the graph of g is higher than the graph of f.



The solution to $f(x)<g(x)$ is $a<x<b$.

(a,b). interval notation

The solutions to $f(x)=g(x)$ are $x=a$ and b.

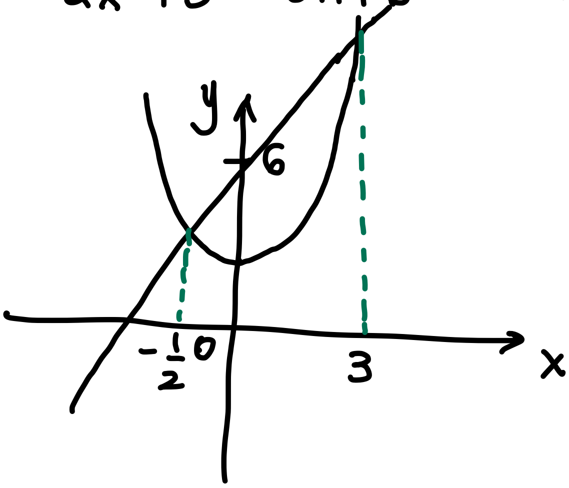
Example.

Solve the following graphically

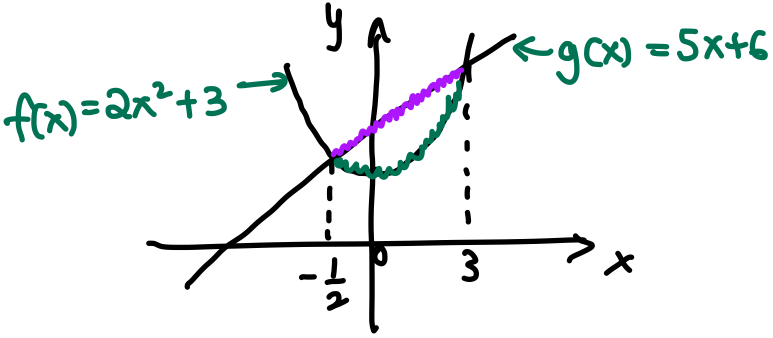
(a) $2x^2+3 = 5x+6 \longrightarrow 2x^2-5x-3=0$

$(2x+1)(x-3)=0$

$\boxed{x=-\frac{1}{2}}$, $\boxed{x=3}$

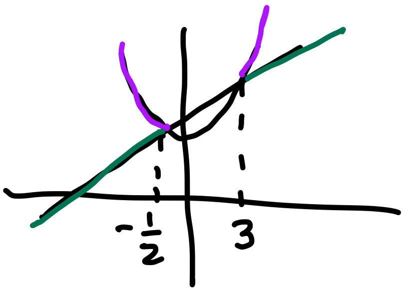


(b) $\underset{\text{parabola}}{\underline{2x^2+3}} \le \underset{\text{straight line}}{\underline{5x+6}}$ $f(x) \le g(x)$



Solution: $[-\frac{1}{2},3]$

(c) $2x^2+3 > 5x+6$



$\boxed{(-\infty,-\frac{1}{2}) \cup (3,\infty)}$

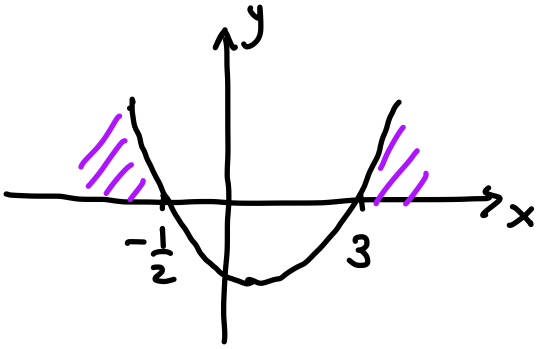
Note. You could rearrange the equation so that all terms are on one side, then graph the function that corresponds to the nonzero side of the equation and then find the solution.

$$2x^2+3 > 5x+6$$

$$2x^2-5x-3 > 0$$

$$(2x+1)(x-3) > 0$$

zeros: $x = -\frac{1}{2}, 3$



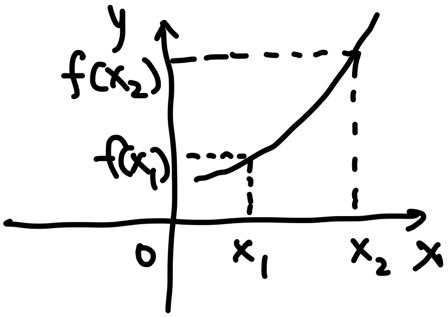
Solution:

$$\boxed{(-\infty, -\tfrac{1}{2}) \cup (3, \infty)}$$

Increasing and decreasing functions

Definition. f is increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

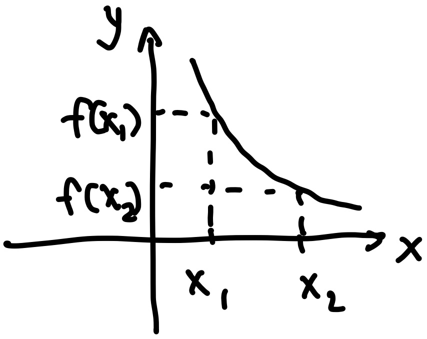
f is decreasing on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.



if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

for an increasing function.

(the graph rises)



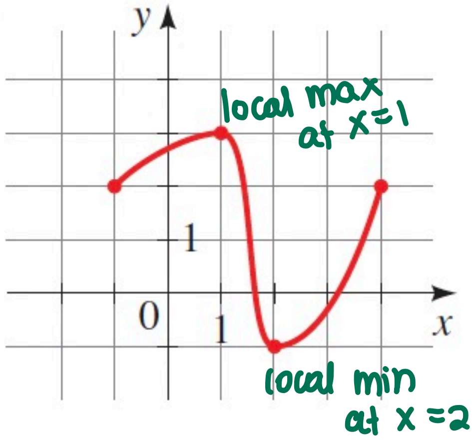
if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

for a decreasing function

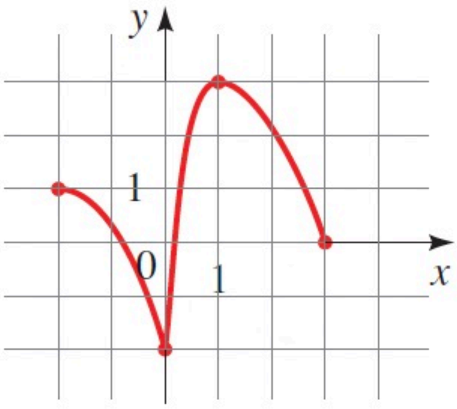
(the graph falls)

31–34 ■ Increasing and Decreasing The graph of a function f is given. Use the graph to estimate the following. **(a)** The domain and range of f. **(b)** The intervals on which f is increasing and on which f is decreasing.

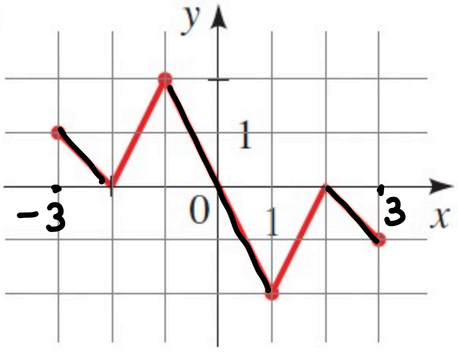
31.



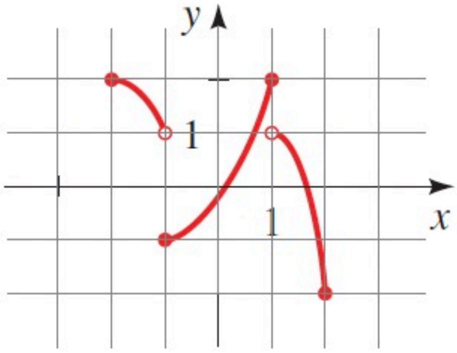
32.



33.



34.



(31) (a) Domain : $[-1,4]$, Range: $[-1,3]$

(b) Increasing : $[-1,1] \cup [2,4]$

Decreasing : $(1,2)$

(33) (a) Domain : $[-3,3]$, Range: $[-2,2]$

(b) Increasing: $[-2,-1] \cup [1,2]$ Decreasing: $(-3,-2) \cup (-1,1) \cup (2,3)$

Local maxima and local minima of a function

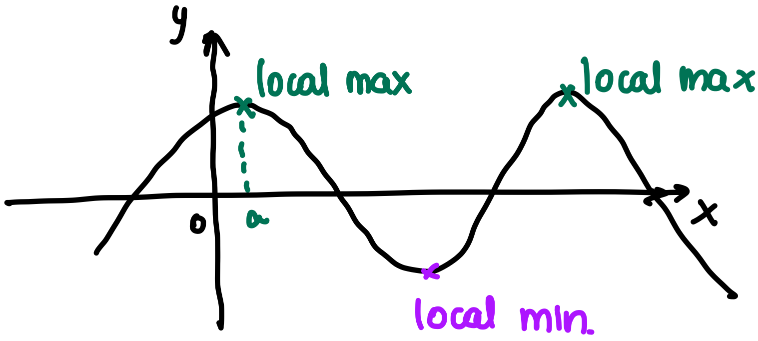
① The function value $f(a)$ is a local minimum value of f if

$$f(a) \leq f(x) \quad \text{for } x \text{ near } a$$

f has a local min at $x=a$.

② The function value $f(a)$ is a local max value of f if

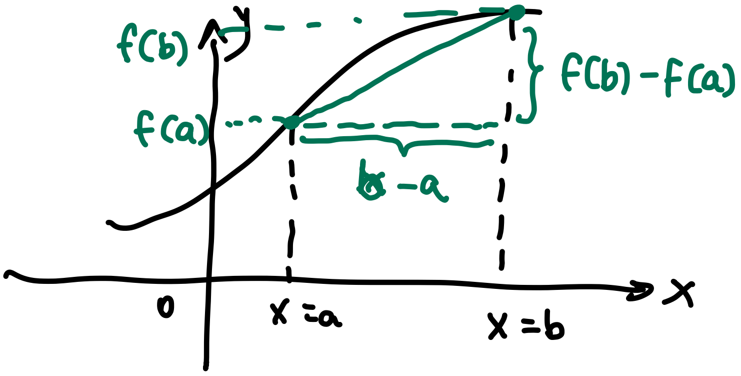
$$f(a) \geq f(x) \text{ when } x \text{ is near } a.$$



Section 2.4: Average rate of change

Definition. The average rate of change of a function $y = f(x)$ between $x = a$ and $x = b$ is

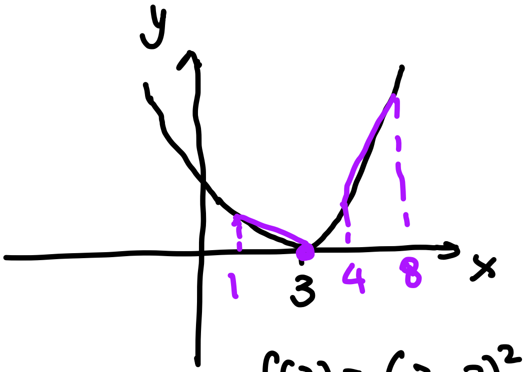
$$\boxed{\text{aver. rate of change} = \frac{f(b) - f(a)}{b-a}} \qquad \left(\frac{\text{change in } y}{\text{change in } x}\right).$$



Example. Given $\boxed{f(x) = (x-3)^2}$

Find the average rate of change between

(a) $x = 1$ and $x = 3$



$$f(3) = (3-3)^2 = 0$$

$$f(1) = (1-3)^2 = 4$$

$$\text{average rate of change} = \frac{f(3) - f(1)}{3-1}$$

$$= \frac{0-4}{3-1} = -2$$

(b) $x = 4$ and $x = 8$

$$\text{average rate of change} = \frac{f(8)-f(4)}{8-4}$$

$$f(8) = (8-3)^2 = 25$$

$$f(4) = (4-3)^2 = 1$$

$$= \frac{25-1}{4}$$

$$= \frac{24}{4}$$

$$= 6.$$

Example. An object is dropped from a cliff and the distance it travels after t seconds is given by $\boxed{d(t) = 16t^2.}$

Determine the average rate of change between $t=a$ and $t=a+h$.

$$\text{average rate of change} = \frac{d(a+h)-d(a)}{(a+h)-a}$$

← change in output

← change in input

$$= \frac{16(a+h)^2 - 16a^2}{\cancel{a}+h-\cancel{a}}$$

$$= \frac{16(a^2+2ah+h^2)-16a^2}{h}$$

$$= \frac{\cancel{16a^2}+32ah+16h^2-\cancel{16a^2}}{h}$$

$$= \frac{16\cancel{h}(2a+h)}{\cancel{h}}$$

$$= 16(2a+h).$$

Note

★ $h(t)$

↑

use the appropriate notation.

Average rate of change.

Example

let $\boxed{f(x) = 4x - 7}$ ★

(a) Avg. rate of change between $x = 2$ and $x = 6$

$$\overset{\Downarrow}{a} \qquad \overset{\Downarrow}{b}$$

$$\text{avg. rate of change} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(6) - f(2)}{6 - 2} \quad 6-2 = 4$$

$f(6) = 4(6) - 7$

$f(2) = 4(2) - 7$

$$= \frac{[4(6) - 7] - [4(2) - 7]}{4} \rightarrow f(6) - f(2)$$

$$= \frac{24 - 7 - 1}{4}$$

$$= \frac{16}{4}$$

$$= 4$$

(b) Avg. rate of change from $x = 3$ $(= a)$ and $x = 3 + h$ $(= b)$

$$\frac{f(b) - f(a)}{b - a} = \frac{f(3+h) - f(3)}{\cancel{3} + h - \cancel{3}}$$

$$= \frac{f(3+h) - f(3)}{h}$$

difference quotient

$f(x) = 4x - 7$

$f(3+h) = 4(3+h) - 7$

$= 12 + 4h - 7$

$= \boxed{5 + 4h}$

$$= \frac{\cancel{5} + 4h - \cancel{5}}{h}$$

$$= \frac{4\cancel{h}}{\cancel{h}}$$

$f(3) = 4(3) - 7 = 12 - 7 = \boxed{5}$ $\quad = \underline{\underline{4}}$

Section 2.5: Linear functions

A <u>linear function</u> is of the form $\boxed{f(x) = ax+b}$

b here is the y-intercept.

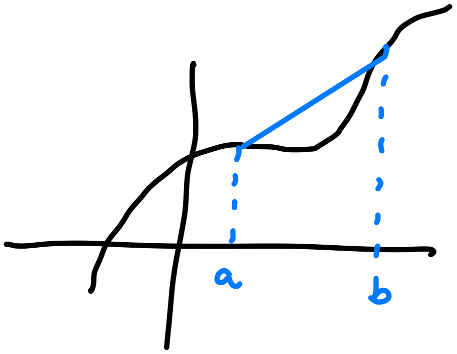
a here is the SLOPE

Question: Which of the following is a linear function?

(A) $f(x) = 1 - 2x$ ✓

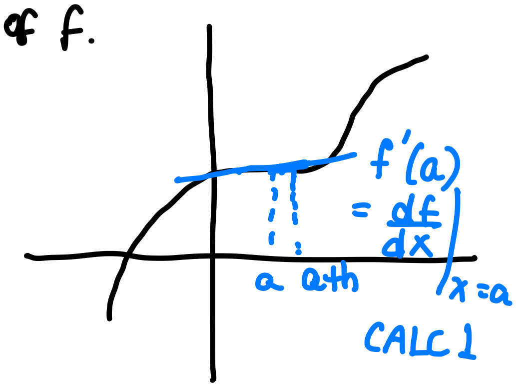
(B) $g(t) = t(3+5t)$ ← Quadratic

(C) $h(w) = \frac{2-4w}{3}$ ✓



Note. For a function $f(x) = ax+b$

Slope of $f = a$ = rate of change of f.



Section 2.6. Transformations of functions

VERTICAL SHIFTS.

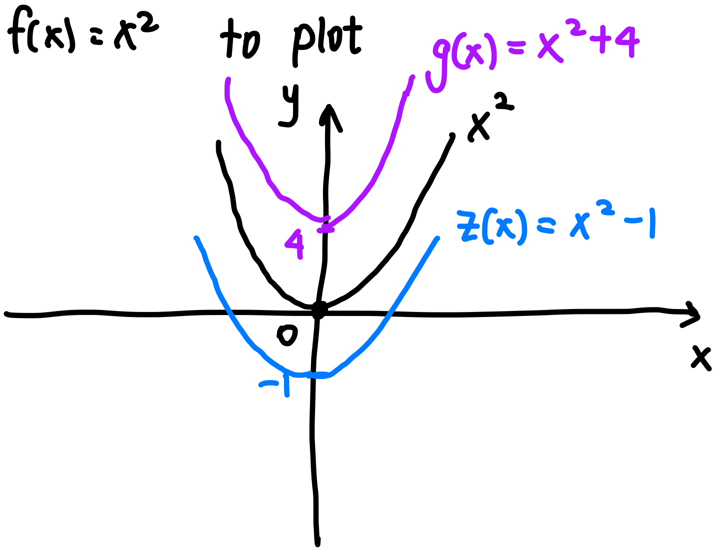
Suppose $\boxed{h>0}$

- To graph $\boxed{y = f(x)+h}$, shift the graph of $y=f(x)$ <u>upward</u> by h units.
- To graph $\boxed{y = f(x)-h}$, shift the graph of $y=f(x)$ <u>downward</u> by h units.

Example. Use the graph of $f(x)=x^2$ to plot

(a) $g(x) = x^2+4$

(b) $z(x) = x^2-1$



HORIZONTAL SHIFTS

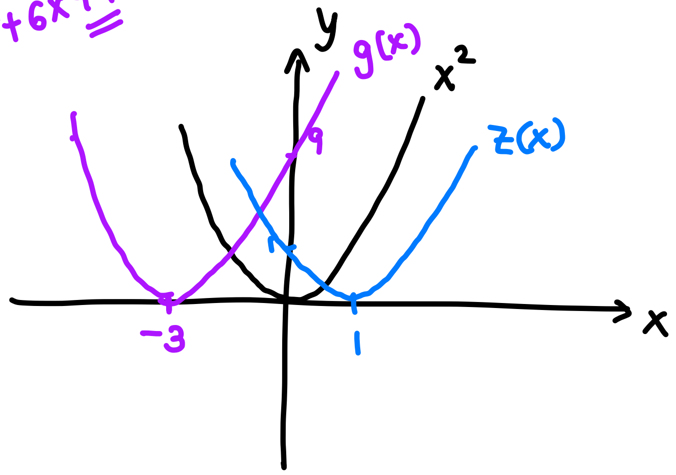
Suppose $\boxed{c>0}$

- To graph $\boxed{y=f(x-c)}$, shift the graph of f(x) to the right c units
- To graph $\boxed{y=f(x+c)}$, shift the graph of f(x) to the left c units.

Example. Let $f(x)=x^2$. Plot

(a) $g(x) = (x+3)^2 \rightarrow \underline{\underline{x^2+6x+9}}$

(b) $z(x) = (x-1)^2 \rightarrow x^2-2x+1$



y-intercept when x=0

x=0

$g(0)=(0+3)^2=9$

$z(0)=(0-1)^2=1$

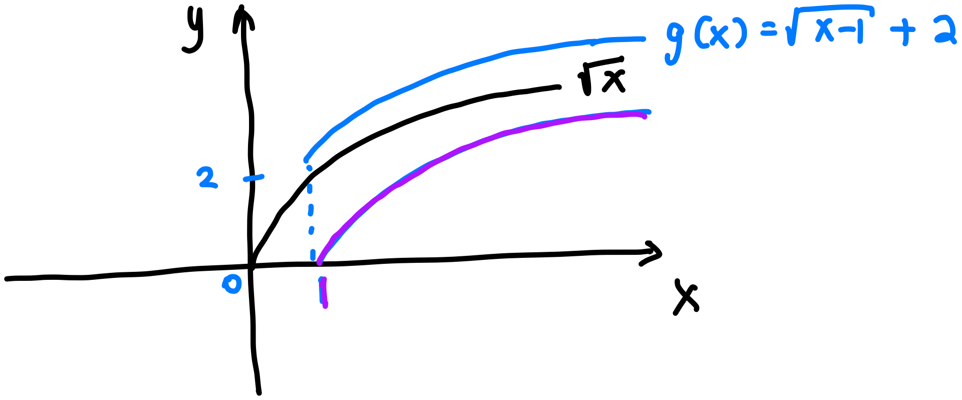
x-intercept when y=0

$0=(x+3)^2 \Rightarrow \underline{x=-3}$

Example Given $f(x)=\sqrt{x}$, sketch $\boxed{g(x)=\sqrt{x-1}+2}$

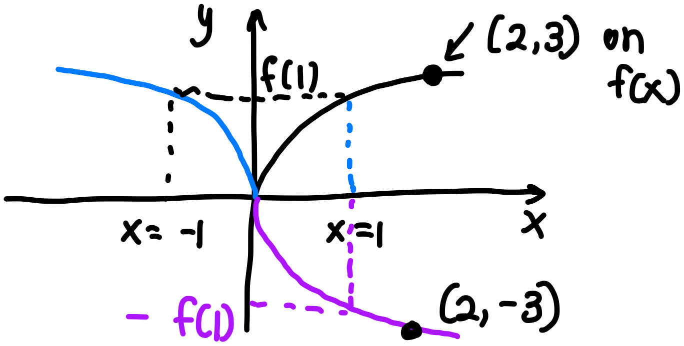
shift to the right by 1 unit

shift up by 2 units



Reflections

- To graph $\boxed{y = -f(x)}$, reflect the graph of f(x) in the x-axis
- To graph $\boxed{y = f(-x)}$, reflect the graph of f(x) in the y-axis



(2,3)

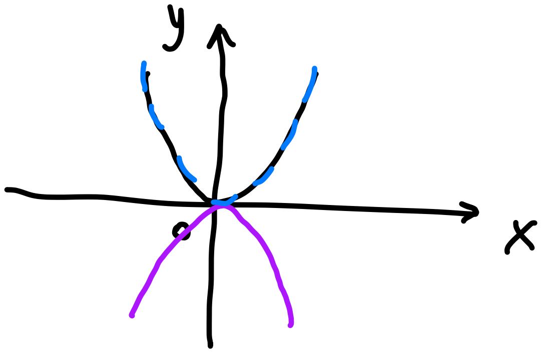
$\Rightarrow f(2) = 3$

$-f(2)$

$= -3$

Example ① Consider $f(x) = x^2$. Graph $g(x) = -x^2 = -f(x)$

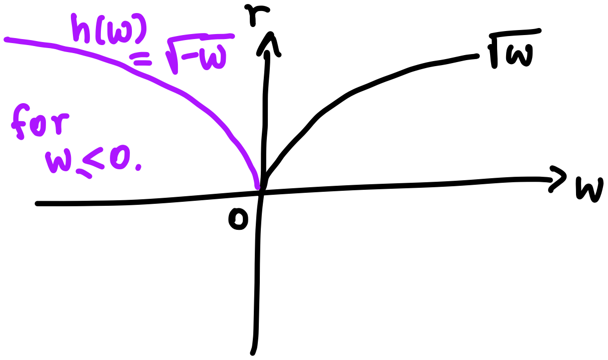
reflection on x-axis



$\boxed{g(x) = f(-x)} = (-x)^2 = x^2 = \underline{\underline{f(x)}}$

② Consider $r(w) = \sqrt{w}$

$h(w) = \sqrt{-w}$

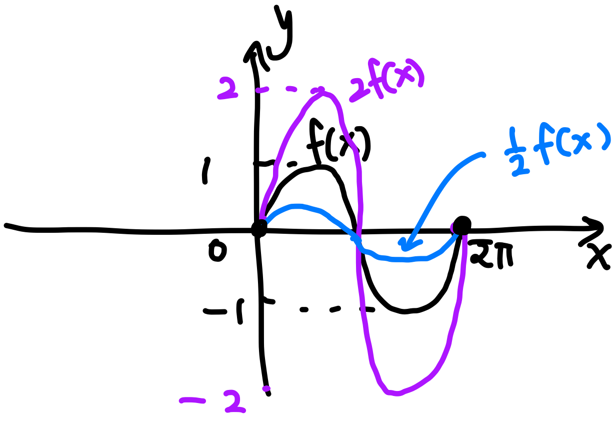


VERTICAL STRETCHES / COMPRESSIONS

To graph $\boxed{y = cf(x)}$

- If $\boxed{c > 1}$, stretch the graph of $y = f(x)$ vertically by a factor of c
- If $\boxed{0 < c < 1}$, shrink / compress the graph of $f(x)$ vertically by a factor of c

Example



$\boxed{g(x) = 2 \cdot f(x)}$

For f(x):

Domain: $[0, 2\pi]$

Range: $[-1, 1]$

For g(x):

Domain: $[0, 2\pi]$

Range: $[-2, 2]$

---

all x, $x \neq 3$ Interval notation: $(-\infty, 3) \cup (3, \infty)$

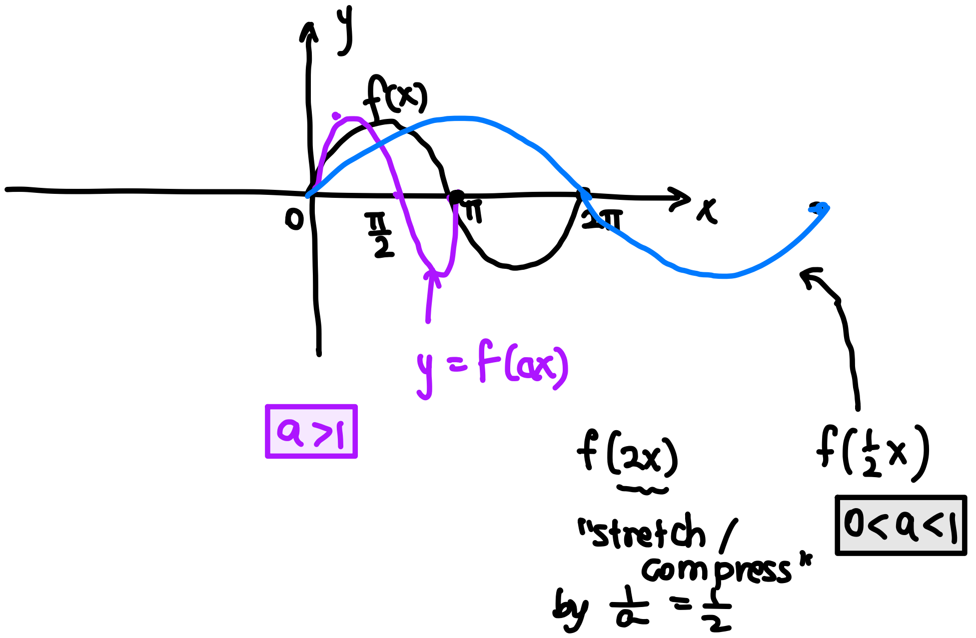
$x \geq 0, x \neq 2$ Interval notation: $[0, 2) \cup (2, \infty)$

all x, $x \neq -5$, $x \neq 1$ // $(-\infty, -5) \cup (-5, 1) \cup (1, \infty)$

HORIZONTAL STRETCHES / COMPRESSIONS

To graph $\boxed{y=f(ax)}$

- If $\boxed{a>1}$, shrink / compress the graph of $y=f(x)$ horizontally by a factor of $1/a$.
- If $\boxed{0<a<1}$, stretch the graph of $y=f(x)$ horizontally by a factor of $1/a$



$\frac{1}{(\frac{1}{2})} = 2$

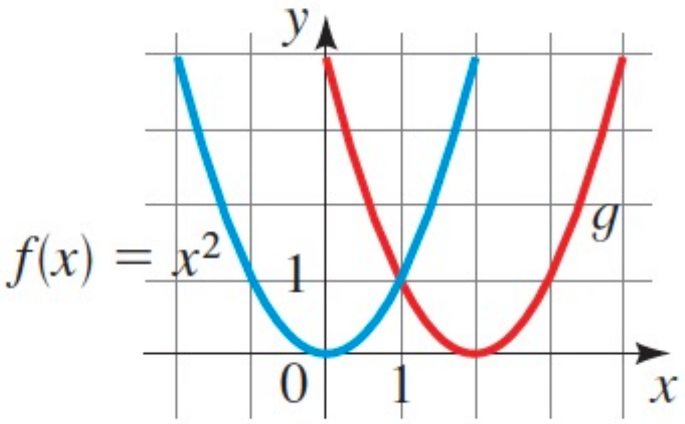
✪ Order of transformations.

choose either Horizontal or Vertical.

1) reflection
2) stretch / compression] the order of these 2 can be switched
3) shift ← this is ALWAYS last.

63–68 ■ Finding Formulas for Transformations The graphs of f and g are given. Find a formula for the function g.

63.



shift to the right by 2

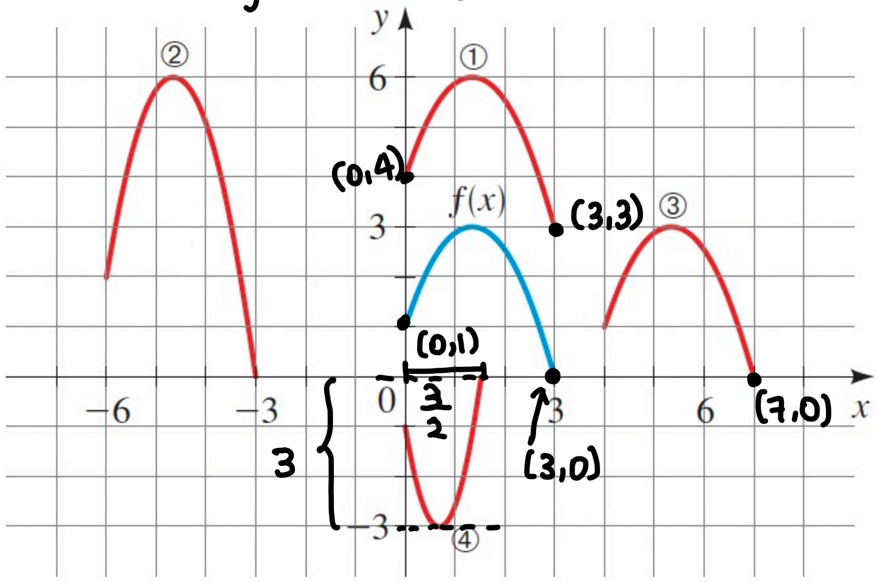
$f(x) = x^2$

$g(x) = f(x-2) = \underline{(x-2)^2}$

69–70 ■ Identifying Transformations The graph of $y = f(x)$ is given. Match each equation with its graph.

69. **(a)** ③ $y = f(x - 4)$ → shift to the right by 4 **(b)** $y = f(x) + 3$ ① shift up by 3

(c) ② $y = 2f(x + 6)$ stretch by 2 vertically and shift to the left by 6. **(d)** $y = -f(2x)$ ④ reflection along x-axis and compression by $\frac{1}{2}$ horizontally



Blue (f(x))

Domain: [0,3]

Range: [0,3]

(d) $y = -f(2x)$

Domain: $[0, \frac{3}{2}]$

Range: $[-3,0]$

$-f(2x)$

$[0,3] \to [-3,0]$

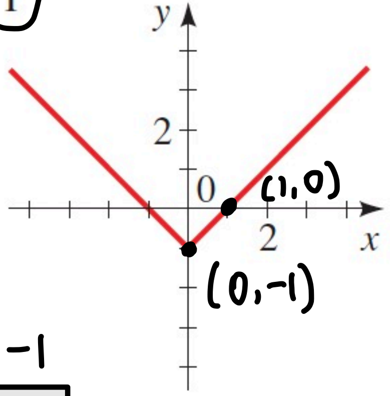
$[0,3] \to [0, \frac{3}{2}]$

25–28 ■ Identifying Transformations Match the graph with the function. (See the graph of $y = |x|$ on page 96.)

25. $y = |x + 1|$ **26.** $y = |x - 1|$

27. $y = |x| - 1$ **28.** $y = -|x|$

Ⓘ

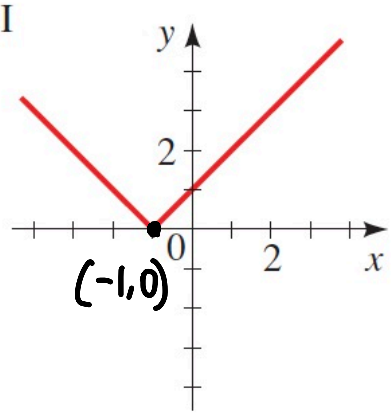


Shift by 1 unit down

$y = f(x) - 1$

$\boxed{y = |x| - 1}$

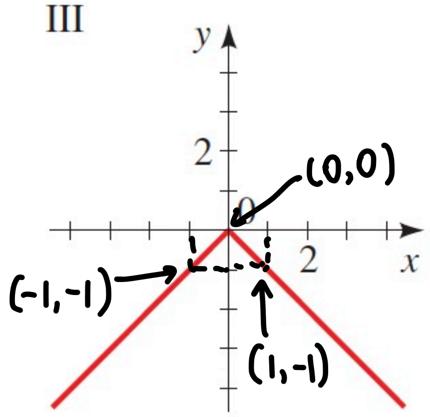
II



$y = f(x+1)$

$\boxed{y = |x+1|}$

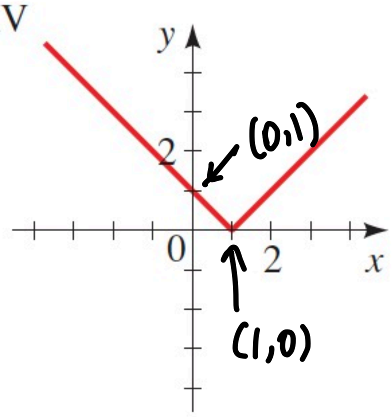
III



$y = -f(x)$

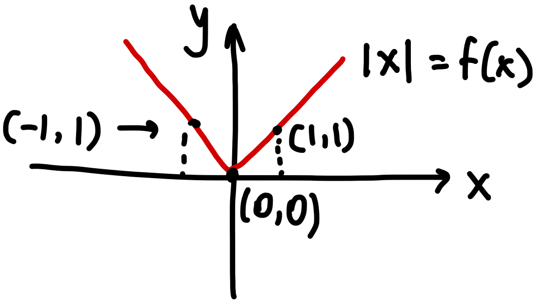
$\boxed{y = -|x|}$

IV



Shift by 1 unit to the right

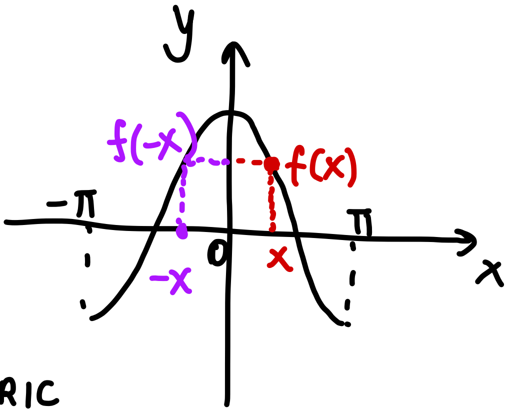
$\boxed{y = |x-1|}$



Even and odd functions

Let f be a function

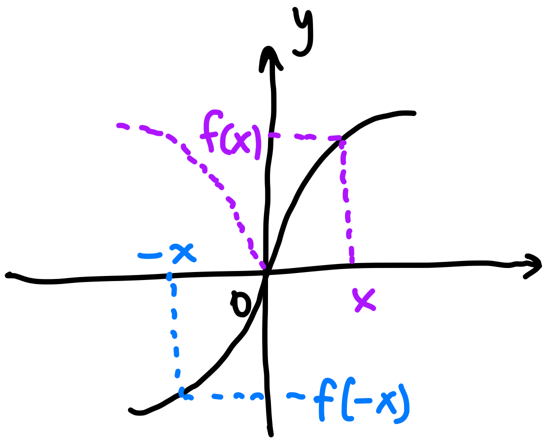
- f is even if $\boxed{f(x) = f(-x)}$



The graph of an even function is symmetric about the y-axis.

- f is odd if $\boxed{f(x) = \underbrace{-f(-x)}}$

here there are two reflections one across the x-axis and one across the y-axis



$-f(-x)$

The graph of an odd function is SYMMETRIC about the origin

Example. Determine whether the functions are odd, even, or neither even or odd.

$-f(x) = f(-x)$

ODD: $f(x) = -f(-x)$

EVEN: $f(x) = f(-x)$.

(a) $f(x) = x^5 + x$.

$$f(-x) = (-x)^5 + (-x)$$
$$= -x^5 - x$$
$$= -(x^5 + x)$$
$$= -f(x)$$

$(-x)^3 = (-1)^3 x^3$

$(ab)^3 = a^3 b^3$

$= -1 \cdot x^3$

$= -x^3$

$\Rightarrow \boxed{f(-x) = -f(x)} \Rightarrow f(x) = x^5 + x$ is odd.

(b) $f(x) = 2x - x^2$

$(-1)^2 x^2 = x^2$

$$f(-x) = 2(-x) - (-x)^2 = -2x - (+x^2)$$
$$= -2x - x^2$$
$$= -(2x + x^2)$$

Since $f(x) \neq f(-x)$ and $f(x) \neq -f(-x)$, the function is neither odd or even

(c) $f(x) = 1 - x^6$

$$f(-x) = 1 - \boxed{(-x)^6} \Rightarrow x^6$$
$$= 1 - x^6$$
$$= f(x)$$

$$\Rightarrow \quad f(-x) = f(x)$$

Thus $f(x)$ is even.

(d) $g(x) = x^3$

$$g(-x) = (-x)^3 = -x^3 = -g(x)$$
$$\Rightarrow g(x) \text{ is odd.}$$

THIS CONCLUDES THE MATERIAL FOR EXAM I.

Section 2.7 Composition of functions / Combinations of functions

---

Let f and g be two different functions with domains A and B. Then the functions $f+g$, $f-g$, $f \cdot g$, $\frac{f}{g}$ are defined as follows

intersection.

$(f+g)(x) = f(x) + g(x)$ Domain $A \cap B$

$(f-g)(x) = f(x) - g(x)$ Domain $A \cap B$

$(fg)(x) = f(x)g(x)$ Domain $A \cap B$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ Domain $\{x \in A \cap B \text{ s.t. } g(x) \neq 0\}$

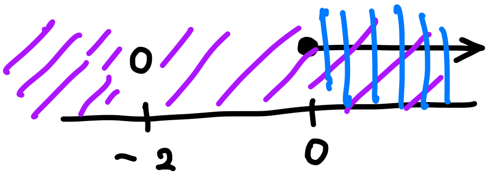
such that

Example. Let $f(x) = \frac{1}{x+2}$ and $g(x) = \sqrt{x}$.

Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$ and their domain.

$(f+g)(x) = \frac{1}{x+2} + \sqrt{x}$ Domain: $x \neq -2$, $x \geq 0$

Interval notation : $[0, \infty)$



Domain of f(x): $x \neq -2$

$(-\infty, -2) \cup (-2, \infty)$

Domain of g(x): $x \geq 0$

$[0, \infty)$

if instead $f(x) = \frac{1}{x-2}$

$x \neq 2$

$[0, 2) \cup (2, \infty)$

$(f-g)(x) = \frac{1}{x+2} - \sqrt{x}$ Domain: $[0, \infty)$

$(fg)(x) = \frac{\sqrt{x}}{x+2}$ Domain: $[0, \infty)$

$\frac{1}{x+2} \cdot \sqrt{x}$

$\left(\frac{f}{g}\right)(x) = \frac{1}{\sqrt{x}\,(x+2)}$ Domain: $\underline{\underline{(}}0, \infty)$

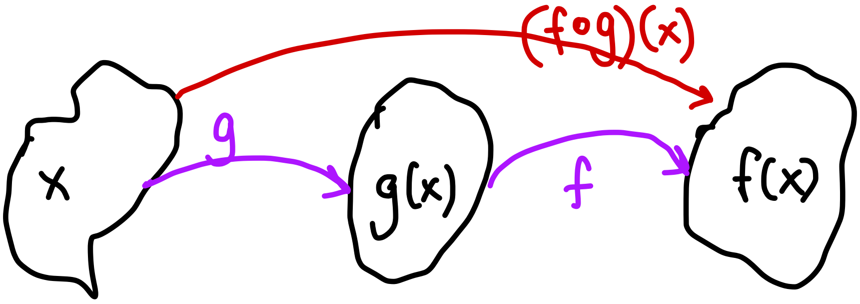
COMPOSITION OF FUNCTIONS

$f(\boxed{g(x)})$

new input for f

Given two functions f and g, the composite function $f \circ g$ is defined as

$$\boxed{(f \circ g)(x) = f(g(x))}$$



<u>Example</u>. Let $f(x) = \sqrt{x+1}$, $g(x) = x^2$

(a) Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= \sqrt{g(x)+1} \\ &= \sqrt{x^2+1}\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= (f(x))^2 \\ &= (\sqrt{x+1})^2 \\ &= x+1\end{aligned}$$

Note. In general $f \circ g \neq g \circ f$.

↗ remember that here g is applied first and f is applied second.

Example Let $f(x)=\sqrt{x}$ and $g(x)=\sqrt{2-x}$

Find $(f\circ f)(x) = f(f(x))$

$$= \sqrt{f(x)}$$

$$= \sqrt{\sqrt{x}}$$

$$= \left((x)^{1/2}\right)^{1/2}$$

$$= x^{1/4}$$

$$= \sqrt[4]{x}$$

Domain: $[0,\infty)$.

Find $(f\circ g)(x) = f(g(x))$

$$= \sqrt{g(x)}$$

$$= \sqrt{\sqrt{2-x}}$$

$$= (2-x)^{1/4}$$

$$= \sqrt[4]{\underbrace{2-x}_{\geq 0}}$$

Domain: $(-\infty, 2]$

$\sqrt{\cdots}$

Solve for $2-x\geq 0$

$+x \quad +x$

$2\geq x \Rightarrow x\leq 2.$

Find $(g\circ f)(x) = g(\underline{f(x)}) = \sqrt{2-f(x)}$

$$= \sqrt{\boxed{2-\sqrt{x}}}$$

↓ Domain is $x\geq 0$

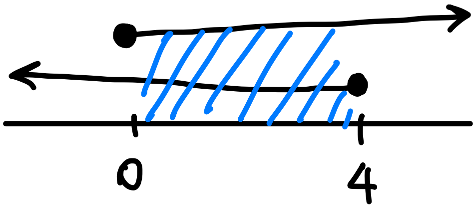
$\sqrt{\boxed{2-\sqrt{x}}}$
≥ 0

Solve $2-\sqrt{x}\ge 0$

$+\sqrt{x} \quad +\sqrt{x}$

$\underline{2}\ge\underline{\sqrt{x}} \Rightarrow (\sqrt{x})^2 \le (2)^2$

$\boxed{x\le 4}$



Domain of $(g\circ f)(x)$ is $[0,4]$

$g(x)=\sqrt{2-x}$

Find $(g\circ g)(x) = g(\boxed{g(x)})$ new input

$= \sqrt{2-\boxed{g(x)}}$ new input

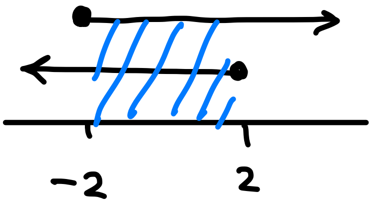
$= \sqrt{\boxed{2-\sqrt{2-x}}}\}$

Domain of $\sqrt{2-x}$ is $2-x\ge 0$

$+x \quad +x$

$2\ge x$

$x\le 2$



$2-\sqrt{2-x}\ge 0$

$+\sqrt{2-x} \quad +\sqrt{2-x}$

$2\ge\sqrt{2-x} \rightarrow 2^2\ge(\sqrt{2-x})^2$

$4\ge 2-x$

$2\ge -x$

$x\ge -2$

Domain is $[-2,2]$.

Compositions of 3 functions

$$\boxed{(f\circ g\circ h)(x) = f\big(g(h(x))\big)}$$

Example. Let $f(x) = \dfrac{x}{x+1}$, $g(x) = x^8$, $h(x) = x-2$

Find $(f\circ g\circ h)(3)$

$$\begin{aligned}(f\circ g\circ h)(x) &= f\big(g(\underbrace{h(x)}))\big) \\ &= f\big(\boxed{g(x-2)}\big) \\ &= f\big((x-2)^8\big) \\ &= \frac{(x-2)^8}{(x-2)^8+1}\end{aligned}$$

$g(x-2)$
$= (x-2)^8$

$$(f\circ g\circ h)(3) = \frac{(3-2)^8}{(3-2)^8+1} = \frac{1}{1+1} = \frac{1}{2}.$$

Recognizing a composition of functions.

1. Given $h(x) = \sqrt[3]{x+9}$ find $f(x)$ and $g(x)$ such that $h(x) = (f\circ g)(x)$.

$$\begin{aligned}h(x) &= (f\circ g)(x) \\ &= f(g(x)) \\ &= f(x+9) \\ &= \sqrt[3]{x+9}\end{aligned}$$

where I used that $g(x) = x+9$
$f(x) = \sqrt[3]{x}$

2. $F(x) = 2+\sqrt{x+1} = f(g(x))$. Find $f(x)$ and $g(x)$.

$$f(x) = 2+\sqrt{x}$$
$$g(x) = x+1$$

Section 2.8 : One-to-one functions and their inverses

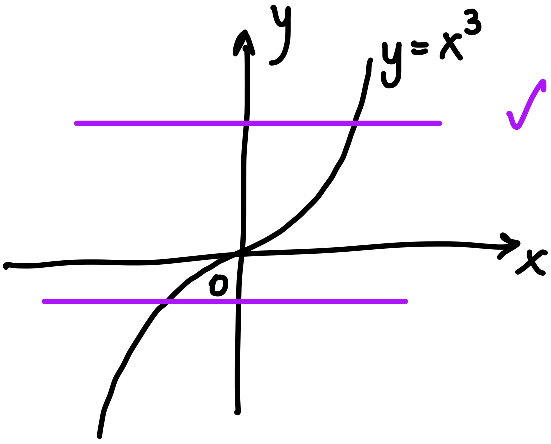
Definition : A function is one-to-one if no two elements in the domain A have the same image (i.e. if no two elements in the domain A have the same output)

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

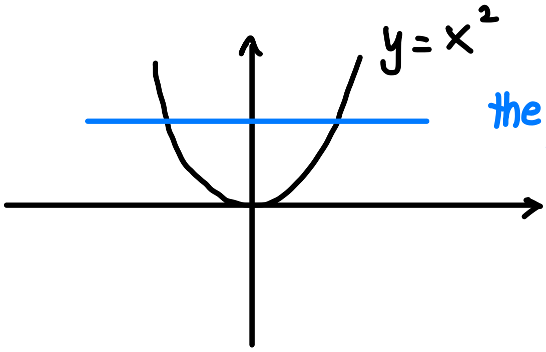
Horizontal line test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example.

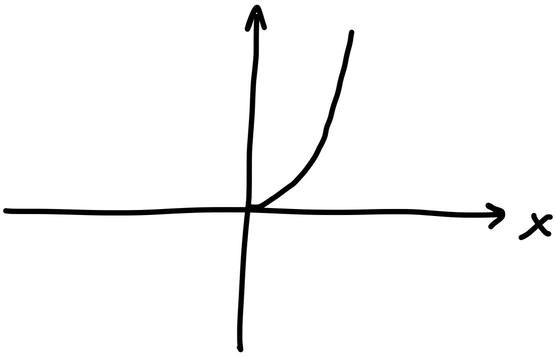


✓ passes the horizontal line test so it's one-to-one.



the horizontal line passes more than once through the graph so this function is <u>not</u> one-to-one.

If you restrict the domain to $x \geq 0$ then $y = x^2$ is



The inverse of a function

Let f be a one-to-one function with domain A and range B. Then the inverse of f is denoted f^{-1} has domain B and range A and is defined by

$$\boxed{f^{-1}(y) = x \text{ if and only if } f(x) = y} \quad \star$$

for any y in B.

Note: $\left[\begin{array}{l} \text{domain of } f^{-1} = \text{range of } f \\ \text{range of } f^{-1} = \text{domain of } f \end{array}\right] \star$

Example: ① $f(3) = 4$, $\boxed{f(4) = 6}$, $f(5) = 2$

$f(x) = y$

$x = f^{-1}(y)$

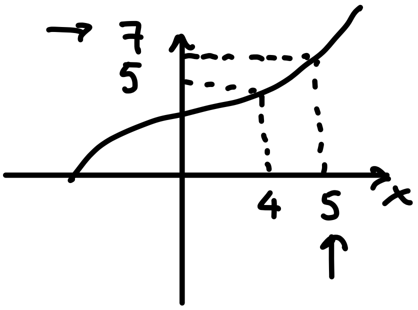
↓

$f(4) = 6$

$4 = f^{-1}(6)$

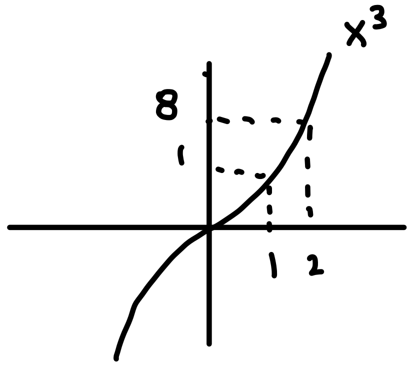
Find $\underbrace{f^{-1}(6)}_{\substack{\shortparallel \\ 4}}$, $f^{-1}(2) = 5.$

②



Find $f^{-1}(\underset{\substack{\uparrow \\ \text{output}}}{5}) = 4$

$f(\underset{\uparrow}{5}) = 7$



$f^{-1}(8) = 2$

Recall from last time what compositions of functions are.

$$f(\overbrace{g(x)})$$

g(x) is the new input for function f.

x is the input of g

In general.
$f(g(x)) \neq g(f(x))$

Property of inverse functions

Let f be a one-to-one function with domain A and range B.
The inverse f^{-1} satisfies

" f^{-1} with f cancel each other "

$f^{-1}(f(x)) = x$ for every x in A.

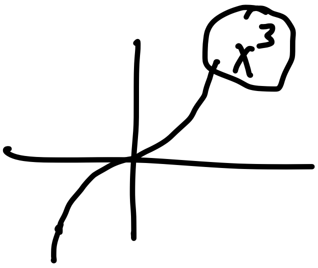
$f(f^{-1}(x)) = x$ for every x in B

note B is the domain of f^{-1}.

Example. Let $f(x) = x^{\frac{1}{3}}$ and let $\overset{f^{-1}(x)}{g(x)} = x^3$

Determine whether f and g are inverses of each other

Domain of f(x) is $(-\infty, \infty)$
Domain of g(x) is $(-\infty, \infty)$.



Check whether $f(g(x)) = x$ and $g(f(x)) = x$.

- $f(g(x)) = f(x^3) = (x^3)^{1/3} = \boxed{x}$

- $g(f(x)) = g(x^{1/3}) = (x^{1/3})^3 = \boxed{x}$

Thus f and g are inverses of each other

Finding the inverse of a function.

STEP 1. Write $y = f(x)$

STEP 2 Solve this equation for x in terms of y.

STEP 3. Interchange x and y. Write the resulting equation as $y = f^{-1}(x)$.

Example. Let $\boxed{f(x) = 4x+5.}$ Find $f^{-1}(x)$.

STEP 1. $y = 4x+5$

STEP 2. $\frac{y-5}{4} = x$

STEP 3. $\frac{x-5}{4} = y$

$$\boxed{f^{-1}(x) = \frac{x-5}{4}}$$

f inverse

check. $f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = x.$

$$f^{-1}(f(x)) = f^{-1}(4x+5)$$
$$= \frac{(4x+\cancel{5})-\cancel{5}}{4}$$
$$= \frac{4x}{4}$$
$$= x \quad ✓$$

$$f(f^{-1}(x)) = f\left(\frac{x-5}{4}\right)$$
$$= \cancel{4}\left(\frac{x-5}{\cancel{4}}\right)+5$$
$$= x-\cancel{5}+\cancel{5}$$
$$= x \quad ✓$$

Example 2 $f(x) = \frac{x^5+3}{2}$. Find $f^{-1}(x)$

STEP 1 $y = \frac{x^5+3}{2}$ Replace f(x) with y

STEP 2 Make x the subject of the formula

$$2y = x^5+3$$
$$2y-3 = x^5$$
$$\rightarrow x = (2y-3)^{1/5} = \sqrt[5]{2y-3}$$

STEP 3 Interchange x with y

$$y = (2x-3)^{1/5} \quad \text{or} \quad y = \sqrt[5]{2x-3}$$

Change y into $f^{-1}(x)$

$$f^{-1}(x) = (2x-3)^{1/5} \quad \text{or} \quad f^{-1}(x) = \sqrt[5]{2x-3}$$

Example 3. Involving rational expressions.

$$g(x) = \frac{2x+5}{x-1}$$

STEP 1.

$$y = \frac{2x+5}{x-1}$$

STEP 2. Make x the subject of the formula

$$y \cdot (x-1) = 2x+5$$

$$\underline{xy} - y = \underline{2x}+5$$

$$xy - 2x = 5+y$$

$$x(y-2) = 5+y$$

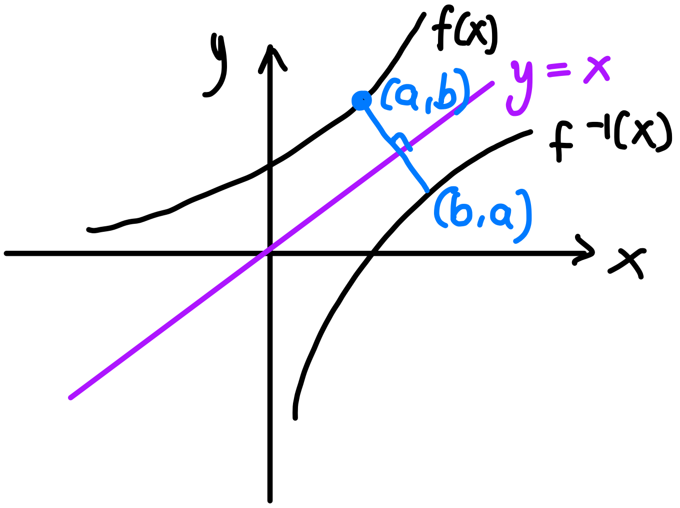
$$x = \frac{5+y}{y-2}.$$

STEP 3.

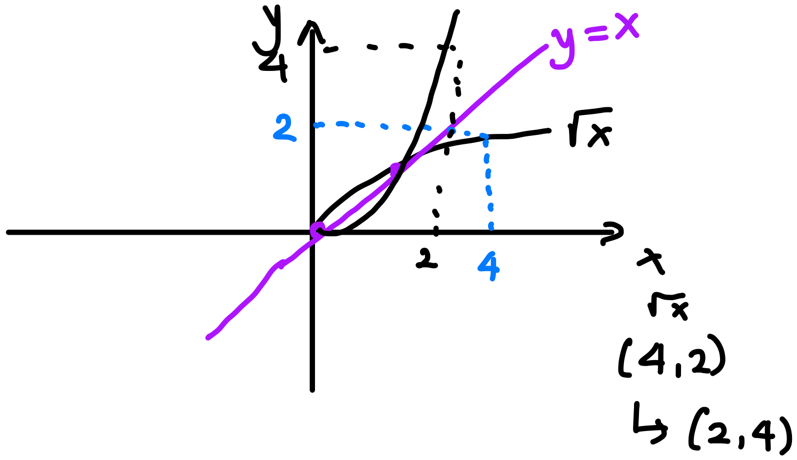
$$y = \frac{5+x}{x-2} \Rightarrow \boxed{g^{-1}(x) = \frac{5+x}{x-2}}$$

Graphing inverse functions

The graph of f^{-1} is found by reflecting the graph of f in the line $y=x$.



Example. Let $f(x) = \sqrt{x}$. Sketch $f^{-1}(x)$



Does $y = x$ intersect with $y = \sqrt{x}$?

$$\boxed{x = \sqrt{x}}$$
$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \quad x = 1$$

(★) Note.

Solve for x.

$$\overset{\text{LHS}}{\boxed{\sqrt{3+x} - 5}} = \overset{\text{RHS}}{\boxed{x+4}} \quad (*)$$
$$\sqrt{3+x} = x+9$$
$$3+x = (x+9)^2$$
$$3+x = x^2+18x+81$$
$$x^2+17x+78 = 0$$
$$x = \frac{-17 \pm \sqrt{17^2-4(1)(78)}}{2}$$
$$\boxed{x_1} = \ldots$$
$$\boxed{x_2} = \ldots$$

Check whether x_1 and x_2 satisfy (*)

Section 3.1 Quadratic functions.

Reminder. A quadratic function is a polynomial of degree 2 and is of the form

$$\boxed{f(x) = ax^2 + bx + c} \quad \text{where } a \neq 0.$$

STANDARD FORM OF A QUADRATIC FUNCTION

$$\boxed{f(x) = a\left(x - \underset{\uparrow}{h}\right)^2 + \underset{\uparrow}{k}}$$

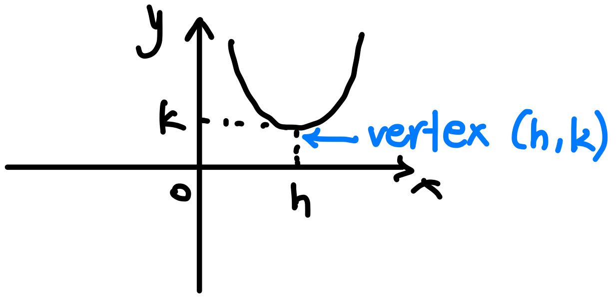
vertex: (h, k)

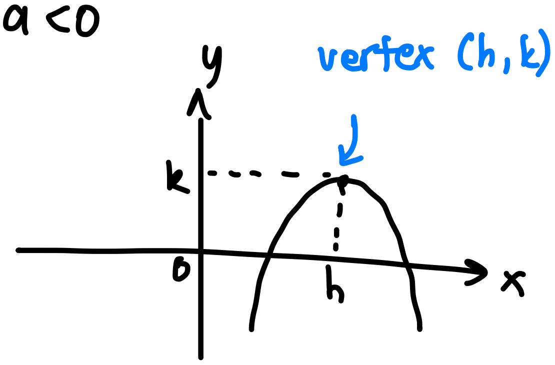
$y = x^2$

To go from $ax^2 + bx + c$ to the standard form use completing the square. The graph of $f(x)$ is a parabola with <u>vertex</u> (h, k).

The parabola opens up when $a > 0$

" opens downward when $a < 0$





$f(x) = a(x-h)^2 + k$ with $\boxed{a > 0}$

$f(x) = a(x-h)^2 + k$ with $\boxed{a < 0}$.

<u>Example.</u> Let $f(x) = 2x^2 - 12x + 13$.

Find what $f(x)$ becomes in standard form.

$$f(x) = 2(x^2 \underset{+}{-} \overset{b}{6}x) + 13$$

$$\hookrightarrow \left(x \overset{+}{-} \frac{b}{2}\right)^2 \overset{\text{always.}}{-} \left(\frac{b}{2}\right)^2$$

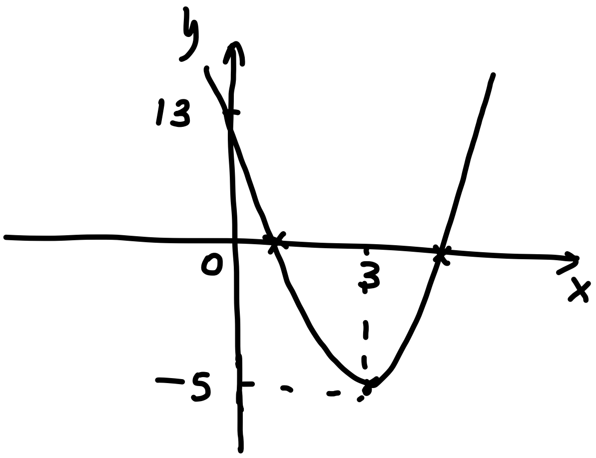
$$= 2\left[(x-3)^2-9\right]+13$$

$$= 2(x-3)^2-18+13$$

$$= 2(\underset{\uparrow}{x-3})^2 \underset{\uparrow}{-5}$$

vertex: $(3, -5)$

$f(x) = a(x-h)^2 + k$



y-intercept if $x=0$

$$f(0) = 2(0-3)^2 - 5$$
$$= 2(9) - 5$$
$$= 13.$$

Range: $[-5, \infty)$
Domain: $(-\infty, \infty)$.

WebAssign 2.7

⑥? Consider $f(x) = \dfrac{x}{x+1}$ and $g(x) = \dfrac{1}{x}$.

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\underset{\overset{\uparrow}{*}}{\frac{1}{x}}\right) = \boxed{\frac{\frac{1}{x}}{\frac{1}{x}+1}} = \frac{\frac{1}{x}}{\frac{1}{x}+\frac{x}{x}}$$

$$= \frac{\frac{1}{x}}{\frac{1+x}{x}} = \frac{1}{\cancel{x}} \cdot \frac{\cancel{x}}{1+x} = \boxed{\frac{1}{1+x}}$$

(b) Domain : $\boxed{(-\infty, -1) \cup (-1, 0) \cup (0, \infty)}$

$x \neq -1, 0$

(c) $(g \circ f)(x) = g\left(\overset{\text{new input}}{\boxed{f(x)}}\right) = g\left(\frac{x}{\underbrace{x+1}_{\uparrow}}\right)$

$$= \frac{1}{\left(\frac{x}{x+1}\right)}$$

$$= \frac{x+1}{x} \qquad x \neq -1, 0$$

(d) Domain: $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$

(e) $\underline{(f \circ f)}(x) = f(f(x)) = f\left(\boxed{\frac{x}{x+1}}\right)$ $\leftarrow \boxed{x \neq -1}$

Recall $f(x) = \frac{x}{x+1}$

$g(x) = \frac{1}{x}$

$$= \frac{\frac{x}{x+1}}{\frac{x}{x+1} + 1}$$

$$= \frac{\frac{x}{x+1}}{\frac{x}{x+1} + \frac{x+1}{x+1}}$$

$$= \frac{\frac{x}{x+1}}{\frac{2x+1}{x+1}}$$

$$= \frac{x}{2x+1} \qquad \boxed{x \neq -\frac{1}{2}}$$

(f) Domain: $(-\infty, -1) \cup \left(-1, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right).$

(g) $(g\circ g)(x) = g\left(\frac{1}{x}\right) = \frac{1}{\left(\frac{1}{x}\right)} = x$

Recall $g(x) = \frac{1}{x}$

$x \neq 0$

Domain: $(-\infty, 0) \cup (0, \infty)$.

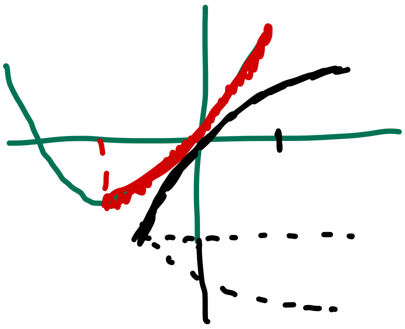
WebAssign 2.8

⑩. Find the inverse of f.

$f(x) = x^2 + 7x$
$= \left(x + \frac{7}{2}\right)^2 - \frac{49}{4}$

$f(x) = x^2 + 7x, \quad \underline{x \geq -\frac{7}{2}}$

Find $f^{-1}(x)$ when $x \geq -\frac{49}{4}$



$y = x^2 + 7x$

Make x the subject of the formula:

$x^2 + 7x - y = 0$

Use the quadratic formula $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1, \quad b = 7, \quad c = -y$

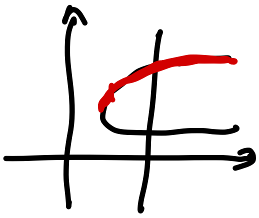
$$x = \frac{-7 \pm \sqrt{7^2 - 4(1)(-y)}}{2(1)}$$

$$= \frac{-7 \pm \sqrt{49 + 4y}}{2}.$$

Use only the + square root.

Final answer:

$$\boxed{f^{-1}(x) = \frac{-7 + \sqrt{49 + 4x}}{2}}$$



WebAssign 27

rate at which the radius is increasing is 3 cm/s.

⑦ (a) $f(t) = 3t$

(b) volume of sphere as a function of the radius.

$$g(r) = V = \frac{4\pi r^3}{3}$$

(c) $$\underline{g \circ f} = g(\boxed{f(t)}) = g(3t) = \frac{4\pi(3t)^3}{3} \leftarrow$$

$$= 36\pi t^3.$$

↑ overall output is the output of g which is the volume of a sphere

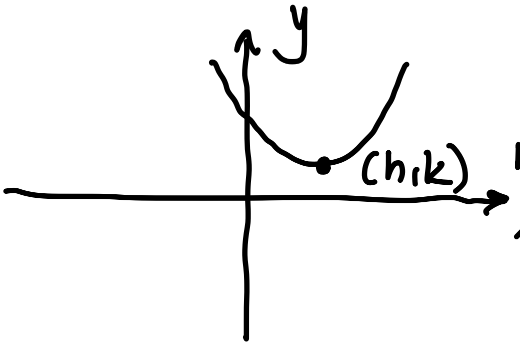
↑ input

The function represents the volume as a function of time.

Section 3 1 continuing Quadratic functions

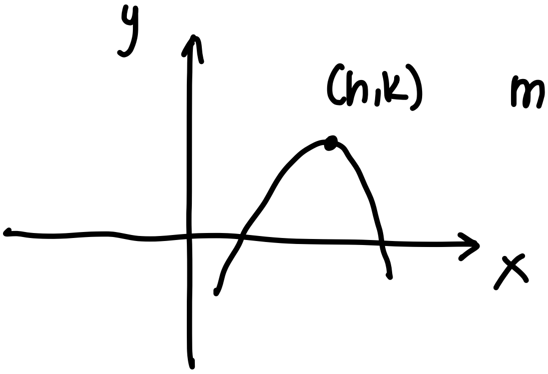
$$f(x) = a(x-h)^2 + k$$

Maximum and minimum value of a quadratic



minimum ocurs at $x = h$ and the $y(h) = k$.

$a > 0$.



maximum occurs at $x=h$

and $y(h)=k$

$a<0$

For any quadratic formula $f(x)=y=ax^2+bx+c$

the maximum/minimum occurs at

$$\boxed{x=-\frac{b}{2a}}$$

and if $a>0$, the minimum value is $f(-\frac{b}{2a})$

if $a<0$, the maximum value is $f(-\frac{b}{2a})$.

Examples

①. Find the max/min value of each quadratic formula

$f(x) = -2x^2+4x-5.$ $\quad = ax^2+bx+c$

max because $a<0$.

$a = -2$

$b = 4$

$c = -5$

Maximum occurs at $x = -\frac{b}{2a} = -\frac{4}{2(-2)} = 1$

$f\left(-\frac{b}{2a}\right) = f(1) = -2(1)^2+4(1)-5 = -2+4-5$

$= -3$

ALTERNATIVE.

Write

$$
\begin{aligned}
&-2x^2+4x-5\\
&= -2\left[x^2-2x\right]-5\\
&= -2\left((x-1)^2-1\right)-5\\
&= -2(x-1)^2+2-5\\
&= -2(x-1)^2 \boxed{-3}\\
&= a(x-h)^2+\boxed{k}
\end{aligned}
$$

vertex at $(h,k) = (1,-3)$

$\uparrow$ x value at which max occurs

$\uparrow$ max y value.

Section 3.2 Polynomial functions and their graphs

Defn.

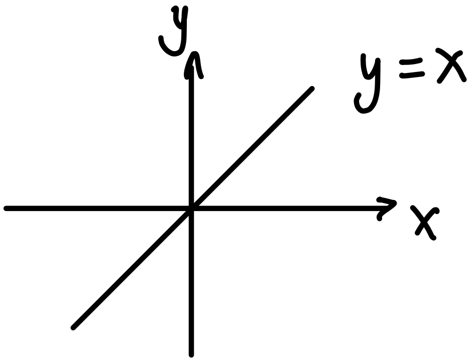
A polynomial function of degree n is a function of the form

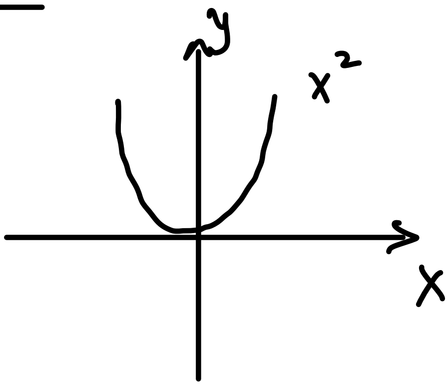
$$f(x) = \overset{\text{leading coefficient}}{\boxed{a_n}}x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

where n is a non-negative integer and $a_n \neq 0$.

- $a_n, a_{n-1}, \dots, a_1, a_0$ are the coefficients
- a_0 is the constant coeff. of the constant term
- $a_n x^n$ is the leading term

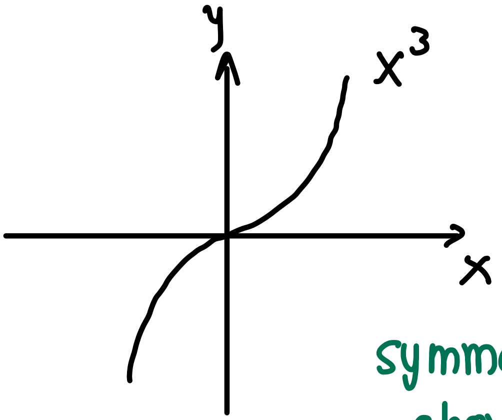
Graphs of basic polynomials

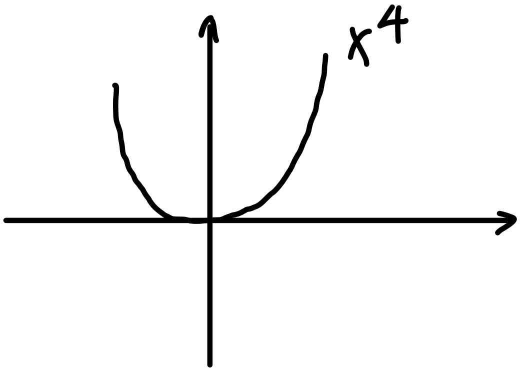




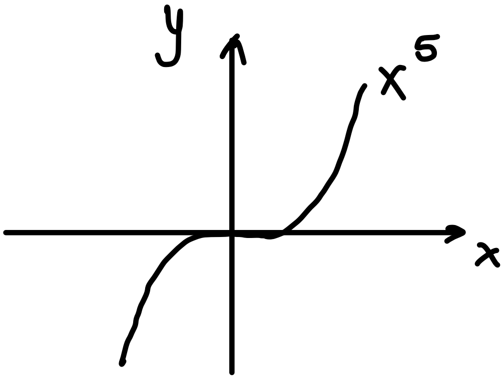
$$y = a_n x^n + a_{n-1}x^{n-1} + \dots + \underset{\sim\sim\sim}{a_1} x + a_0$$

* the n is a non-negative integer.



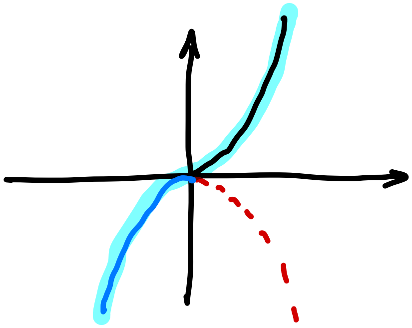


symmetric about the origin



ODD Function: $f(x) = \underbrace{-f(\overset{\downarrow}{-}x)}_{\text{transformations}}$

- reflection about the y-axis
- reflection about the x-axis.



Note $f(x)=x^n$ this has the same general shape as $y=x^2$ when n is even however the larger the n is, the flatter the graph gets around the origin and steeper elsewhere.

- this has the same general shape as x^3 when n is odd.

End behavior of polynomials

It is determined by the degree of the polynomial and the sign of the leading coefficient $\boxed{a_n}$ as $\boxed{x\to\infty}$

f(x) has odd degree

$y=(x+2)\boxed{(x-3)(x+1)}$

$=1\cdot x^3+\dots$

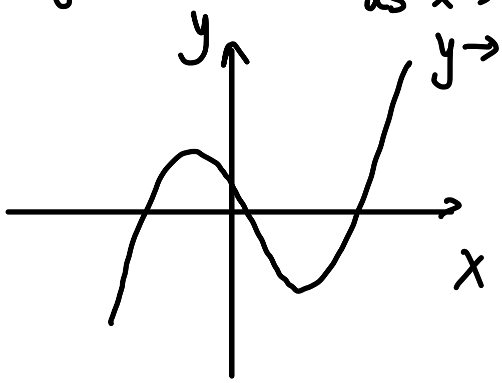
↑ positive

$=(x+2)[x^2-2x-3]$

$=\boxed{x^3}-2x^2+\dots$

e.g.

as $x\to\infty$ $y\to\infty$



as $x\to-\infty$

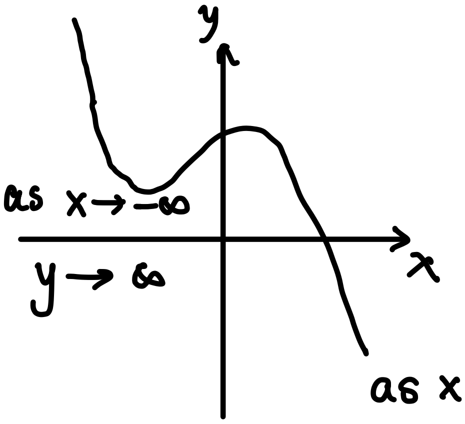
x goes to $-\infty$

tends to

approaches

$y\to-\infty$

LEADING COEFFICIENT IS POSITIVE

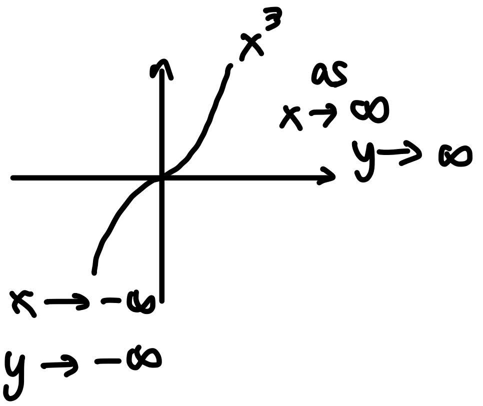


as $x\to+\infty$

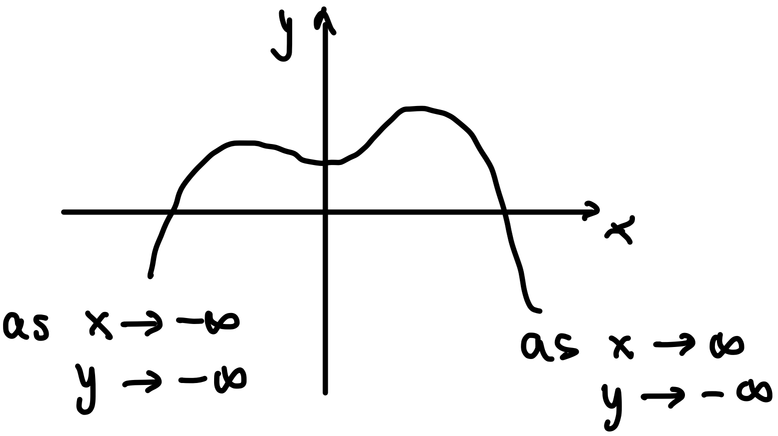
$y\to-\infty$

LEADING COEFFICIENT IS NEGATIVE

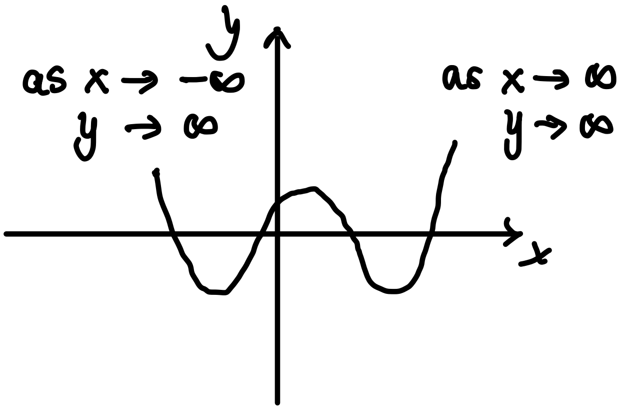
e.g



f(x) has even degree



LEADING COEFFICIENT IS NEGATIVE



LEADING COEFFICIENT IS POSITIVE

Example. Determine the end behavior of a polynomial.

$$f(x) = -2x^4 + 5x^3 + 4x - 7$$

as $x \to -\infty$, $y \to -\infty$

as $x \to +\infty$, $y \to -\infty$

Study inverse functions for quiz.

Zeros of a polynomial

If f(x) is a polynomial and c is a real number then we have the following equivalent statements.

1. c is a zero of f(x)
2. $x = c$ is a solution of the polynomial of f(x)
3. $x - c$ is a factor of f(x)
4. c is an x-intercept of the graph of f(x).

$f(x) = \underbrace{(x-3)}\underbrace{(x+4)} = 0$

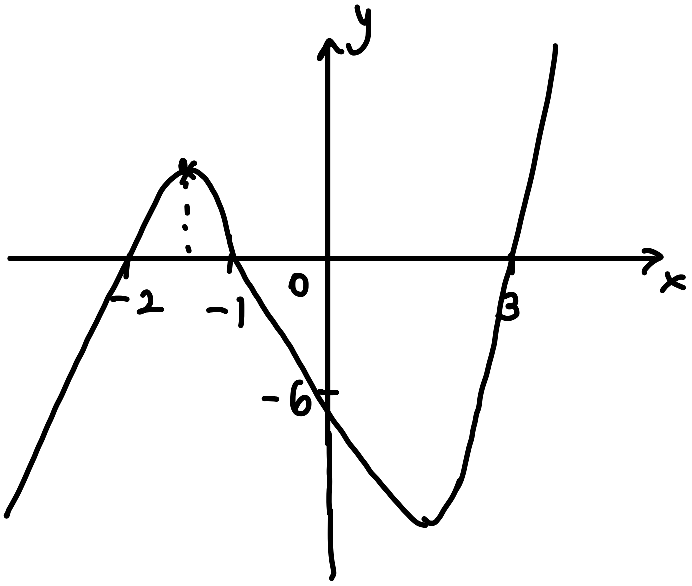
$x = -4, 3$

Graphing a polynomial

1. Find the zeros
2. Test various points
3. Look at the end behavior (as $x \to \pm\infty$, $y \to ?$)
4. Graph.

Examples

Sketch the graph of $f(x) = (x+2)(x-3)(x+1)$



zeros: $x = -2, -1, 3$

y-intercept. when $x=0$

$f(0) = (0+2)(0-3)(0+1)$

$= -6$

Test other points

e.g. when $x = -1.5$ $f(x) = (+)(-)(-) = (+)$

$x = 1$ $f(x) = (+)(-)(+) = (-)$

$= (3)(-2)(2)$

$= -12$

End-behavior $f(x) = (x+2)(x-3)(x+1)$

as $x \to +\infty$ $f(x) \to +\infty$

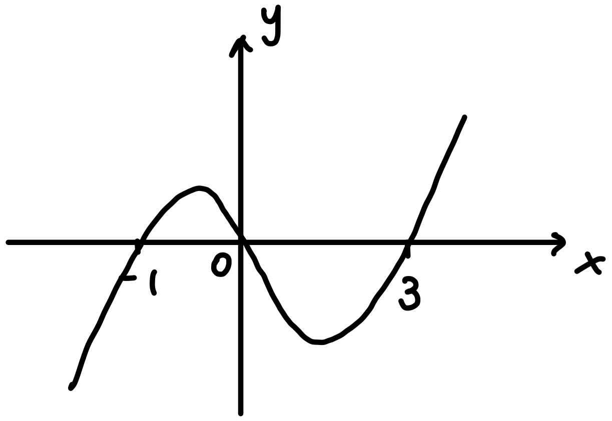
as $x \to -\infty$ $f(x) \to -\infty$

Example. $P(x) = x^3 - 2x^2 - 3x.$

$$= x(x^2 - 2x - 3)$$

$$= x(x+1)(x-3)$$

zeros : $x = -1, 0, 3.$



End-behavior ·
- leading coeff. is positive
- odd degree.

$x \to +\infty \quad y \to +\infty$

$x \to -\infty \quad y \to -\infty$

Example. $f(x) = -2x^4 - x^3 + 3x^2$. Sketch its graph.

$$= -x^2(2x^2 + x - 3)$$

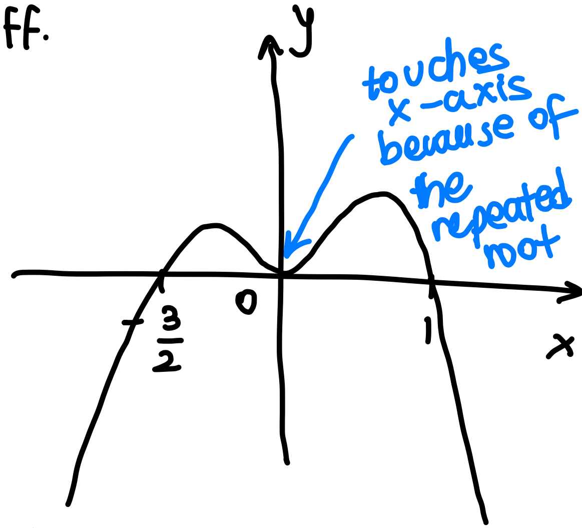
$$= -x^2(2x+3)(x-1)$$

zeros $x = -\frac{3}{2}, 0, 1$

↑ repeated root.

End-behavior: Even degree polynomial.
Negative leading coeff.

$$\left[\begin{array}{l} \text{as } x \to +\infty,\ y \to -\infty \\ \text{as } x \to -\infty,\ y \to -\infty \end{array}\right]$$



as $x \to \pm\infty$

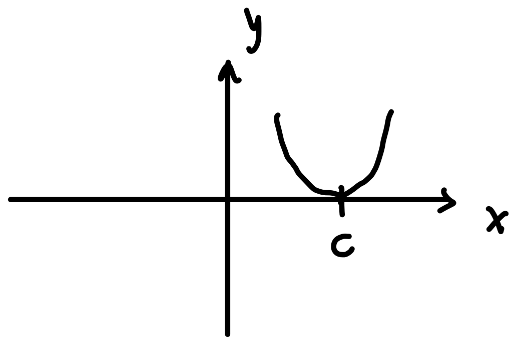
↳ $x \to \infty, \ x \to -\infty$

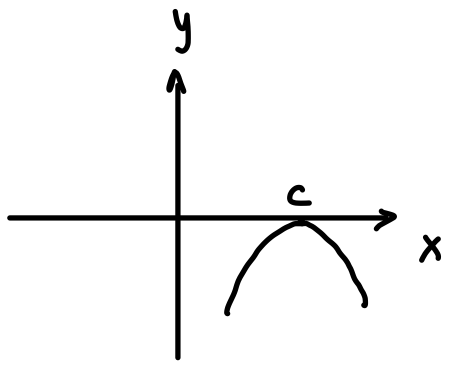
Multiplicities of the roots

Shape of polynomial near a zero of multiplicity m.

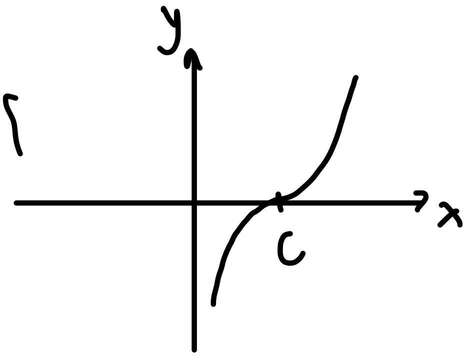
Assume c is a zero of f(x) and has multiplicity m. The shape of the graph of f(x) near c is as follows.

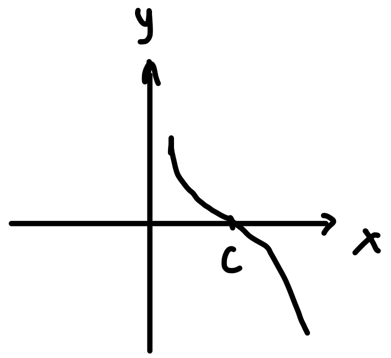
- m is even, m>1





- m is odd, m>1





Compare

$y = x^2$ to $y = x^4$

as $x \to \infty$

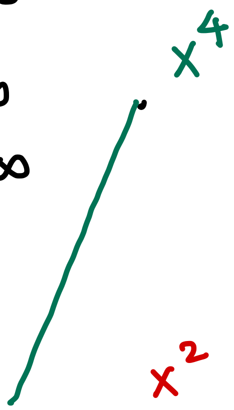
$y \to \infty$

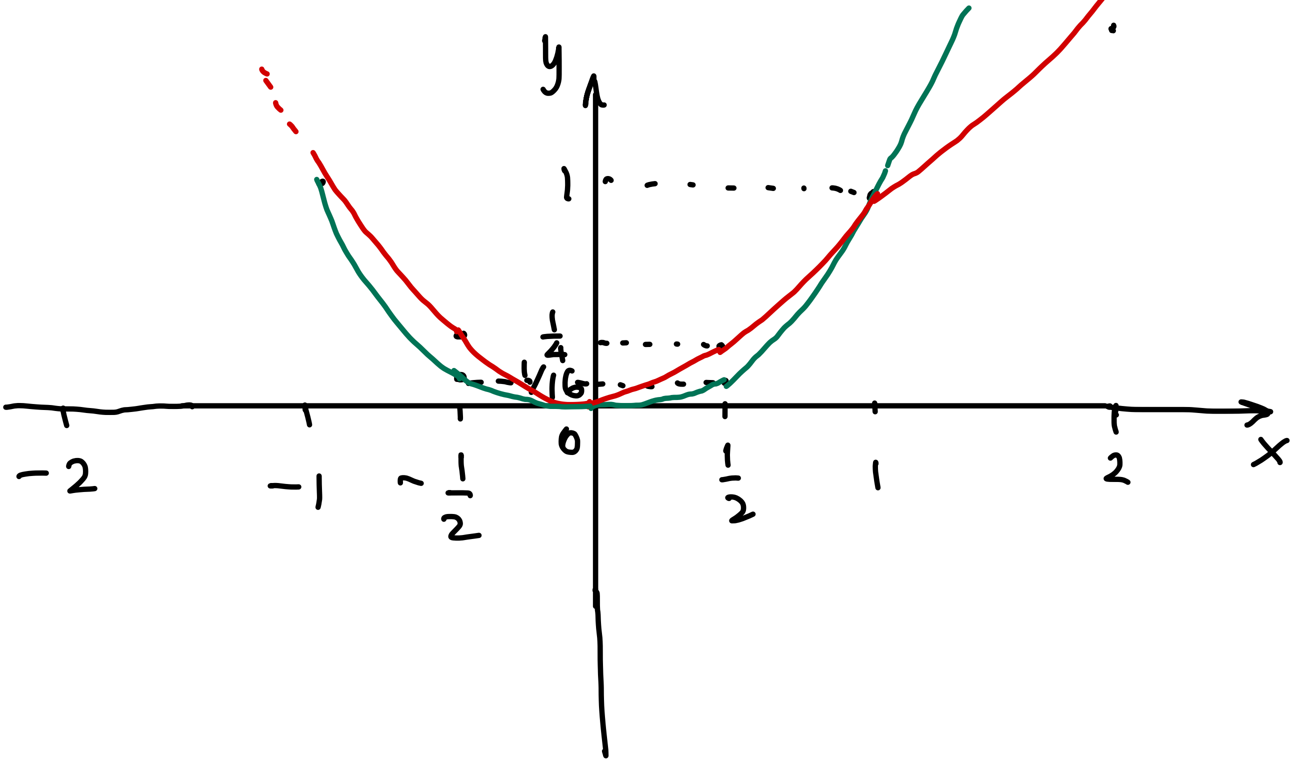
$x = \frac{1}{2}$ $y = \frac{1}{4}$

$x = \frac{1}{2}$ $y = \frac{1}{16}$

as $x \to -\infty$

$y \to +\infty$





Example Sketch $\boxed{f(x) = x^4 (x-2)^3 (x+1)^2}$

$a_n x^n + a_{n-1} x^{n-1} + \dots$

Zeros: $x = -1, 0, 2$

multiplicity: $2 \quad 4 \quad 3$

$$a_n x^{\overset{\downarrow}{n}} + a_{n-1} x^{n-1} + \dots + a_0$$

$x^m \cdot x^n = x^{m+n}$

$$x^4 \underset{\downarrow}{\underline{(x-2)^3}} \underbrace{(x+1)^2} = x^4 (x^3 + \dots)(x^2 + \dots)$$

$x^3 + \dots$

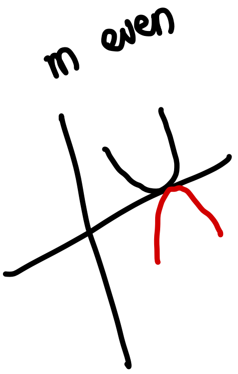
$$= x^{4+3+2} + \dots$$

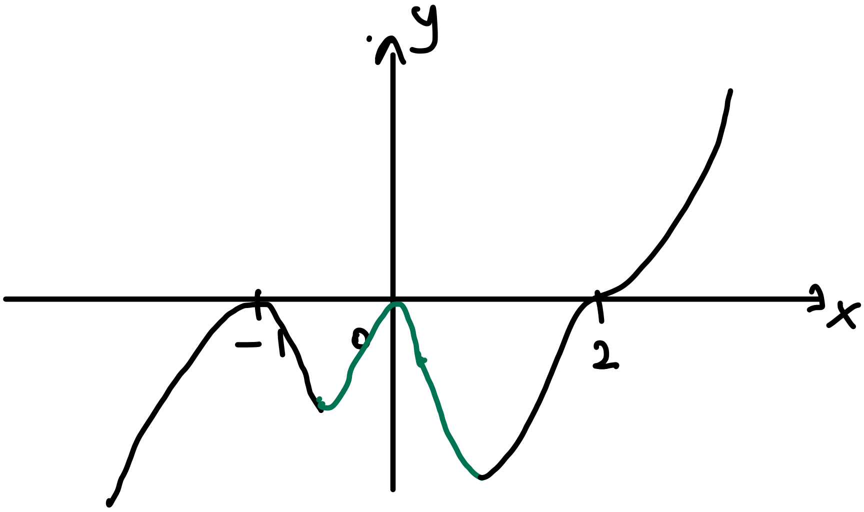
$$= x^9 + \dots$$

End-behavior: odd-degree

positive leading coeft.

as $x \to +\infty$, $y \to +\infty$ and as $x \to -\infty$, $y \to -\infty$



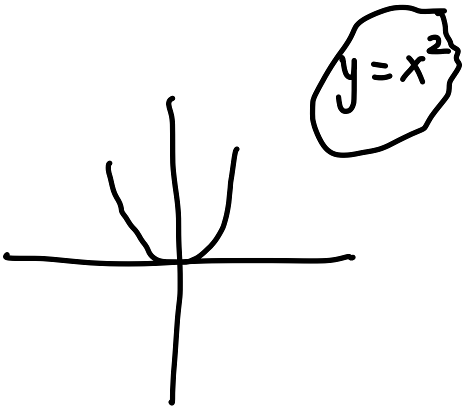


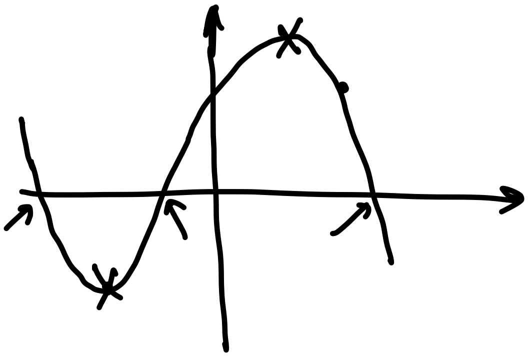
Local extrema of polynomials

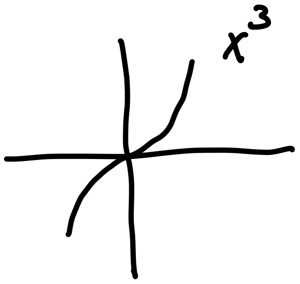
If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is a polynomial of degree n, then the graph of f(x) will have at most n-1 local extrema.

$\downarrow$

min or max







Find the maximum of f(x) . . .

$$f(x) = a(x-h)^2 + k \qquad \underline{f(h)} = a(\overset{0}{\cancel{h-h}})^2 + k = k$$

$$x = h \rightarrow \boxed{y = k} \leftarrow \text{max/min}$$

$$f(x) = ax^2 + bx + c$$

use $\quad x = -\dfrac{b}{2a}$

max/min $\quad f\left(-\dfrac{b}{2a}\right) = \cdots$

- Quiz this week on Section 3.2: Polynomial functions & their graphs
- HW 6 due tonight at 11:59 pm
- HW7 can also be found on Brightspace and Gradescope.

Section 3.3: Dividing polynomials

LONG DIVISION OF POLYNOMIALS

Division algorithm

If $f(x)$ and $D(x)$ are polynomials where $D(x) \neq 0$ then there is a unique polynomial $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$.

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

or, equivalently,

$$\underbrace{P(x)}_{\text{dividend}} = \underbrace{D(x)}_{\text{divisor}} \cdot \underbrace{Q(x)}_{\text{quotient}} + \underbrace{R(x)}_{\text{remainder}}$$

Long division algorithm

Example 1 Let $P(x) = 6x^2 - 26x + 12$. Divide by $x - 4$.

$$\begin{array}{r} 6x - 2 \\ x-4 \, \big) \overline{\, 6\cancel{x^2} - 26x + 12} \\ -\ \underline{6\cancel{x^2} - 24x } \downarrow \\ -\cancel{2x} + 12 \\ -\ \underline{-\cancel{2x} + 8} \\ 4 \end{array}$$

1. $$\frac{6x^2 - 26x + 12}{x - 4} = 6x - 2 + \frac{4}{x-4}$$

2. $$6x^2 - 26x + 12 = (x-4) \cdot (6x - 2) + 4$$

dividend divisor quotient remainder

Example 2. Divide $8x^4 + 6x^2 - 3x + 1$ by $2x^2 - x + 2$

$$\begin{array}{r} 4x^2 + 2x \\ 2x^2 - x + 2 \, \big) \overline{\, 8x^4 + 0x^3 + 6x^2 - 3x + 1} \\ -\ \underline{8x^4 - 4x^3 + 8x^2 } \\ 4x^3 - 2x^2 - 3x \\ -\ \underline{4x^3 - 2x^2 + 4x } \\ -7x + 1 \end{array}$$

$$\frac{8x^4 + 6x^2 - 3x + 1}{2x^2 - x + 2} = 4x^2 + 2x + \frac{(-7x + 1)}{2x^2 - x + 2}$$

OR $8x^4+6x^2-3x+1 = (2x^2-x+2)(4x^2+2x) - 7x+1.$

Example 3. Divide $2x^2-x-3$ by $x-3$

$$\begin{array}{r r}
 & 2x+5 \\ \cline{2-2}
x-3 \,\big) & 2x^2-x-3 \\
- & 2x^2-6x \\ \cline{2-2}
 & 5x-3 \\
- & 5x-15 \\ \cline{2-2}
 & 12
\end{array}$$

$$\frac{2x^2-x-3}{x-3} = 2x+5+\frac{12}{x-3}$$

$$2x^2-x-3 = (x-3)(2x+5)+12$$

Example 4 $\dfrac{x^6+x^4+x^2+1}{x+1}$

$$\begin{array}{r l}
 & x^5-x^4+2x^3-2x^2+3x-3 \\ \cline{2-2}
x+1 \,\big) & x^6+0x^5+x^4+0x^3+x^2+0x+1 \\
- & x^6+x^5 \\ \cline{2-2}
 & \quad -x^5+x^4 \\
- & \quad -x^5-x^4 \\ \cline{2-2}
 & \qquad 2x^4+0x^3 \\
- & \qquad 2x^4+2x^3 \\ \cline{2-2}
 & \qquad\quad -2x^3+x^2 \\
- & \qquad\quad -2x^3-2x^2 \\ \cline{2-2}
 & \qquad\qquad 3x^2+0x \\
- & \qquad\qquad 3x^2+3x \\ \cline{2-2}
 & \qquad\qquad\quad -3x+1 \\
- & \qquad\qquad\quad -3x-3 \\ \cline{2-2}
\end{array}$$

4

$$\frac{x^6+x^4+x^2+1}{x+1} = x^5 - x^4 + 2x^3 - 2x^2 + 3x - 3 + \frac{4}{x+1}$$

SYNTHETIC DIVISION

Example. Divide $2x^3 - 7x^2 + 5$ by $x-3$

the coefficients of each term in the original polynomial

result from the multiplication

$$\begin{array}{c|cccc} \textcircled{3} & 2 & -7 & 0 & 5 \\ & & 6 & -3 & -9 \\ \hline & \textcircled{2} & -1 & -3 & \boxed{-4} \end{array}$$

coefficients of the quotient

remainder

$$\underset{\text{dividend}}{2x^3 - 7x^2 + 5} = \underset{\text{divisor}}{(x-3)}\cdot(2x^2 - x - 3) - 4$$

$$\begin{array}{r} 2x^2 - x - 3 \\ x-3 \overline{\smash{)}\, 2x^3 - 7x^2 + 0x + 5} \\ -\ 2x^3 - 6x^2 \\ \hline -x^2 + 0x \\ -\ \ -x^2 + 3x \\ \hline -3x + 5 \\ -\ \ -3x + 9 \\ \hline -4 \end{array}$$

* Use synthetic division to divide $P(x) = \underline{3x^5+5x^4-4x^3+7x+3}$ by $\underline{x+2}$. $\rightarrow x - \boxed{(-2)}$

Find $P(\underset{\substack{\uparrow \\ c}}{-2}) = \underset{\substack{\uparrow \\ \text{remainder}}}{5}$

$$\begin{array}{r|rrrrrr} -2 & 3 & 5 & -4 & 0 & 7 & 3 \\ & & -6 & 2 & 4 & -8 & 2 \\ \hline & 3 & -1 & -2 & 4 & -1 & \boxed{5} \end{array}$$

quotient coeff.

remainder

$$3x^5+5x^4-4x^3+7x+3 = (3x^4-x^3-2x^2+4x-1)(x+2) + 5$$

Exercise. Use synthetic division for $\dfrac{4x^2-3}{x-2}$.

$$\begin{array}{r|rrr} 2 & 4 & 0 & -3 \\ \hline & & 8 & 16 \\ \hline & 4 & 8 & 13 \end{array}$$

$$\frac{4x^2-3}{x-2} = (4x+8) + \frac{13}{x-2}$$

$$4x^2-3 = (x-2)\cdot(4x+8) + 13$$

Note Synthetic division can only be used if the divisor is of the form $(x-c)$.

Remainder theorem.

If the polynomial $P(x)$ is divided $x-c$, then the remainder is the value $P(c)$.

optional **Proof**:

$$P(x) = (x-c)\cdot Q(x) + r$$

$$x=c \qquad P(c) = \overset{0}{\cancel{(c-c)\cdot Q(c)}} + r = r$$

$P(c)$ is the remainder r.

Factor theorem

c is a zero of P if and only if $x-c$ is a factor of $P(x)$.

optional **Proof**. If $P(x)$ factors as $P(x) = (x-c)\cdot Q(x)$ then

$$P(c) = (\underbrace{c-c}_{\overset{\parallel}{0}})\cdot Q(c) = 0$$

Conversely, if $P(c)=0$ then by the remainder theorem

$$P(x) = (x-c)\,Q(x) + 0$$

$$= (x-c)\cdot Q(x)$$

$\Rightarrow$ $x-c$ is a factor of $P(x)$

... continuing Section 3.3 "Dividing polynomials" from last time.

WebAssign 3.3.

⑧ Find a polynomial of degree 3 that has zeros 1, −6, 7 and in which the coefficient of x^2 is 3.

From the factor theorem we know that

$x-1$, $x-(-6)$, and $(x-7)$

are factors of the desired polynomial

$$P(x) = \underset{\uparrow}{a}(x-1)(x-(-6))(x-7)$$

constant to be found.

$$\Rightarrow \quad P(x) = a(x-1)\underbrace{(x+6)(x-7)}$$

$$= a(x-1)(x^2-x-42)$$

$$= a(x^3 - \underline{x^2} - \underline{42x} - \underline{x^2} + \underline{x} + 42)$$

$$= a(x^3 - 2x^2 - 41x + 42)$$

$$= ax^3 - 2ax^2 - 41ax + 42a$$

coeff of x^2: $\quad -2a = 3 \quad \Rightarrow \quad \boxed{a = -\frac{3}{2}}$

Therefore $\quad P(x) = -\frac{3}{2}(x^3 - 2x^2 - 41x + 42)$

OR $\quad = -\frac{3}{2}x^3 + 3x^2 + \frac{123}{2}x - 63$

Example. Find a polynomial of degree 4 that has zeros $-3, 0, 1$, and 5 and has coefficient -6 in front of x^3.

Factors $\quad x-(-3),\ x,\ x-1,\ x-5$

$$P(x) = a(x+3)(x)\underbrace{(x-1)(x-5)}$$

$$= a(x+3)\underbrace{(x)(x^2-6x+5)}$$

$$= a\underbrace{(x+3)(x^3-6x^2+5x)}$$

$$= a(x^4 \underline{-6x^3} + \underline{5x^2} + \underline{3x^3} \underline{-18x^2} + 15x)$$

$$= a(x^4 - 3x^3 - 13x^2 + 15x)$$

$$= ax^4 - 3ax^3 - 13ax^2 + 15ax$$

x^3: $\quad -3a = -6 \Rightarrow \boxed{a = 2}$

So $P(x) = 2x^4 - 6x^3 - 26x^2 + 30x.$

Example: Consider $f(x) = \underline{x^3 - 7x + 6}$. Show that $\underline{f(1) = 0}$ and using this factor $f(x)$ completely.

$f(1) = 1^3 - 7(1) + 6 = 1 - 7 + 6 = 0$ ✓

Because $f(1) = 0 \Rightarrow x = 1$ is a zero of $f(x)$

$\Rightarrow x - 1$ is a factor of $f(x)$ (by the factor theorem).

$$f(x) = x^3 - 7x + 6 = (x-1)\cdot Q(x) + R(x).$$

Long Division

$$\begin{array}{r|l}
 & \boxed{x^2 + x - 6} \\ \hline
x-1 & x^3 + 0x^2 - 7x + 6 \\
- & \underline{x^3 - x^2} \quad \downarrow \\
 & x^2 - 7x \\
- & \underline{x^2 - x} \quad \downarrow \\
 & -6x + 6 \\
- & \underline{-6x + 6} \\
 & \underline{\underline{0}}
\end{array}$$

Synthetic division

$$\begin{array}{c|cccc}
1 & 1 & 0 & -7 & 6 \\ \hline
 & \downarrow & 1 & 1 & -6 \\ \hline
 & 1 & 1 & -6 & \underset{\uparrow \text{ remain.}}{0}
\end{array}$$

$$\Rightarrow f(x) = (x-1)(x^2 + x - 6)$$

$$= (x-1)(x+3)(x-2)$$

* Section 3.6 : Rational functions.

Definition : A rational function is of the form

$$\boxed{r(x) = \frac{P(x)}{Q(x)}}$$

Where $P(x)$ and $Q(x)$ are polynomials and have no common factor

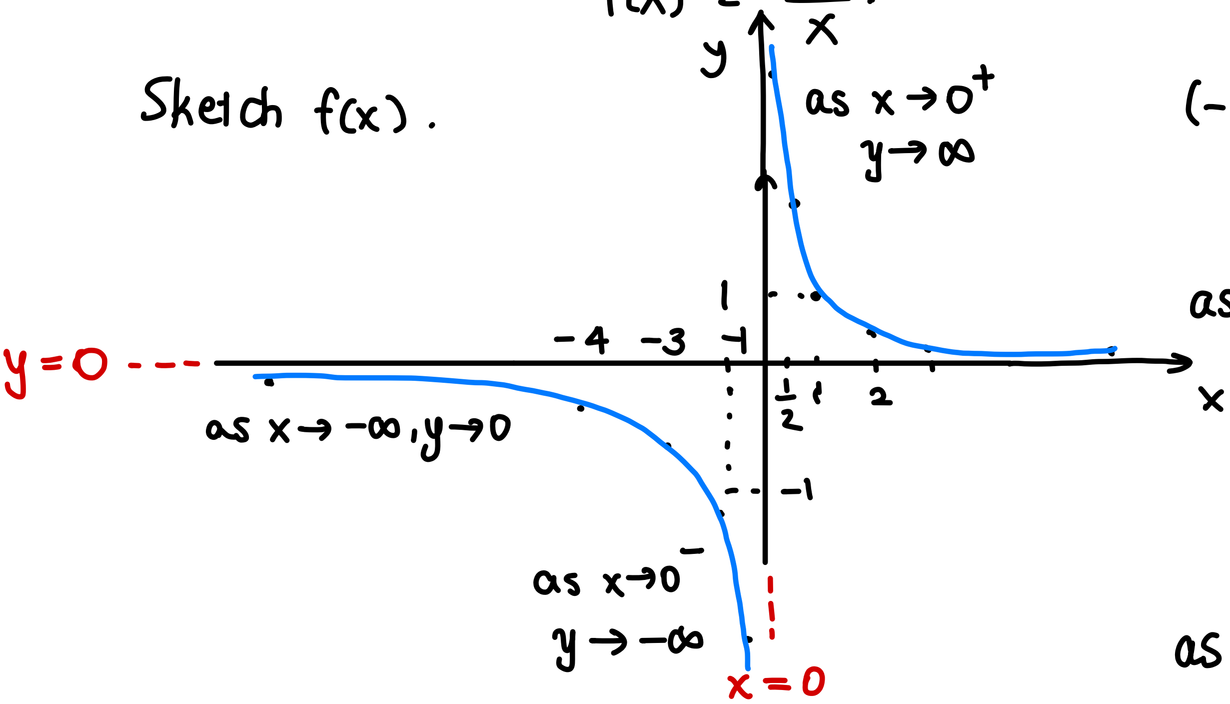
Recall

$y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ is a polynomial when the exponents are non-negative integers

Example: One of the simplest rational functions is

$$f(x) = \frac{1}{x}.$$

Sketch f(x).



as $x \to 0^+$
$y \to \infty$

as $x \to -\infty, y \to 0$

as $x \to 0^-$
$y \to -\infty$

Domain:
$(-\infty, 0) \cup (0, \infty)$
$\boxed{x \neq 0}$

as $x \to \infty, y \to 0$

as $x \to \infty$
$\frac{1}{x} \to 0$

Note. $\frac{1}{\text{big number}}$ = small number

$\frac{1}{\text{small number}}$ = big number

$$\frac{1}{0.01} = \frac{1}{\left(\frac{1}{100}\right)} = 100$$

Symbols

$x \to a^-$ x tends to a from the left

$x \to a^+$ x tends to a from the right

$x \to \infty$ x tends to infinity (i.e. x increases without bound)

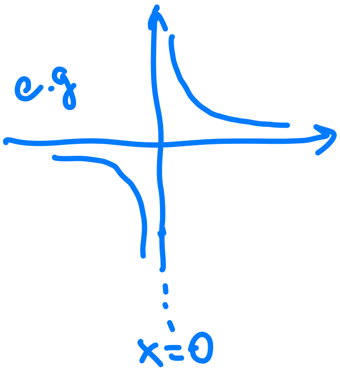
$x \to -\infty$ x tends to negative infinity (i.e. x decreases without bound)

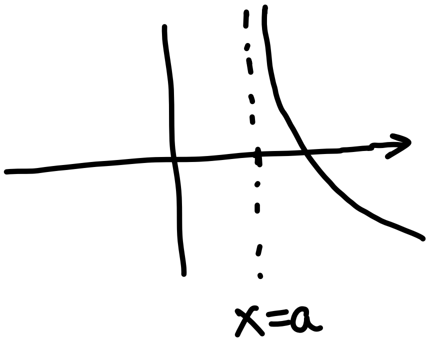
Definitions

Vertical asymptotes. The line $\boxed{x=a}$ is a vertical asymptote of $y=f(x)$ is y approaches $\pm\infty$ as x approaches a either from the right or from the left

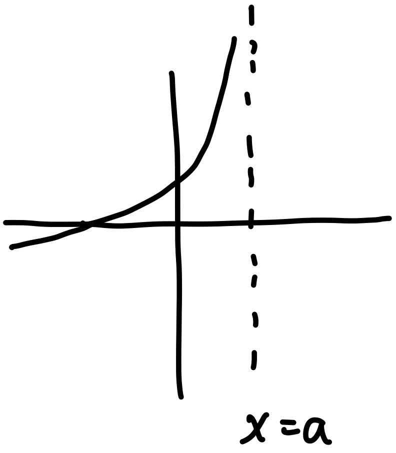
∞, or $-\infty$

e.g

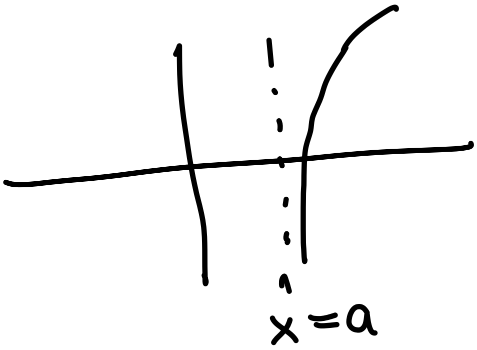




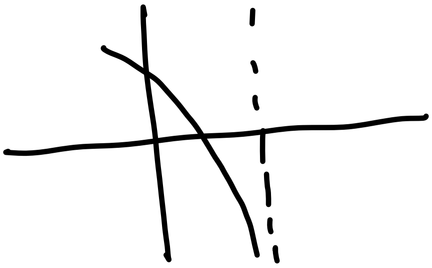
$y \to \infty$ as $x \to a^+$



$y \to \infty$ as $x \to a^-$

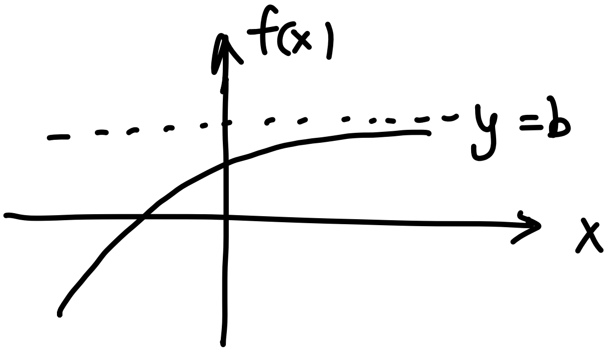


$y \to -\infty$ as $x \to a^+$

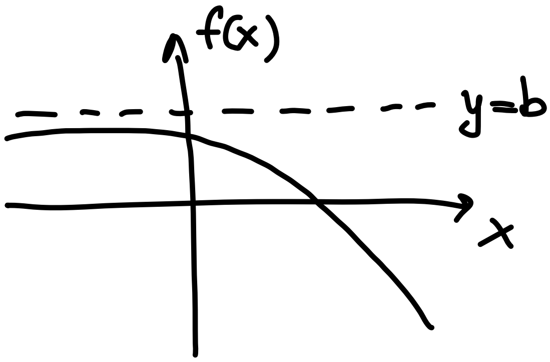


$y \to -\infty$ as $x \to a^-$

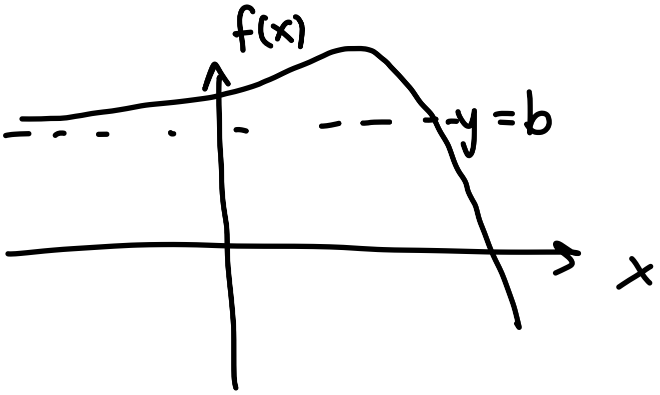
Horizontal asymptotes The line $\boxed{y=b}$ is a horizontal asymptote of $y=f(x)$ if y approaches b as $x \to \pm\infty$



y → b as x → ∞

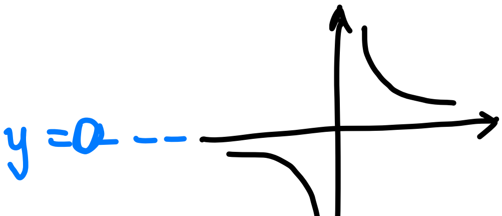


y → b as x → -∞



y → b as x → -∞

Graphing rational functions



Good to start with transformations of $y = \boxed{\frac{1}{x} = f(x)}$

Ex 1. Graph $r(x) = \frac{2}{x-3}$.

$$= 2\left(\frac{1}{x-3}\right)$$

$$= 2f(x-3)$$

$y=0 \Rightarrow 0 = \frac{2}{x-3}$

solve for x?

$0 \neq 2$

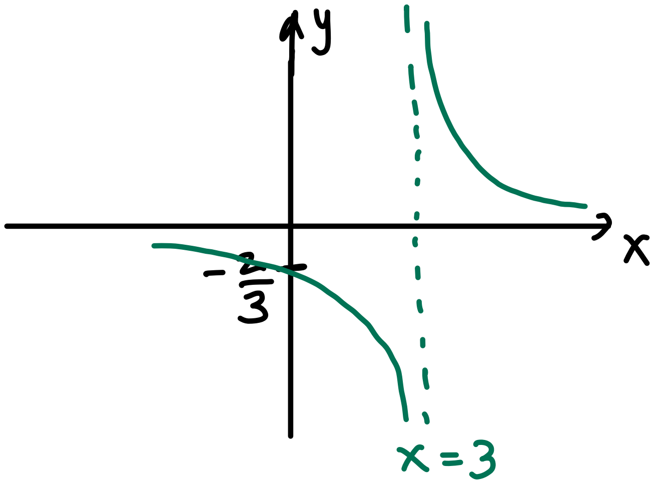
no x-intercept.

Vertical stretch by a factor of 2

Horizontal shift to the right by 3

vertical asymptote at $x=3$ (x-value for which the denominator is zero).

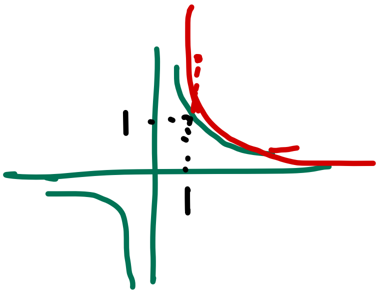
horizontal asymptote at $y=0$



y-intercept

$x=0$

$r(0) = \frac{2}{0-3} = -\frac{2}{3}$



In general $\boxed{r(x) = \frac{ax+b}{cx+d}}$ linear fractional transformations

(use $\frac{1}{x}$ as your guide).

e.g. $r(x) = \frac{3x+5}{x+2}$. Graph $r(x)$.

$$\begin{array}{r|l} & \;\;3 \\ x+2 & \;3x+5 \\ & -\;3x+6 \\ \hline & \quad\;\; -1 \end{array}$$

$$\boxed{r(x) = 3 - \frac{1}{x+2}}$$

$$r(x) = \underline{Q(x)} + \frac{R(x)}{D(x)}$$

$f(x) = \frac{1}{\boxed{x}}$

$$r(x) = - \frac{1}{\boxed{x+2}} + 3$$

$$= - f(\boxed{x+2}) + 3$$

$f(x) = \frac{1}{x}$

$f(x+2) = \frac{1}{x+2}$

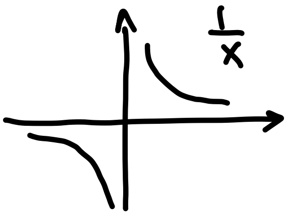
Transformations:

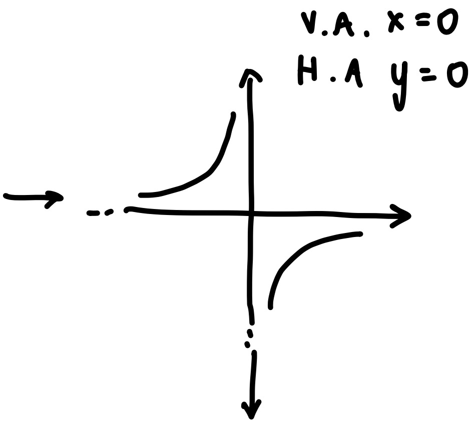
- Reflections about x-axis
- Shift up by 3 units
- Horiz. shift to the left by 2.

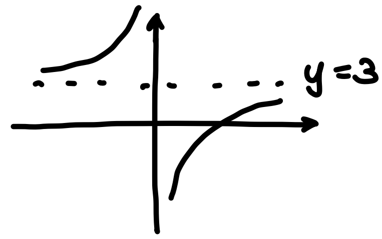
$f\left(\frac{1}{x+2}\right) = \frac{1}{\left(\frac{1}{x+2}\right)} = x+2$

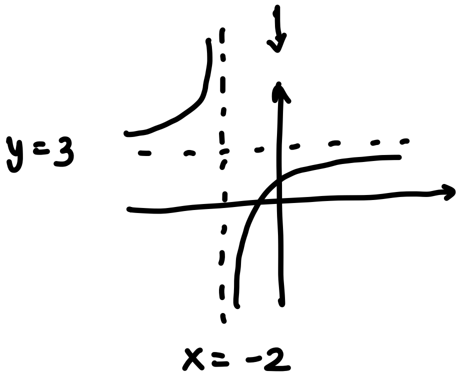
$f(x+2) = \frac{1}{x+2}$ ✓

From last time...









$r(x) = 3 - \frac{1}{x+2}$

$f(x) = \frac{1}{x}$

$f\left(\frac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)} = 2$

$f(2) = \frac{1}{2}$

$\boxed{f(x+2) = \frac{1}{x+2}}$

$f(x-3) = \frac{1}{x-3}$

$f\left(\frac{1}{x+1}\right) = \frac{1}{\left(\frac{1}{x+1}\right)} = x+1$

$r(x) = \boxed{\frac{1}{x+2} = f(x+2)}$

- Quiz this week: Long division and synthetic division
- HW 7 due tonight at 11:59pm
- WebAssign
- Office hours today on Zoom at 3:30-4:30pm

Example. Write $r(x) = \frac{2x-9}{x-4}$ as a transformation of $f(x) = \frac{1}{x}$.

$$\left[\begin{array}{r} 2 \\ x-4\,\overline{)\,2x-9} \\ -\ 2x-8 \\ \hline -1 \end{array} \qquad r(x) = 2 - \frac{1}{x-4}\right]$$

OR

$$r(x) = \frac{(2x-8)+(8-9)}{x-4} = \frac{2(x-4)-1}{x-4} = 2 - \frac{1}{x-4} = -\frac{1}{x-4} + 2$$

$$= -f(x-4) + 2$$

ASYMPTOTES

To find the vertical asymptote (V.A.) you set the denominator to zero and solve for x. The V.A. is an equation of the form $x=a$ for some constant a.

Horizontal asymptotes (H.A.)

Let r be a rational function of the form

$$r(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0}$$

1. If $n > m$, then r has no horizontal asymptote.
2. If $n < m$, then r has a H.A. at $y=0$.
3. If $n=m$, then r has a H.A. at $y = \frac{a_n}{b_m}$

if $r(x) = \frac{x^2}{x+2}$

NO H.A.

Example ① Find the vertical and horizontal asymptotes of

$$r(x) = \frac{3x^2-2x-1}{2x^2+3x-2}.$$

$$r(x) = \frac{3x^2-2x-1}{(2x-1)(x+2)}$$

$$\boxed{\text{V. A.} \quad x = \frac{1}{2}, -2}$$

$$\boxed{\text{H.A.} \quad y = \frac{3}{2}}$$

② $r(x) = \frac{2x-3}{x^2-1} = \frac{2x-3}{(x-1)\cdot(x+1)}$

V.A. $x = -1, 1$

H.A. $y = 0$

③ $r(x) = \frac{6x^3-2}{2x^3+5x^2+6x} = \frac{6x^3-2}{x(2x^2+5x+6)}$

$$\boxed{\begin{array}{ll} \text{V. A.} & x=0 \\ \text{H. A.} & y=3 \end{array}}$$

Example. Graphing rational functions

$$r(x) = \frac{x^2-4}{2x^2+2x}$$

STEP 1. Try to factor $\boxed{r(x) = \frac{(x-2)(x+2)}{2x(x+1)}}$ ★

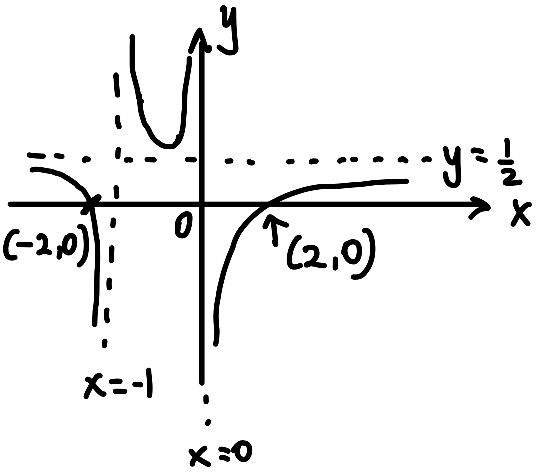
STEP 2. x-intercepts : Set $y=0$ (the $\overset{\text{when}}{\text{numerator}}$ is 0) $(-2,0)$, $(2,0)$

y-intercepts : set $x=0$ $r(0) \overset{?}{=} \frac{(-2)(2)}{0(1)}$ $\boxed{\text{No y-intercept}}$

STEP 3. Find V.A. and H.A.

V.A: $\boxed{x = -1, 0}$

H.A: $y = \frac{1}{2}$.



STEP 4. Find the behavior close to the V.A.

As $x \to -1^-$ $y \to \frac{(-)(+)}{(-)(-)} = (-)$
e.g. $x=-1.1$

As $x \to -1^+$ $y \to \frac{(-)(+)}{(-)(+)} = (+)$
e.g $x = -0.9$

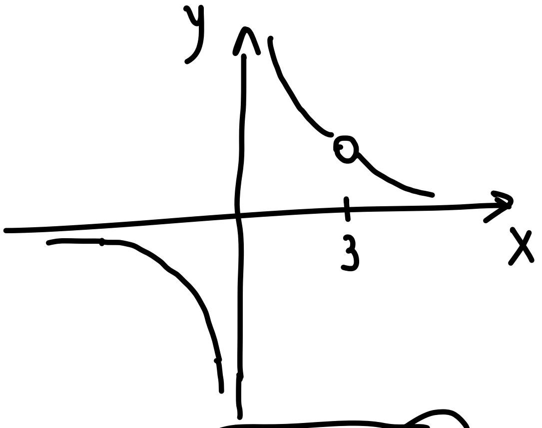
As $x \to 0^-$ $y \to \frac{(-)(+)}{(-)(+)} = (+)$
e.g. $x = -0.1$

As $x \to 0^+$ $y \to \frac{(-)(+)}{(+)(+)} = (-)$
e.g. $x = 0.1$

HOLES in rational functions.

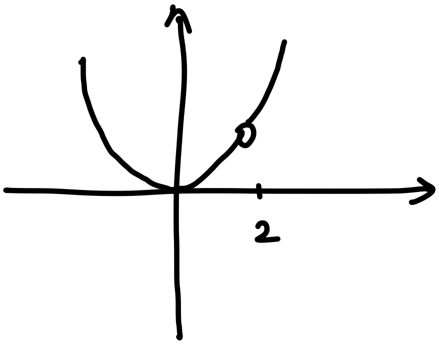
1. Consider $r(x) = \frac{x-3}{x^2-3x} = \frac{\cancel{x-3}}{x(\cancel{x-3})}$ x = 3 is a hole.

$$= \frac{1}{x} \quad \text{for } x \neq 3.$$



2. $r(x) = \boxed{\frac{x^3-2x^2}{x-2}} = \frac{x^2(\cancel{x-2})}{\cancel{x-2}}$ Hole $x=2$

$= x^2$ for $x \neq 2$



SLANT ASYMPTOTES

If $r(x) = \frac{P(x)}{Q(x)}$ is a rational function in which the degree of the numerator is one more than the degree of the denominator, we can use long division to write it in the form

$$\boxed{r(x) = ax+b+\frac{R(x)}{Q(x)}}$$

where the degree of $R(x)$ is less than the degree of Q and $a \neq 0$.

Example. Consider $r(x) = \frac{x^2-4x-5}{x-3}$. Graph.

$$r(x) = \frac{(x+1)(x-5)}{x-3}$$

✓ x-intercepts: $(-1,0)$, $(5,0)$

✓ y-intercepts: $r(0) = \frac{1(-5)}{(-3)} = \frac{5}{3}$ $\quad (0, \frac{5}{3})$.

✓ V.A. $x=3$

H.A. No H.A.

e.g. $x=2.9$

Behavior close to the V.A. as $x \to 3^-$ $y \to \frac{(+)(-)}{(-)} = (+)$

as $x \to 3^+$ $y \to \frac{(+)(-)}{(+)} = (-)$.

e.g. $x=3.1$

SLANT ASYMPTOTE.

$$r(x) = \frac{x^2-4x-5}{x-3}$$

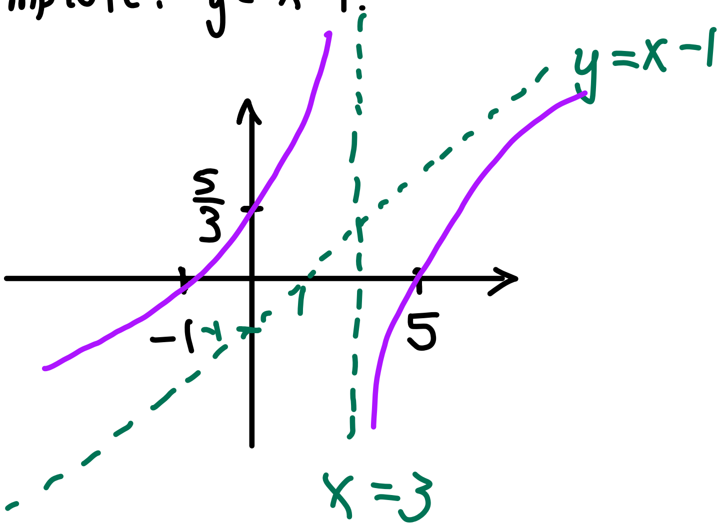
$$\begin{array}{rr} & x-1 \\ x-3 \big) & x^2-4x-5 \\ - & \underline{x^2-3x} \\ & -x-5 \\ - & \underline{-x+3} \\ & -8 \end{array}$$

Wanted: $r(x)$ $ax+b+\frac{R(x)}{Q(x)}$

$$\boxed{r(x) = x-1-\frac{8}{x-3}}$$

Since the denominator is one degree less than the numerator, the function has a slant asymptote

Slant asymptote: $y = x-1$.



Section 4.1 Exponential functions

Definition: The exponential function with base a is defined for all real values x by

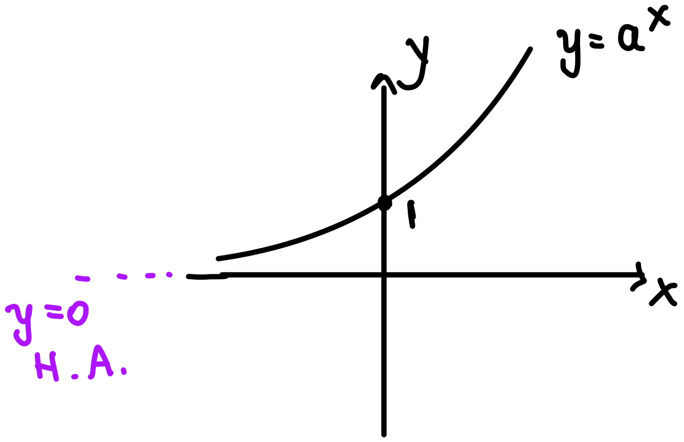
$$\boxed{f(x)= a^x}$$

where $a>0$ and $a\neq 1$.

e.g. 2^x

as $x \to -\infty$

$2^{-1000000} = \frac{1}{2^{1000000}} \approx 0$



y-intercepts set $x=0$

$f(0)= a^0 = 1.$

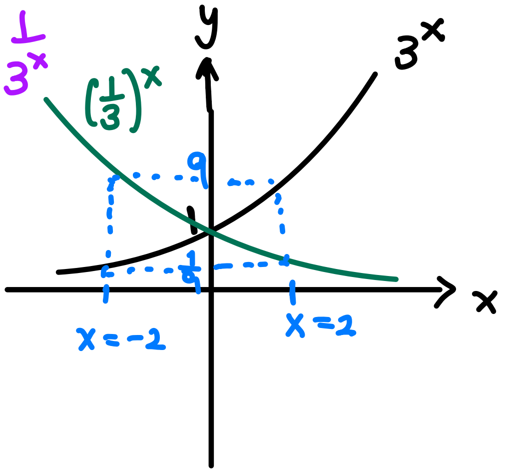
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

$\frac{1^x}{3^x} = \frac{1}{3^x}$

e.g. Graph 3^x and $\left(\frac{1}{3}\right)^x = \frac{1}{\boxed{3^x}}$

as $x \to \infty$
$y \to 0$

as $x \to -\infty$
$y \to \infty$

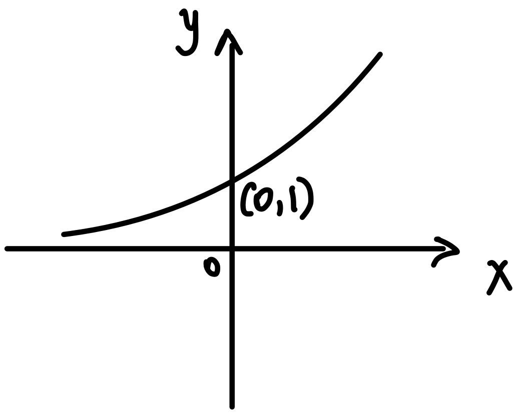


$\left(\frac{1}{3}\right)^{-2} = \frac{1}{3^{-2}}$

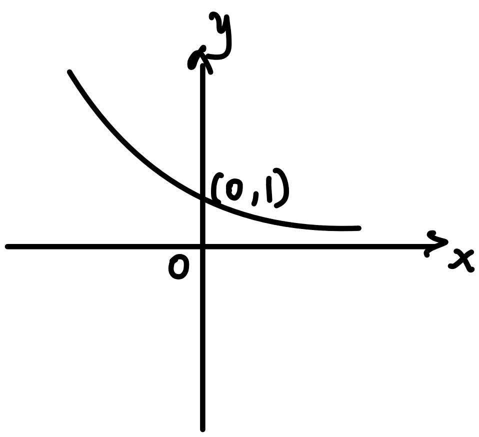
$= 3^2 = 9$

$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

In general, we summarize it like this

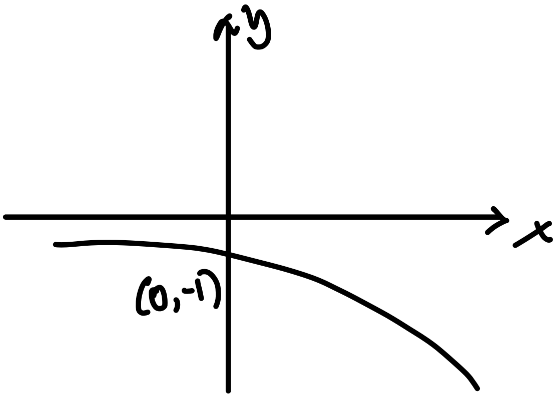


$f(x) = a^x, \ a>1$



$f(x) = a^x \ , \ 0<a<1$

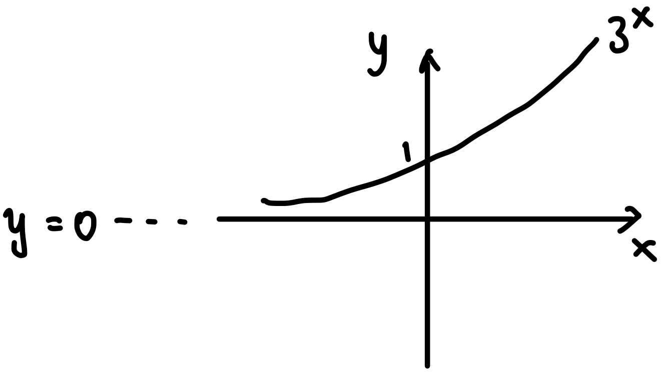
Example. Graph $f(x) = -2^x$.



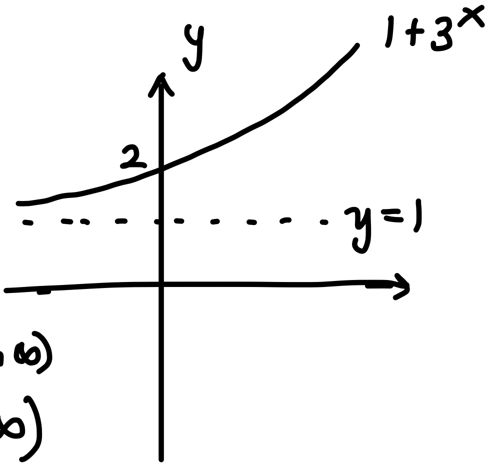
Office hours today at 5:30-6:30pm in WWH Room 1025.

Transformations of exponential functions

e.g. Graph $f(x) = 1 + 3^x$.



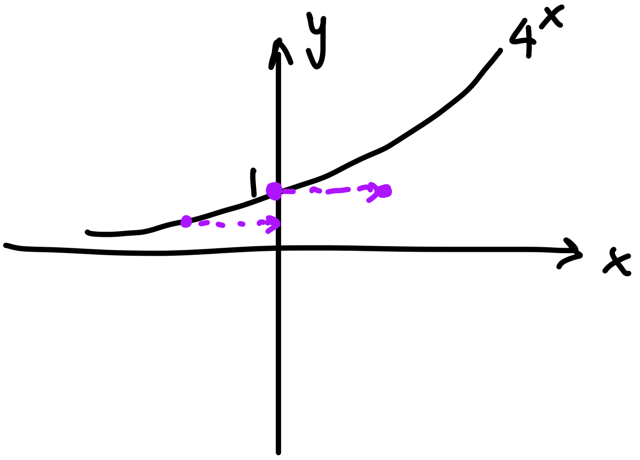
shift up by 1 unit $\longrightarrow$

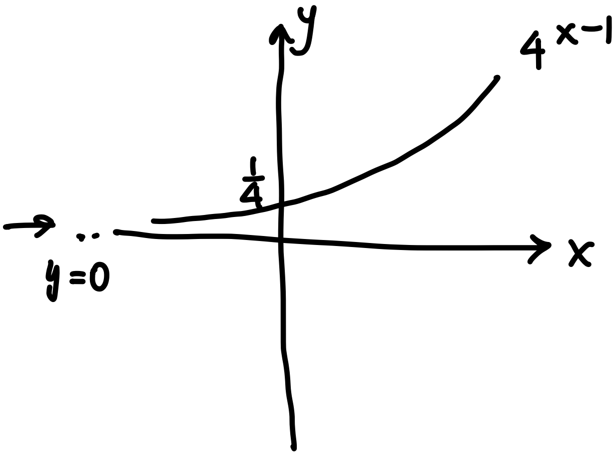


Domain : $(-\infty, \infty)$

Range : $(1, \infty)$

e.g Graph $h(x) = 4^{x-1}$ shift to the right by 1.





Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

when $x = 0 \Rightarrow h(0) = 4^{0-1}$

$= 4^{-1}$

$= \frac{1}{4}$

Upcoming HW: STUDY FOR MIDTERM!

Ex Find the exponential function of the form $y = C \cdot a^x$ which passes through $(-1, 2)$ and $(4, 5)$.

want to find a and C.

STEP 1: Use one of the coordinates.

$$(\underset{x}{-1}, \underset{y}{2})$$

$$2 = C \cdot a^{-1} \Rightarrow 2 = \frac{C}{a} \Rightarrow \boxed{C = 2a}$$

STEP 2: Use the other point $(\underset{x}{4}, \underset{y}{5})$

$$5 = C \cdot a^4$$

Subst. $C = 2a$ into $5 = C \cdot a^4$

$$5 = 2a \cdot a^4 \Rightarrow 5 = 2a^5$$

$$\frac{5}{2} = a^5$$

$$a = \sqrt[5]{\frac{5}{2}}$$

or $\boxed{a = \left(\frac{5}{2}\right)^{\frac{1}{5}}}$

We also have $C = 2a$

$$\Rightarrow \boxed{C = 2 \cdot \left(\frac{5}{2}\right)^{1/5}}$$

Now subst. into $y = C \cdot a^x \Rightarrow y = 2 \cdot \left(\frac{5}{2}\right)^{\frac{1}{5}} \cdot \left[\left(\frac{5}{2}\right)^{\frac{1}{5}}\right]^x$

$(a^m)^n = a^{m \cdot n}$

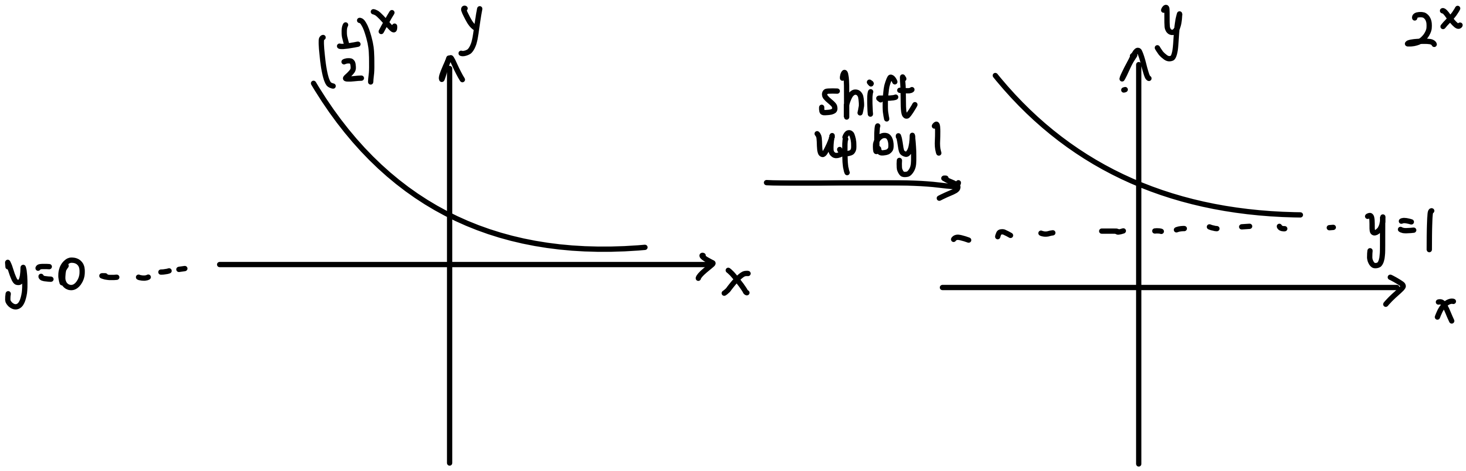
$$\boxed{y = 2\left(\frac{5}{2}\right)^{1/5} \cdot \left(\frac{5}{2}\right)^{x/5}}$$

WebAssign 4.1

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

12. Graph $y = 2^{-x} + 1 = \frac{1}{2^x} + 1 \overset{\downarrow}{=} \left(\frac{1}{2}\right)^x + 1$

$$\left(\frac{1}{2}\right)^x = \frac{1^x}{2^x} = \frac{1}{2^x}$$



Domain: $(-\infty, \infty)$

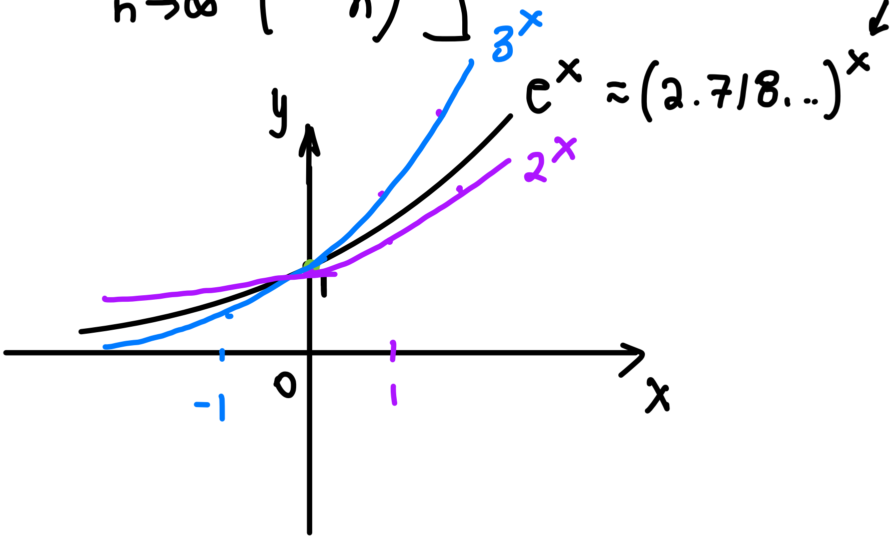
Range: $(1, \infty)$.

State the asymptote: $\boxed{y=1}$

Section 4.2 : The natural exponential function.

The number e: $\boxed{e \approx 2.71828...}$ ←

[Aside: $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$.]



Sketch on the same axis $y = 2^x$ and $y = 3^x$.

$y = 2^x$, $x=1$ $y=2$

$y = e^x$, $x=1$ $y=e$

$y = 3^x$, $x=1$ $y=3$

$y = 2^x$ $\quad x = -1,\ y = 2^{-1} = \frac{1}{2}$

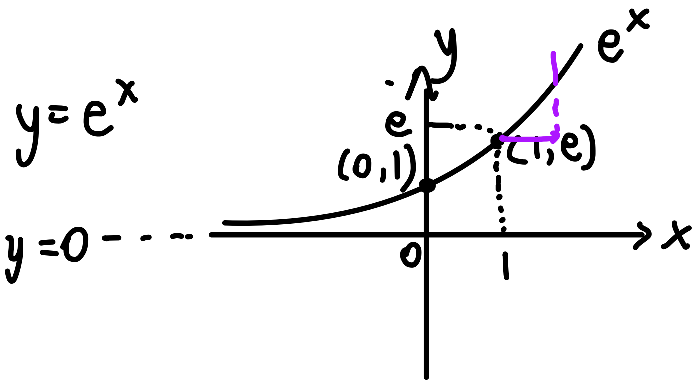
$y = e^x$ $\quad x = -1,\ y = \frac{1}{e}$

$y = 3^x$ $\quad x = -1,\ y = \frac{1}{3}$

Transformations of the natural exponential function

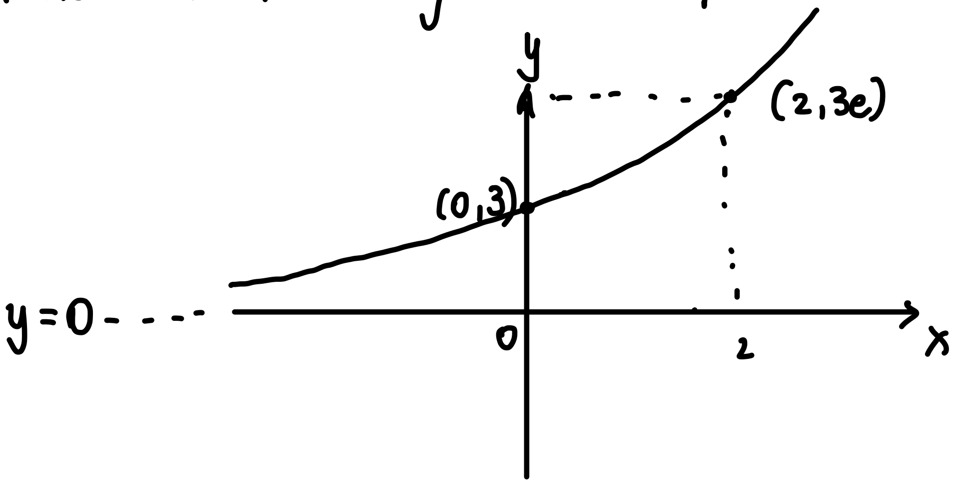
e.g Graph $y = 3e^{\frac{1}{2}x}$. Find the domain, the range, and asymptote

Sketch first $y = e^x$



Vertical stretch by a factor of 3.

Horizontal stretch by a factor of 2.



$(1, e) \rightarrow (2, 3e)$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

An infectious disease begins to spread in a city with population 10 000. After t days, the number of people who have the virus is given by

$$v(t) = \frac{10\ 000}{5 + 1245e^{-0.97t}}$$

$$e^{-0.97t} = \frac{1}{e^{0.97t}}.$$

as $t \to \infty$
$e^{-0.97t} \to 0$

(a) How many people had the virus initially? [Find V(t) when t=0]

$$V(0) = \frac{10\,000}{5 + 1245\underbrace{e^{-0.97(0)}}_{1}} = \frac{1000\cancel{0}}{125\cancel{0}} = 8 \text{ people.}$$

$$C \cdot a^x \Leftrightarrow C \cdot e^{kx} = C \cdot (e^k)^x \quad \text{using} \quad a^{mn} = (a^m)^n$$

$$\Rightarrow \quad a = e^k$$

(b) Find the number of people that have the virus by day 5.

$$V(5) = \frac{10000}{5 + 1245 e^{-0.97(5)}} \text{ people.}$$

COMPOUND INTEREST

(not going to ask about this in Midterm 2 but might be in the final).

This is calculated by the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

r: interest rate per year

t number of years

amount after t years.

principal

n: number of times the interest is compounded per year.

Example. A sum of \$1000 is invested at a rate of 12% per year. Find the amount in the account after 3 years if it's compounded annually, monthly and daily.

$P = \$1000$
$r = 0.12$
$t = 3$

annually $n=1$ $\longrightarrow$ $A(3) = 1000\left(1+\frac{0.12}{1}\right)^{1(3)}$

$= 1000(1.12)^3$

$= \$1404.93$

monthly $n=12$ $\longrightarrow$ $A(3) = 1000\left(1+\frac{0.12}{12}\right)^{12(3)}$

$= \$1430\ 77$

daily $n=365$ $\longrightarrow$ $A(3) = 1000\left(1+\frac{0.12}{365}\right)^{365(3)}$

$= \$1433.24.$

Note that exponential functions grow faster than polynomial or power functions

<u>Note</u>

Exponential functions : $y = C\cdot a^x$ e.g. $y = 2^x$

Power functions : $y = C\cdot x^a$ e.g. $y = x^2$

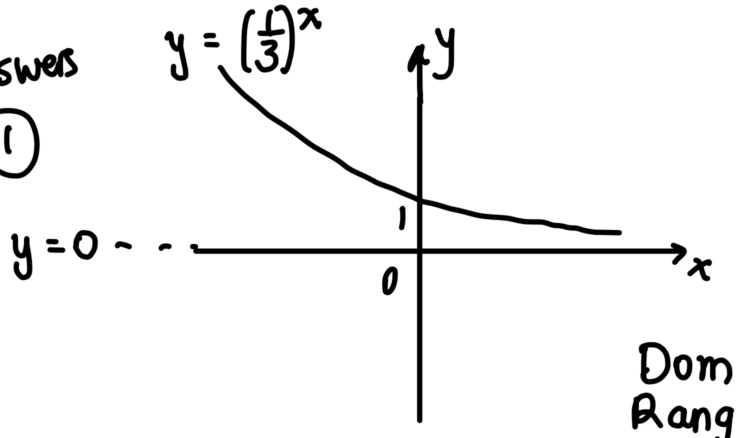
Try the following exercises: (Quiz next week).

① Sketch $h(x) = 4+\left(\frac{1}{3}\right)^x$. State domain, range, asymptote

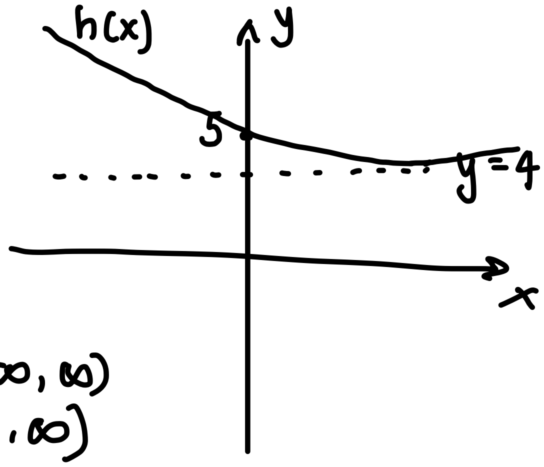
② Sketch $\boxed{g(x) = 2^{x-4}-1}$. State domain, range, asymptote.

Answers

①



$\longrightarrow$

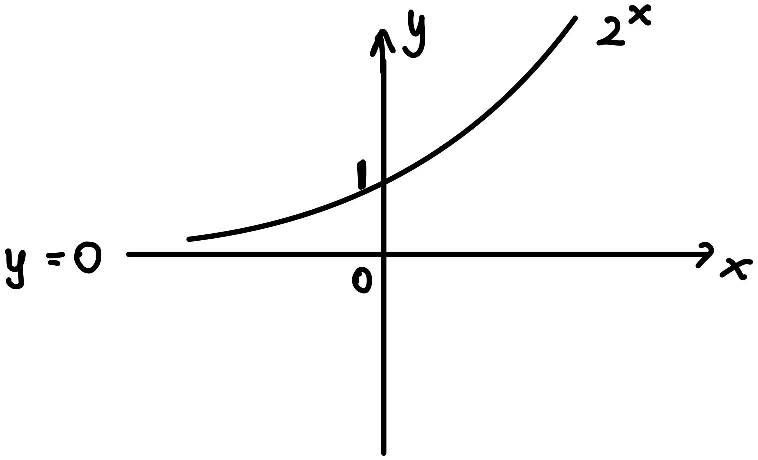


Domain : $(-\infty, \infty)$

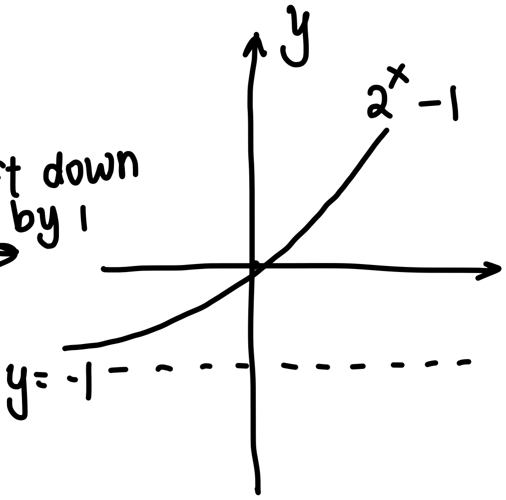
Range : $(4, \infty)$

Asymptote: $y=4$.

②



shift down by 1 →



↓ shift to the right by 4

$$y = 2^{x-4} - 1$$

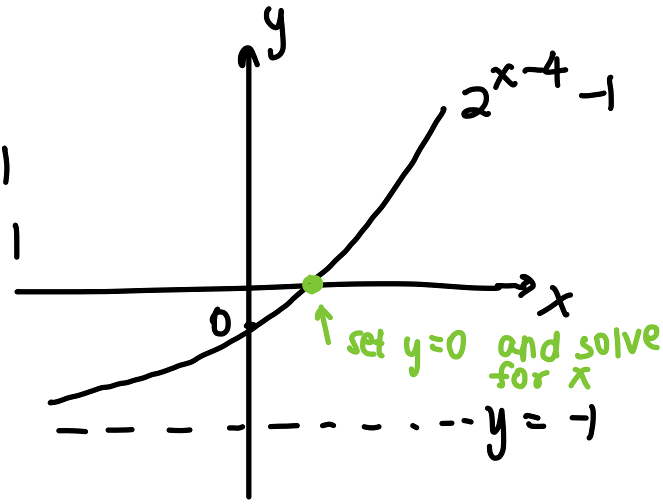
y-intercept: $x = 0$

$$2^{0-4} - 1 = 2^{-4} - 1$$

$$= \frac{1}{2^4} - 1$$

$$= \frac{1}{16} - 1$$

$$= -\frac{15}{16}$$



Domain: $(-\infty, \infty)$

Range: $(-1, \infty)$

Asymptote: $y = -1$.

Recall.

Difference quotient. $\dfrac{f(x+h) - f(x)}{h}$

e.g. If $f(x) = 3^{x-1}$, show that $\dfrac{f(x+h) - f(x)}{h} = 3^{x-1}\left(\dfrac{3^h - 1}{h}\right)$.

$f(x+h) = 3^{x+h-1}$, $f(x) = 3^{x-1}$

$$\frac{f(x+h)-f(x)}{h} = \frac{3^{x+h-1} - 3^{x-1}}{h}$$

$a^{x+y} = a^x a^y$

$3^{x+h-1} = 3^{x-1+h}$

$= 3^{x-1}3^h$

$$= \frac{\boxed{3^{x-1}}3^h - \boxed{3^{x-1}}}{h}$$

$$= \frac{3^{x-1}(3^h-1)}{h}$$

$$\frac{2(3)}{6} = 2\cdot\left(\frac{3}{6}\right)$$

$$= 3^{x-1}\cdot\left(\frac{3^h-1}{h}\right)$$

//

Midterm 2: $\boxed{\text{Sections } 2.7, 2.8, 3.1-3\;3, 3.6, 4.1, 4.2}$

Layout for the exam: 10 multiple choice questions (4 pts each)
5 free response questions (6 pts each)

Section 4.3: Logarithmic functions

Definition: Let a be a positive number ($a \neq 1$), then the logarithmic function with base a (which we denote by $\log_a$) is defined by

$$\log_{\underset{\text{base}}{a}} \overset{\text{result}}{x} = \underset{\text{exponent}}{y} \iff a^y = x$$

$\log_2 8 = 3 \iff 2^3 = 8$

$\log_2\left(\frac{1}{8}\right) = ? \iff 2^? = \frac{1}{8}$ where $? = -3$

$$\log_{10}(1\,000\,000) = 6$$

$$\log_2 x = 5 \iff 2^5 = x \Rightarrow x = 32.$$

Properties

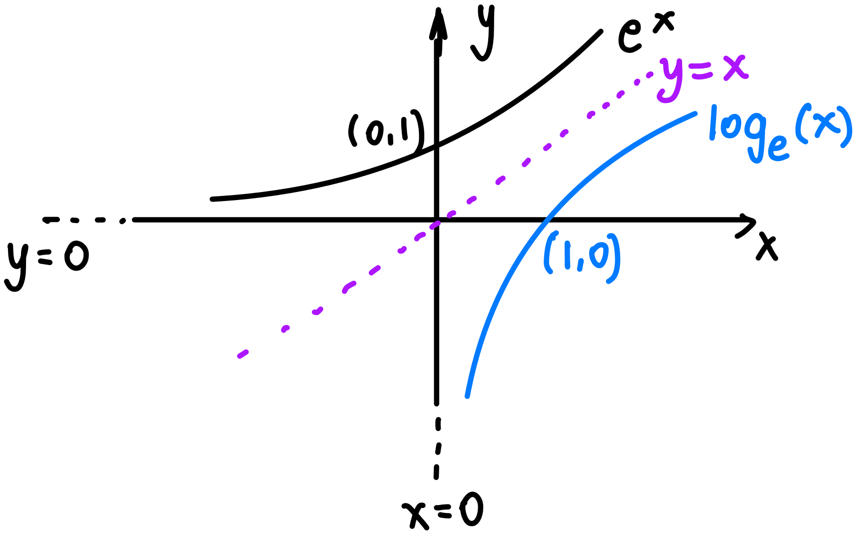
1. $\log_a 1 = 0$ $\quad (a^0 = 1)$
2. $\log_a a = 1$ $\quad (a^1 = a)$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$

$\log_a x$ and a^x are inverses of each other

$$f(f^{-1}(x)) = x$$

$$f^{-1}(f(x)) = x$$

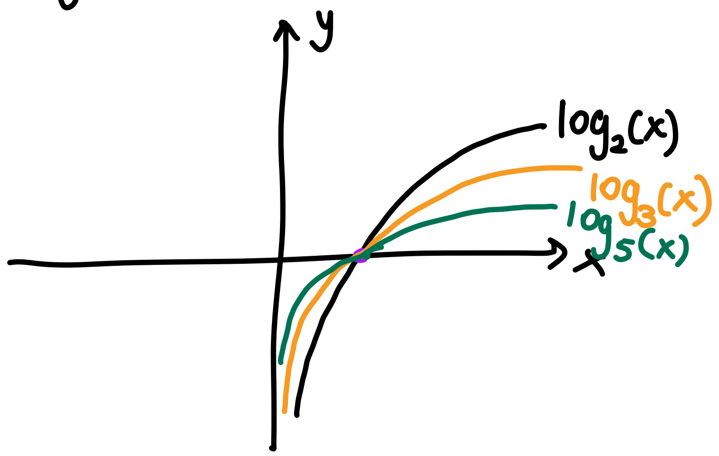
Graphs of logarithmic functions



Domain of $\log_a(x)$: $(0, \infty)$

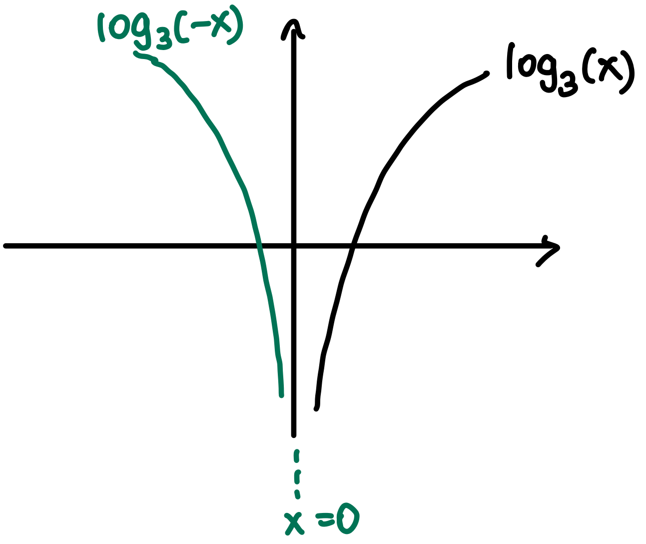
Range of $\log_a(x)$: $(-\infty, \infty)$

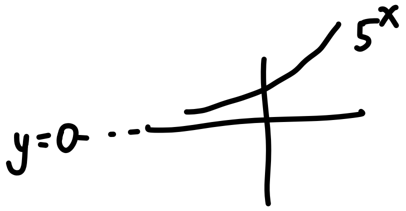
In general, logarithmic functions with different bases look as follows



Example Sketch the following graphs using transformations of $\log_a(x)$.

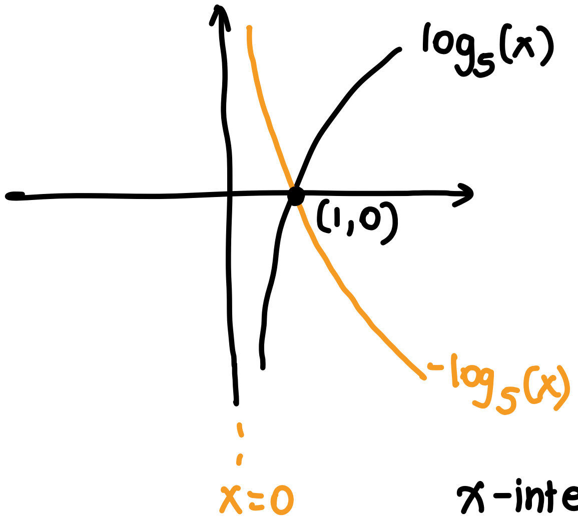
(a) $y = \log_3(-x)$ Domain: $(-\infty, 0)$, Range: $(-\infty, \infty)$





(b) $y = -\log_5(x)$

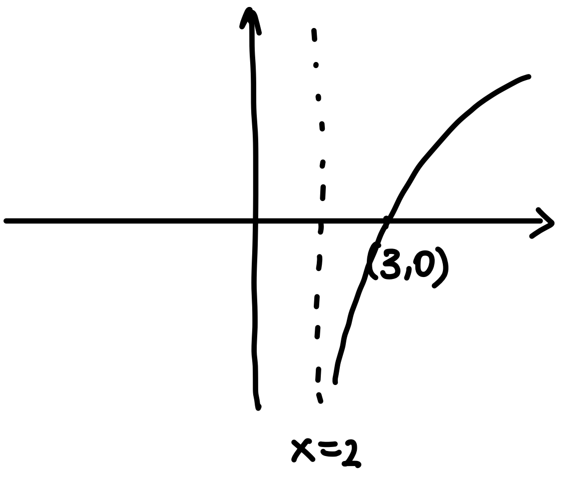
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$



x-intercept:
$y=0$
$y = -\log_5(x)$
$0 = -\log_5 x$
$5^0 = x$
$x = 1$

(c) $y = \log_4(x-2)$

shift $\log_4 x$
to the right by 2



Common logarithms

The logarithm with base 10 is called the common logarithm and usually we omit the base:

$$\boxed{\log x = \log_{10} x}$$

e.g

$\log 100 = 2$

$\log 10 = 1$

$\log_{10} 100 = 2$

$\log 0.001 = -3$

Example. The loudness in a room is measured in dB and is given by

intensity of the sound

$$B = 10 \log\left(\frac{I}{I_0}\right)$$

constant.

Find the loudness level in dB when $\boxed{I = 100 I_0}$.

$$B = 10 \log\left(\frac{100\cancel{I_0}}{\cancel{I_0}}\right)$$

$$= 10 \log(100)$$

$$= 10(2)$$

$$= 20 \text{ dB}$$

NATURAL LOGARITHM

The logarithm with base e is called the natural logarithm and is written as

$$\boxed{\ln(x) = \log_e(x)}$$

Note: $\ln(x) = y \iff e^y = x$

$\ln(x) = y$

$\log_e(x) = y$

$e^y = x$

Properties of the natural logarithm

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

] e^x and $\ln(x)$ are inverses of each other so by $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ we get properties 3. and 4.

e.g. $\ln\left(\frac{1}{e^2}\right) = \ln(e^{-2}) = -2$ (by 3.)

e.g. $e^{\ln 8} = 8$

Exercises.

1. Express the following equations in exponential form

(a) $\log_8 4 = \frac{2}{3} \iff \boxed{8^{2/3} = 4}$ $\quad \underbrace{((\sqrt[3]{8})^2 = 2^2 = 4)}$ $\quad \log_a b = x$

$a^x = b$

(b) $\log_{10} 3 = 2t \iff 10^{2t} = 3$

(c) $\ln(x-1) = 4 \iff e^4 = x-1$

2. Evaluate the following.

(a) $\log_6 1 = 0$

(b) $e^{\ln(\frac{1}{\pi})} = \frac{1}{\pi}$ $\qquad e^{\ln x} = x$

(c) $\log_4 \sqrt{2} = x$. Find x. $\quad 4^x = \sqrt{2} \Rightarrow (2^2)^x = \sqrt{2}$

$2^{2x} = 2^{1/2}$

$2x = \frac{1}{2} \Rightarrow x = \frac{1}{4}$

(d) $\log_5 125 = 3$.

Thus $\log_4 \sqrt{2} = \frac{1}{4}$.

Section 4.4. Laws of logarithms

Laws

1. $\log_a(AB) = \log_a A + \log_a B$

2. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$

3. $\log_a(A^C) = C \cdot \log_a(A)$

Examples

Evaluate each of the following

1. $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$

$$= \log_4(64)$$
$$= 3$$

2. $-\frac{1}{3}\log 8 = \log\left(8^{-1/3}\right)$

$$= \log\left(\frac{1}{8^{1/3}}\right)$$
$$= \log\left(\frac{1}{\sqrt[3]{8}}\right)$$
$$= \log_{10}\left(\frac{1}{2}\right)$$
$$\approx -0.301 \text{ (using calculator)}$$

-0.301

$10^{x} = \frac{1}{2}$

$$a^{-m} = \frac{1}{a^m}$$

Common mistakes to AVOID:

- $\log_a(A+B) \neq \log_a(A) + \log_a(B)$
- $\log_a(A-B) \neq \log_a(A) - \log_a(B)$
- $\log_a\left(\frac{A}{B}\right) \neq \frac{\log_a(A)}{\log_a(B)}$

$(\log_a A)^C \neq C\log_a A$

Expanding and combining logarithms . Use the laws to expand these .

e.g. (a) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6 = 3\log_5 x + 6\log_5 y$

Law 1

$\log_a(AB) = \log_a(A) + \log_a(B)$

Law 3: $\log_a(A^C) = C\log_a A$

(b) $\ln\left(\frac{xy^{1/2}}{\sqrt[3]{z}}\right) = \ln(xy^{1/2}) - \ln(\sqrt[3]{z})$

$= \ln(x) + \ln(y^{1/2}) - \ln(\sqrt[3]{z})$

$= \ln(x) + \frac{1}{2}\ln(y) - \frac{1}{3}\ln(z)$

Law 2

$\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$

Expressing logarithms as a single logarithm

Ex. (a) $4\ln(s) + \frac{1}{3}\ln(w^2) - \ln(p^2-1)$

$= \ln(s^4) + \ln(w^{2/3}) - \ln(p^2-1)$

using law 3

$\log_a(A^C) = C\log_a A$

$= \ln(s^4 . w^{2/3}) - \ln(p^2-1)$

$= \ln\left(\frac{s^4 w^{2/3}}{p^2-1}\right)$

(b) $\frac{1}{3}\log((x+2)^3) + \frac{1}{2}\log x^4 - \log(x-1)$

$\log[(x+2)^{3 \cdot 1/3}]$

law 3 law 3

$= \log(x+2) + \log(x^2) - \log(x-1)$

law 1

$= \log(x^2(x+2)) - \log(x-1)$

$= \log\left(\frac{x^2(x+2)}{x-1}\right)$ law 2

$(A+B)^2 \neq A^2+B^2$

$= (A^2+2AB+B^2)$

$= (A+B)(A+B)$

$= A^2 + \boxed{AB+AB} + B^2$

$\sqrt{x^2+4} = (x^2+4)^{1/2} \neq x+2$

$\swarrow$ $(A+B)^{1/2} \neq A^{1/2} + B^{1/2}$

Expanding : $\log\sqrt{\frac{x^2+4}{(x^2+1)(x^3-7)^2}} = \log\left(\left(\frac{x^2+4}{(x^2+1)(x^3-7)^2}\right)^{1/2}\right)$

$$= \overset{\text{law 3}}{\frac{1}{2}\log\left(\frac{x^2+4}{(x^2+1)(x^3-7)^2}\right)} = \frac{1}{2}\left[\log(x^2+4) - \overset{\text{law 3}}{\underbrace{\log\left((x^2+1)(x^3-7)^2\right)}}\right]$$

$$= \frac{1}{2}\left[\log(x^2+4) - \left(\underbrace{\log(x^2+1) + \log(x^3-7)^2}_{\text{law 1}}\right)\right]$$

$$= \frac{1}{2}\left[\log(x^2+4) - \log(x^2+1) - \log((x^3-7)^2)\right]$$

$$= \frac{1}{2}\log(x^2+4) - \frac{1}{2}\log(x^2+1) - \log(x^3-7)$$

$$\log\left(\left(\frac{x^2+4}{(x^2+1)(x^3-7)^2}\right)^{1/2}\right) = \log\left(\frac{(x^2+4)^{1/2}}{(x^2+1)^{1/2}(x^3-7)^1}\right)$$

$$= \log(x^2+4)^{1/2} - \left(\log\left((x^2+1)^{1/2}\right) + \log(x^3-7)\right)$$

Change of base formula

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$$

Suppose you are given $\log_a x$ and you want to find $\log_b x$

A special case of this is $\log_a b = \frac{1}{\log_b a}$.

e.g.

Use the formula to evaluate ↙ use $b=8$ and $a=10$.

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} = \dots \quad \text{(use calculator)}$$

$$\log\left(\sqrt{x\sqrt{y\sqrt{z}}}\right) = \log\left(\left(x\sqrt{y\sqrt{z}}\right)^{1/2}\right)$$

$$= \frac{1}{2}\log\left(\underset{A}{x}\underbrace{\sqrt{y\sqrt{z}}}_{B}\right) \qquad \log(AB) = \log A + \log B$$

$$= \frac{1}{2}\left[\log(x) + \log\left(\sqrt{y\sqrt{z}}\right)\right]$$

$$= \frac{1}{2}\left[\log(x) + \log\left(\left(y\sqrt{z}\right)^{1/2}\right)\right]$$

$$= \frac{1}{2}\left[\log(x) + \frac{1}{2}\log\left(y\sqrt{z}\right)\right]$$

$$= \frac{1}{2}\left[\log x + \frac{1}{2}\left[\log y + \overset{\frac{1}{2}\log z}{\log\sqrt{z}}\right]\right]$$

$$= \frac{1}{2}\left[\log x + \frac{1}{2}\left(\log y + \frac{1}{2}\log z\right)\right)$$

$$= \frac{1}{2}\log x + \frac{1}{4}\log y + \frac{1}{8}\log z$$

- New Homework (#9) on Gradescope

Section 5.1: The Unit Circle

The unit circle is a circle with radius 1 and center at the origin given the equation

$$x^2 + y^2 = 1$$

In general, the equation of any circle with radius r and center at (a, b) is given by

$$(x-a)^2 + (y-b)^2 = r^2$$

Terminal points

Given some number t. if $t \geq 0$ then you measure the distance t along the unit circle starting at $\boxed{(1,0)}$ and moving in the counterclockwise direction

If $t < 0$, then you move $|t|$ in the clockwise direction starting at $(1,0)$.

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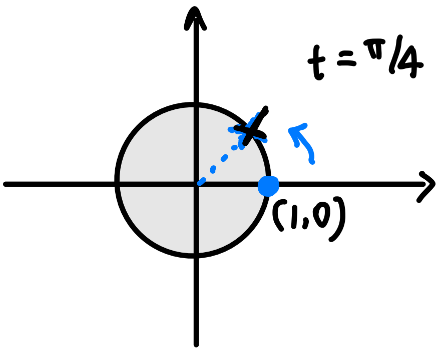
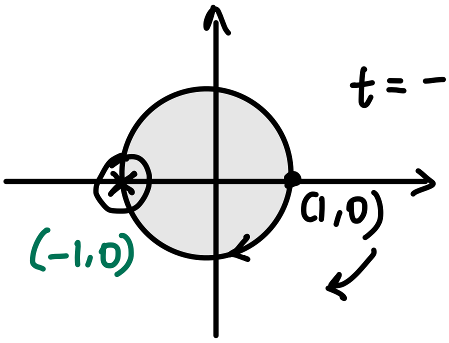




Finding the terminal points.

① Assume we are on the unit circle.

(a) $t = 3\pi$

(b) $t = -\frac{\pi}{2}$.

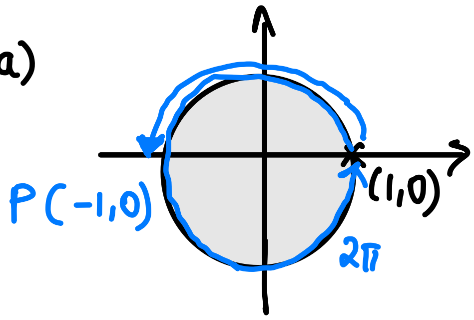
(c) $t = -\frac{3\pi}{2}$

Note

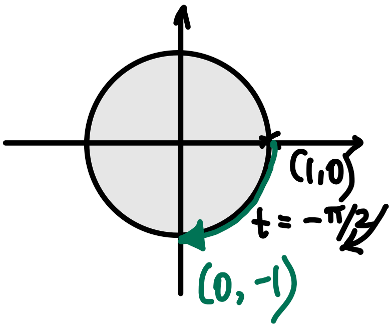
Circumference of a unit circle is 2π.

$(C = 2\pi r = 2\pi(1) = 2\pi)$

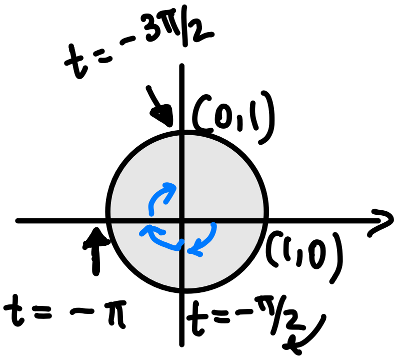
(a)



(b)



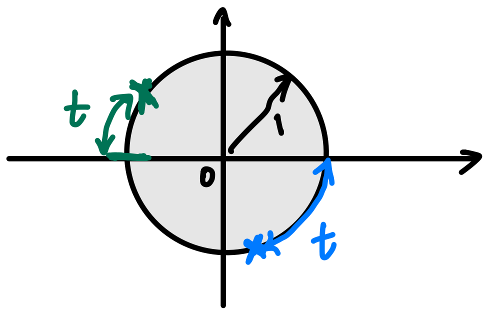
(c)



$t = \pi/2$ and $t = -\frac{3\pi}{2}$ give us the same terminal point.

The reference number

Let t be a real number. The t is the shortest distance along the unit circle between the terminal point determined by the value t and the x-axis.

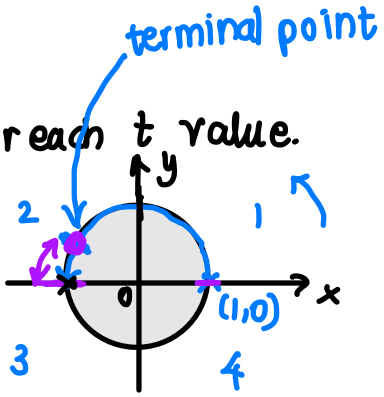


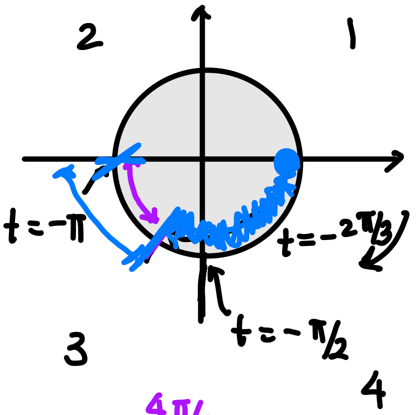
e.g. Find the reference number for each t value.

(a) $t = \frac{5\pi}{6} < \pi$

(b) $t = -\frac{2\pi}{3}$

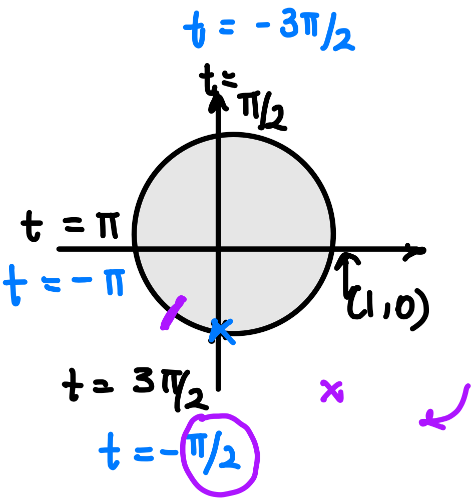
(c) $t = \frac{7\pi}{4}$





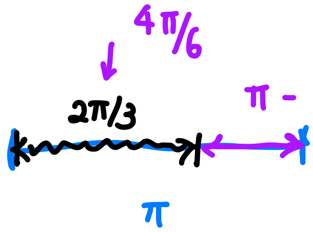
(a) Reference number $= \pi - \frac{5\pi}{6} = \frac{\pi}{6}$

(b) reference number $= \pi - \frac{2\pi}{3} = \frac{\pi}{3}$



$C = 2\pi$

$t = -2\pi/3$



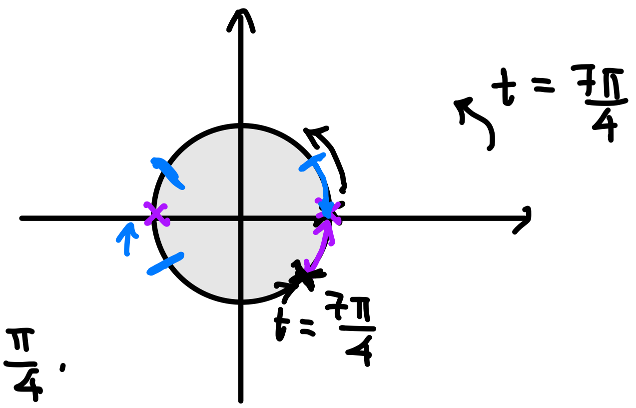
$\pi - \frac{4\pi}{6} = \frac{2\pi}{6}$

$= \frac{\pi}{3}$

$\frac{2\pi}{3} \; \overset{>}{\underset{<}{\times}} \; \frac{\pi}{2}$

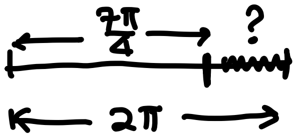
$\frac{2\pi}{3} = \frac{4\pi}{6} > \frac{3\pi}{6} = \frac{\pi}{2}$

(c) $t = \frac{7\pi}{4}$



Reference number

$= 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$.

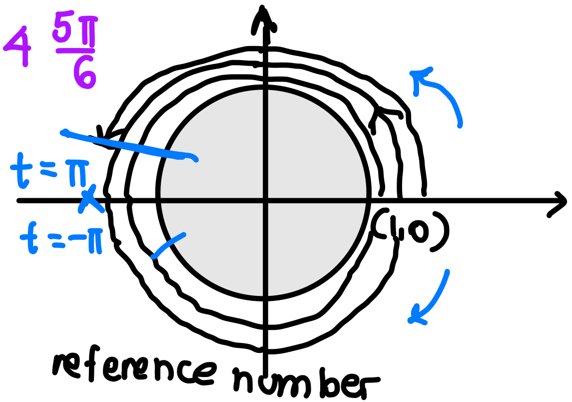


(d) $t = \frac{29\pi}{6} = 4\frac{5\pi}{6}$

$= n \cdot 2\pi + R$

$= \boxed{4\pi} + \frac{5\pi}{6}$

two full revolutions of the circle



reference number

$\pi - \frac{5\pi}{6} = \frac{\pi}{6}$

$t = -\frac{7\pi}{8}$

$\frac{5\pi}{6} < \frac{6\pi}{6} = \pi$

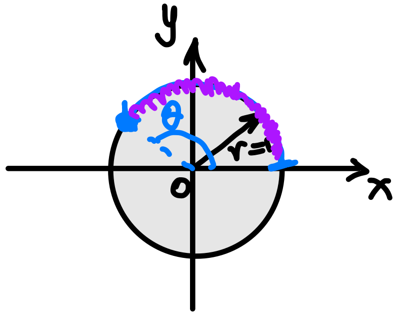
e.g.

$2\frac{1}{3} = \frac{7}{3}$

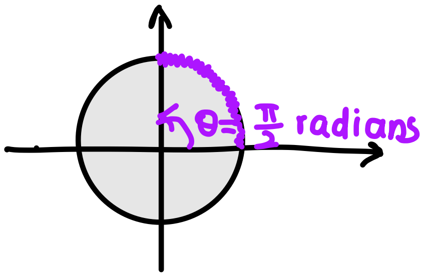
Section 6.1 Angle measures

Note. In general if the angle measure is not specified it means it's in radians.

Radian measure If you are given the unit circle (i.e. radius 1) then the measure of the angle is the length of the arc that subtends the angle.



$\theta = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$



Degrees → Radians

$360^\circ \rightarrow 2\pi$ (circumference)

$180^\circ \rightarrow \pi$

In general, to convert from degrees to radians you multiply by $\frac{\pi}{180}$

e.g. $\theta = 45^\circ$. What is θ in radians?

$45^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{4}$ radians.

Radians → Degrees

$2\pi \rightarrow 360^\circ$

To convert from radians to degrees multiply by $\frac{180^\circ}{\pi}$

Find the angle in degrees.

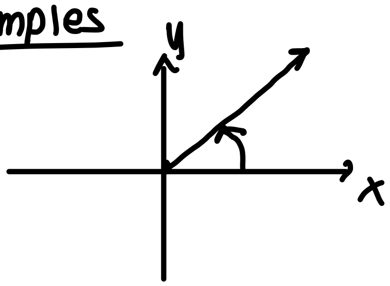
$\theta = \frac{\pi}{3}$.

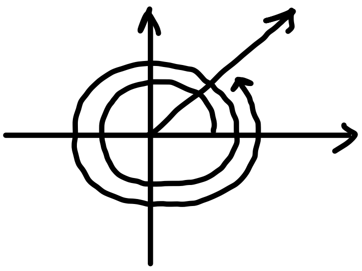
$\frac{\cancel{\pi}}{3} \cdot \frac{180^\circ}{\cancel{\pi}} = 60^\circ$

Angles in standard position / coterminal angles

An angle is in standard position if it is drawn in the x-y plane with its vertex at the origin and initial side on the positive x-axis.

Examples



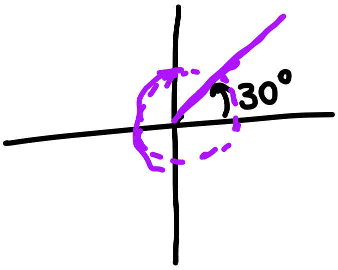


Coterminal angles: Two angles that are in standard position are coterminal if their sides coincide

Exercises ① Find two coterminal angles with the angle $\theta = 30^\circ$ in standard position.

$$\theta = 30^\circ + \overset{1\text{ rev.}}{360^\circ} = 390^\circ$$

$$\theta = 30^\circ + \underset{2\text{ rev.}}{720^\circ} = 750^\circ$$



$$\theta = 30^\circ - 360^\circ = -330^\circ$$

② Find two coterminal angles with the angle $\theta = \frac{\pi}{4}$ $\left(\frac{\pi}{4} \cdot \frac{180}{\pi} = 45^\circ\right)$

$$\theta_1 = \frac{\pi}{4} + 2\pi = \frac{9\pi}{4}$$

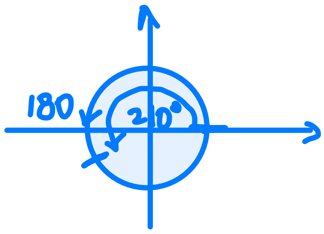
$$\theta_2 = \frac{\pi}{4} + \underset{3\cdot 2\pi}{6\pi} = \frac{25\pi}{4}$$

③ Find an angle with measure between 0° and 360° that is coterminal with the angle of $\boxed{1290^\circ}$ in standard position.

$3(360^\circ) = 1080^\circ$ (3 revolutions around the circle)

$1290^\circ - 1080^\circ = 210^\circ$.

coterminal to 1290°



Length of an arc of a circle

$$\boxed{\theta = \frac{s}{r}}$$

s ← arc length

r ← radius of the circle

θ ← angle in radians

$$\Rightarrow \boxed{s = r\theta}$$

(a) Find the length of an arc of a circle with radius 3 m that subtends an an angle of 60°.

$$\theta = 60^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{3} \text{ radians}, \quad r = 3$$

$s = r\theta = 3 \cdot \frac{\pi}{3} = \pi$ arc length.

(b) Given that the radius is 6 m and the arc length is 5m find the angle θ in degrees

$$\theta = \frac{s}{r} = \frac{5}{6} \text{ radians}$$

$$\theta = \frac{5}{\cancel{6}} \cdot \frac{\cancel{180}^{30}}{\pi} = \left(\frac{150}{\pi}\right)^\circ.$$

$\pi \rightarrow 180$

$\frac{5}{6} \rightarrow ?$

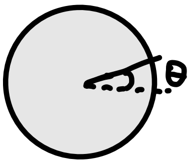
$\pi ? = 180 \cdot \frac{5}{6}$

$? = \frac{180}{\pi} \frac{5}{6}$

Area of a sector of a circle

$$\text{Area of a sector} = \left(\frac{\theta}{2\cancel{\pi}}\right) \cdot \cancel{\pi} r^2 = \frac{1}{2}\theta r^2$$

↑ always in radians



e.g. Find the area of a sector of a circle with angle $\theta = 50^\circ$ if the radius is 4m

$\downarrow$

$\theta = 5\cancel{0} \cdot \frac{\pi}{18\cancel{0}}$

$= \frac{5\pi}{18}$

$$A = \frac{1}{2}\theta r^2 = \frac{1}{2}\left(\frac{5\pi}{18}\right) \cdot 4^2 = \frac{1}{\cancel{2}}\left(\frac{5\pi}{18}\right)\cancel{16}^{8}$$

$$= \frac{5\pi(\cancel{8})^{4}}{\cancel{18}_{9}}$$

$$= \frac{20\pi}{9} \text{ m}^2$$

Note: **Both** $s = \theta r$ (arclength)

$A = \frac{1}{2}\theta r^2$ (area) $\Big\}$ are given with θ = angle in RADIANS

SO Remember to convert degrees to radians by multiplying by $\frac{\pi}{180^\circ}$.

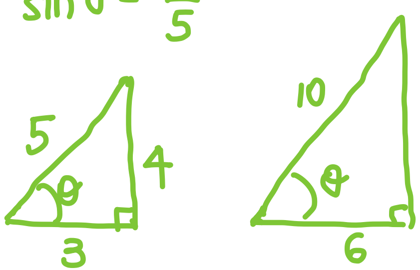
Final exam information

Location : Room Cantor 101

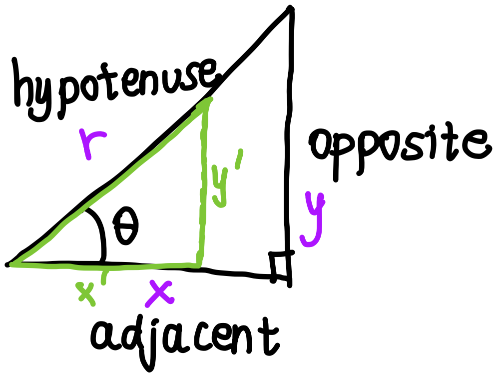
Date : 12/19/2022 (Monday)

Time : 10 ~ 11:50 am

$\sin\theta = \frac{4}{5}$ $\sin\theta = \frac{8}{10} = \frac{4}{5}$



Trigonometric functions



Pythagoras' theorem : $r^2 = x^2 + y^2$

$r = \sqrt{x^2+y^2}$

$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$ SOH

$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$ CAH

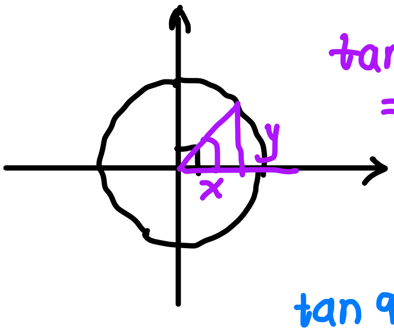
$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$ TOA

$\text{co sec}(\theta) = \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hypotenuse}}{\text{opposite}}$

$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{hypotenuse}}{\text{adjacent}}$

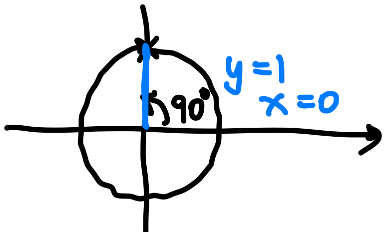
$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adjacent}}{\text{opposite}}$

Note :
be careful
not to divide
by zero.

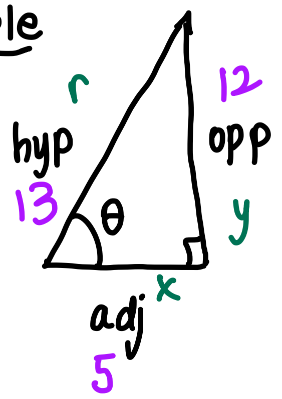


$\tan 45 = \frac{y}{x}$

$\tan 90 \neq \frac{1}{0}$



Example



Consider $\boxed{\cos\theta = \frac{5}{13}}$. Find the other 5 trigonometric functions for this triangle

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{5}{13}$$

$$r^2 = x^2 + y^2$$

Unknown y

$$13^2 = 5^2 + y^2$$

$$y = \sqrt{13^2 - 5^2}$$

$$= \sqrt{169 - 25}$$

$$= \sqrt{144}$$

$$\approx 12$$

$$\cos\theta = \frac{5}{13}$$

$$\sin\theta = \frac{12}{13}$$

$$\tan\theta = \frac{12}{5}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{13}{12}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{13}{5}$$

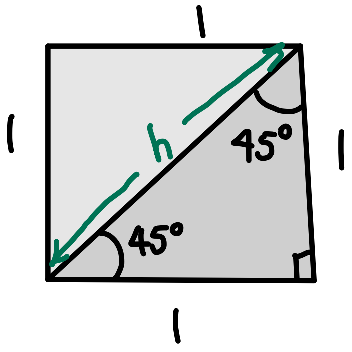
$$\cot\theta = \frac{1}{\tan\theta} = \frac{5}{12}.$$

$$45^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{4}$$

$$\pi^\circ \cdot \frac{\pi}{180} = \frac{\pi^2}{180}$$

$$\frac{1}{\pi}^\circ \cdot \frac{\pi}{180^\circ} = \frac{1}{180}$$

Note: SPECIAL RATIOS OF SIDES.



$$h = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\sin 45^\circ = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

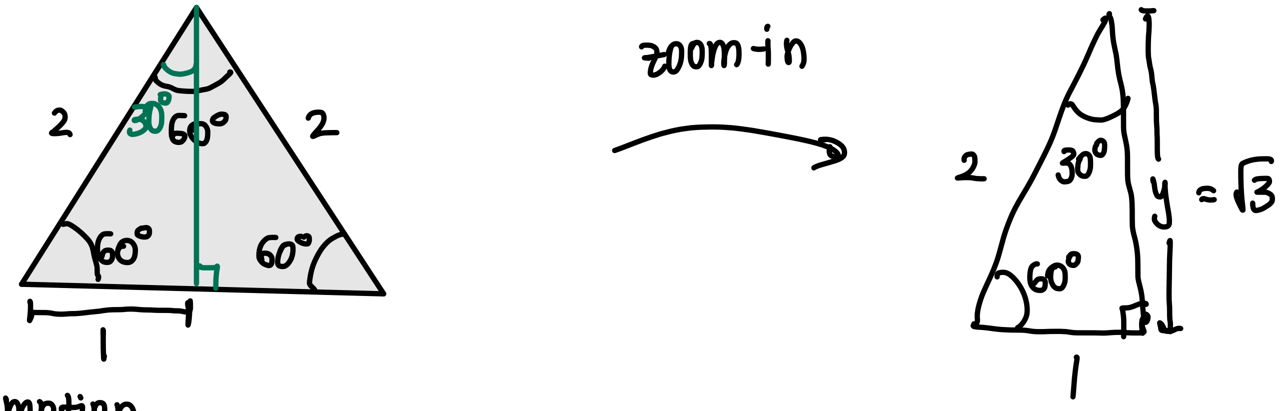
$$\cos 45^\circ = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \tan\left(\frac{\pi}{4}\right) = 1$$

Note $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$$= \frac{\sqrt{2}}{2}$$

Equilateral triangles



Assumption

Each side has a length of 2

Find y

Pythagoras: $2^2 = 1^2 + y^2$

$4 = 1 + y^2$

$3 = y^2$

$y = \sqrt{3}$

Remember

$\sin 30° = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ ✓

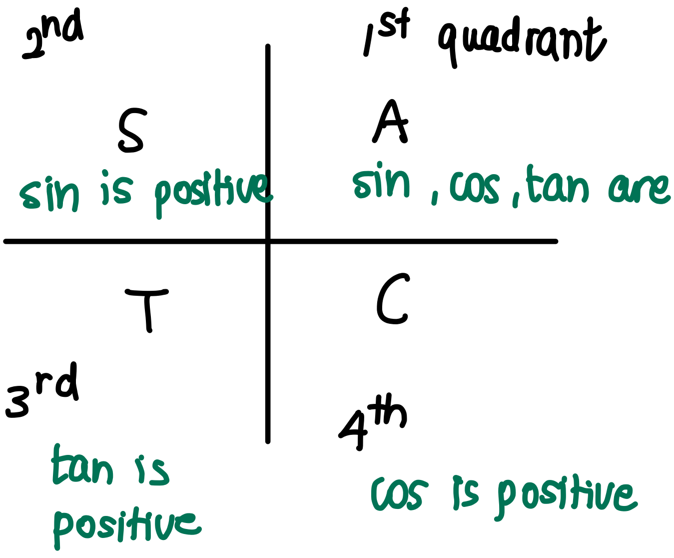
$\cos 30° = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ ✓

$\tan 30° = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ ✓

$\sin 60° = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

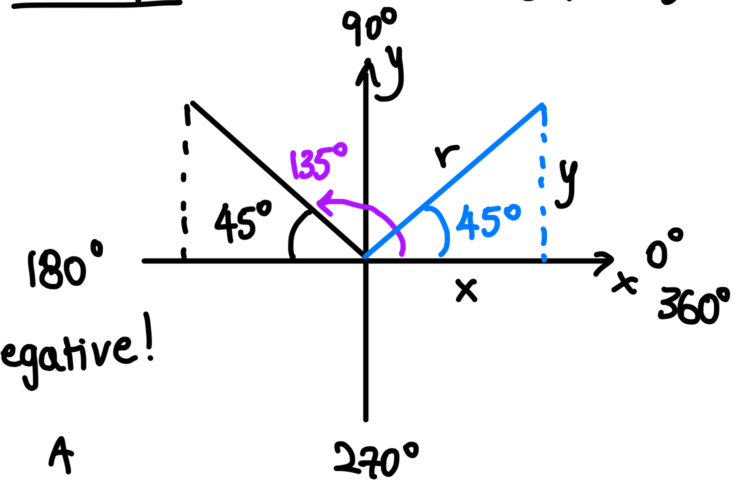
$\cos 60° = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\tan 60° = \tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{1} = \sqrt{3}$

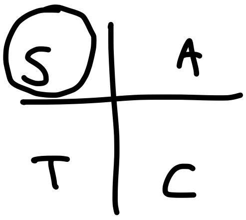


Note: Ⓐll
Ⓢtudents
Ⓣake
Ⓒalculus

Example. ① Find $\cos(135°)$.



cos is negative!



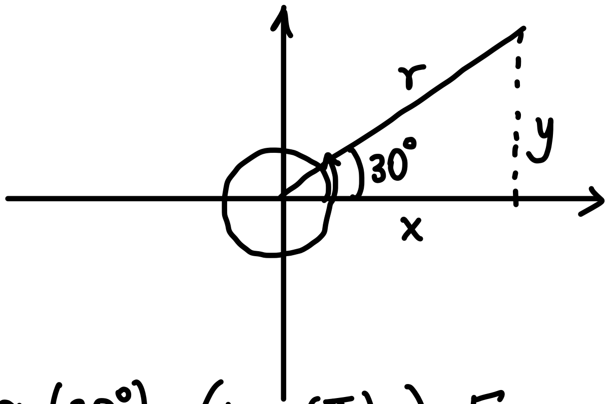
$$\cos 45° = \frac{\sqrt{2}}{2} = \frac{x}{r}$$

$$\cos 135° = -\frac{x}{r} = -\frac{\sqrt{2}}{2}$$

$$\sin 45° = \frac{\sqrt{2}}{2}$$

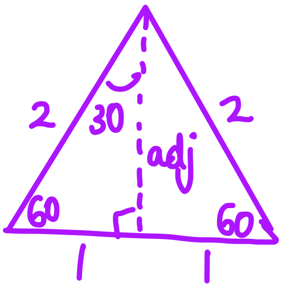
$$\sin 135° = \frac{\sqrt{2}}{2}$$

② Find $\tan(390°)$



$$\tan(390°) = \tan(30°) = \left(\tan\left(\frac{\pi}{6}\right) = \right) \frac{\sqrt{3}}{3}$$

$$= \frac{1}{\sqrt{3}}$$



$$\tan 30° = \frac{\text{opp}}{\text{adj}}$$

$$= \frac{1}{\sqrt{3}}$$

$$\text{adj} = \sqrt{2^2 - 1^2}$$

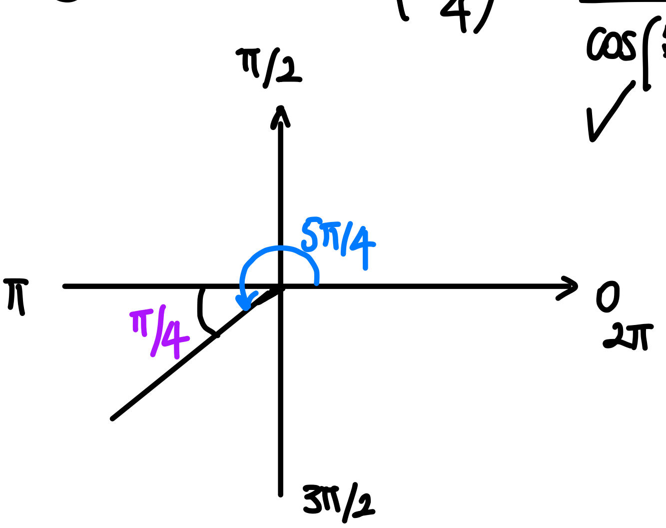
$$= \sqrt{3}$$

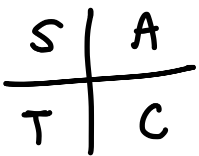
③ Find $\sec\left(\frac{5\pi}{4}\right) = \frac{1}{\cos\left(\frac{5\pi}{4}\right)} \left[= \frac{\text{hyp}}{\text{adj}}\right]$ ✓

$$= \frac{1}{-\cos\left(\frac{\pi}{4}\right)}$$

$$= \frac{1}{-\frac{\sqrt{2}}{2}}$$

$$= -\frac{2}{\sqrt{2}}$$





cos is negative

$$\cos\left(\frac{\pi}{4}\right) = \cos(45°)$$

$$= \frac{\sqrt{2}}{2}$$

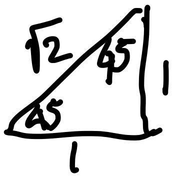
$$\cos\left(\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$$

Find $\tan\left(\frac{5\pi}{4}\right)$, $\sin\left(\frac{5\pi}{4}\right)$, $\operatorname{cosec}\left(\frac{5\pi}{4}\right)$

$$\tan\left(\frac{5\pi}{4}\right) = +\tan\left(\frac{\pi}{4}\right) = 1$$

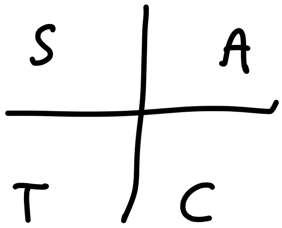
$$\sin\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\operatorname{cosec}\left(\frac{5\pi}{4}\right) = \frac{1}{\sin\left(\frac{5\pi}{4}\right)} = \frac{1}{\left(-\frac{1}{\sqrt{2}}\right)} = -\sqrt{2}$$



negative positive negative

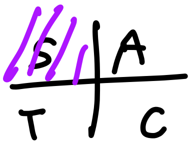
④ Find $\tan(870^\circ)$, $\sin(870^\circ)$, $\cos(870^\circ)$

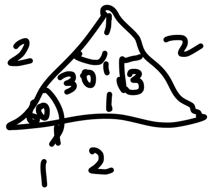


✓ **Step 1:** Determine the quadrant the angle lies in

✓ **Step 2:** Find the small angle you know the sin, cos, tan of

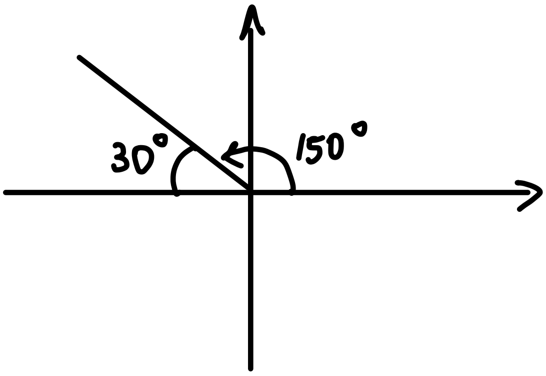
✓ **Step 3:** Determine the sign based on





$$2(360^\circ) = 720^\circ$$

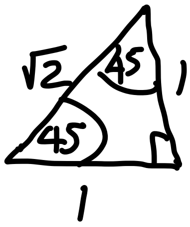
Remaining $= 870 - 720 = 150^\circ$



$$\tan(870^\circ) = -\tan(30^\circ) = -\frac{1}{\sqrt{3}}$$

$$\sin(870^\circ) = \sin(30^\circ) = \frac{1}{2}$$

$$\cos(870^\circ) = -\cos(30^\circ) = -\frac{\sqrt{3}}{2}$$



(Sections 5.2, 6.2, 6.3)

Trigonometric Graphs

The period of sine and cosine is $\boxed{2\pi}$. (This tells you every how many units in x the shape of the graph repeats itself).

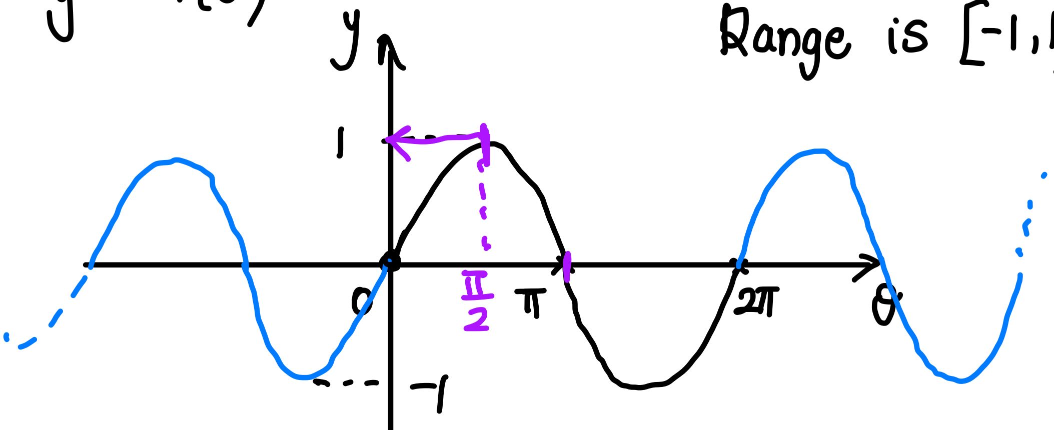
$$\cos(\theta + 2\pi \cdot n) = \cos\theta \quad , \quad \sin(\theta + 2\pi n) = \sin\theta$$

$$\left[\cos(\theta^\circ + 360^\circ n) = \cos\theta \right]$$

The period of tan is $\boxed{\pi}$

Sketch of $y = \sin(\theta)$

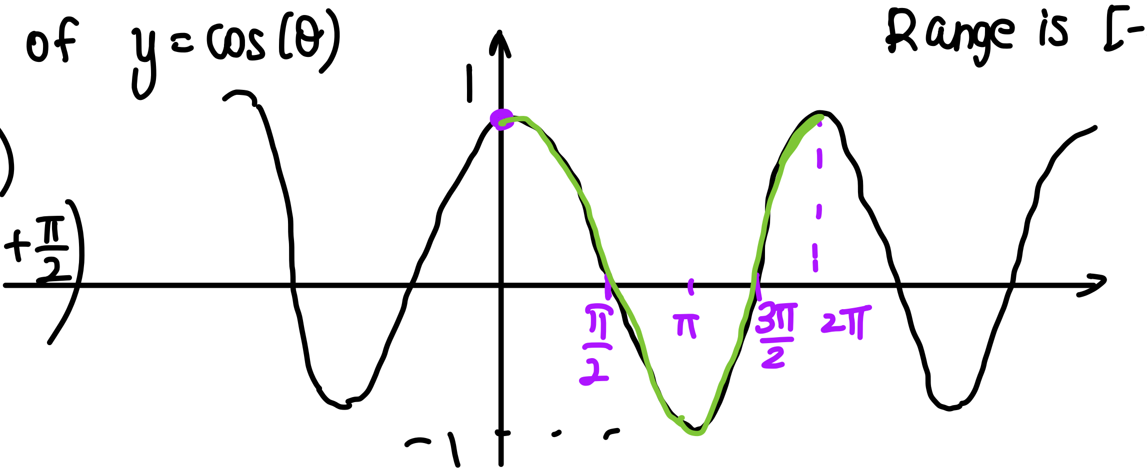
Domain is $(-\infty, \infty)$
Range is $[-1, 1]$



Domain is $(-\infty, \infty)$
Range is $[-1, 1]$

Sketch of $y = \cos(\theta)$

$$y = \cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$$



How to remember special angles:.

radians → degrees
$\times \frac{180°}{\pi}$

t (radians)	t (degrees)	sin t	cos t	tan t
0	0	0	1	0
$\frac{\pi}{6}$	30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90	1	0	undefined.

From the book.

t	sint	cos t
0	$\sqrt{0}/2 = 0$	$\sqrt{4}/2 = 1$
$\frac{\pi}{6}$	$\sqrt{1}/2 = 1/2$	$\sqrt{3}/2$
$\frac{\pi}{4}$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\frac{\pi}{3}$	$\sqrt{3}/2$	$\sqrt{1}/2 = 1/2$
$\frac{\pi}{2}$	$\sqrt{4}/2 = \frac{2}{2} = 1$	$\sqrt{0}/2 = 0$

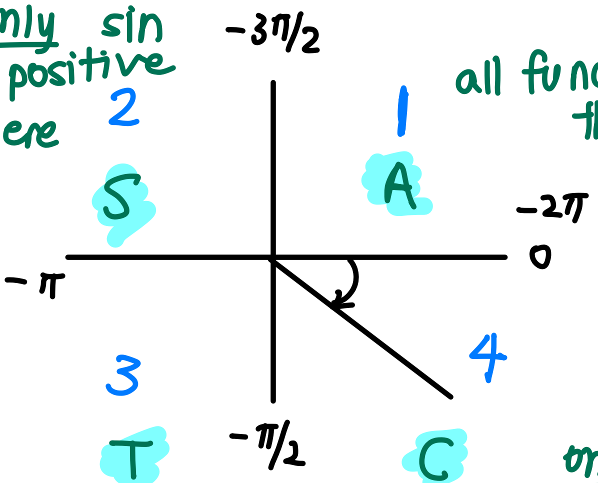
HW 9

Q6 (a) $\theta = - \pi/4$

move clockwise starting at the x-axis

only sin is positive here

all functions in this quadrant are positive



only tan is positive here

only cos is positive here

Quadrant 4

Reference angle = $\pi/4$

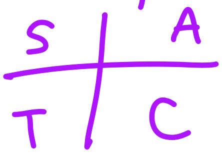
reference angle

$$\cos(\theta) = \cos\left(-\frac{\pi}{4}\right) = +\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\sin(\theta) = \sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$$

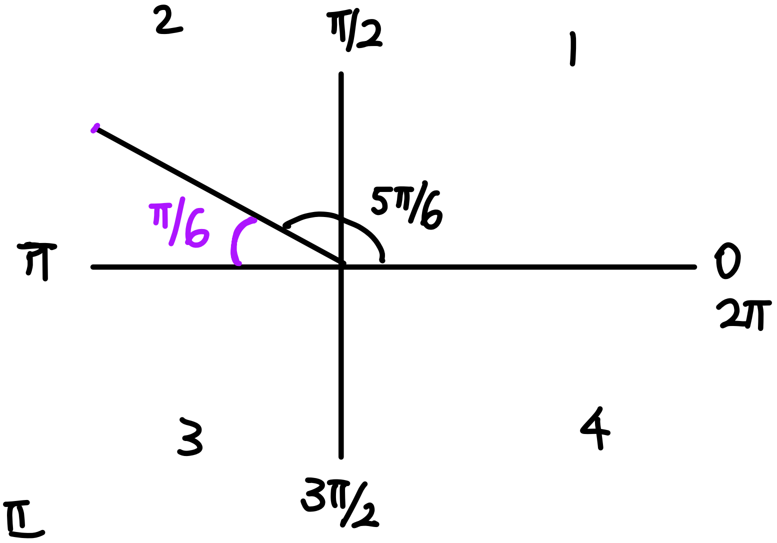
$$\tan(\theta) = \tan\left(-\frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$$

signs determined from the quadrant



(c) $\theta = \frac{5\pi}{6}$

(positive ⇒ counterclockwise)



Quadrant 2

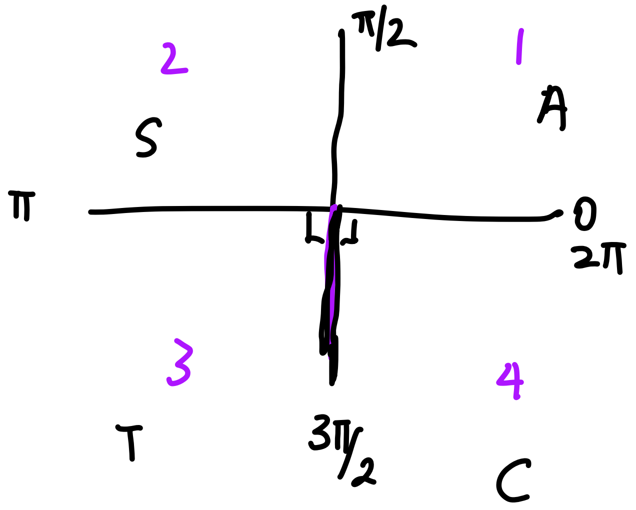
Reference angle = $\frac{\pi}{6}$

$$\cos\left(\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{5\pi}{6}\right) = +\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\tan\left(\frac{5\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

(j) $\theta = \frac{3\pi}{2}$



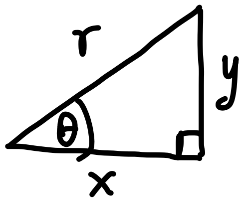
The angle lies on the negative y-axis between quadrant 3 and 4

Reference angle $= \frac{\pi}{2}$

$\cos\left(\frac{3\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

$\sin\left(\frac{3\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$

$\tan\left(\frac{3\pi}{2}\right) = \tan\left(\frac{\pi}{2}\right) =$ undefined.

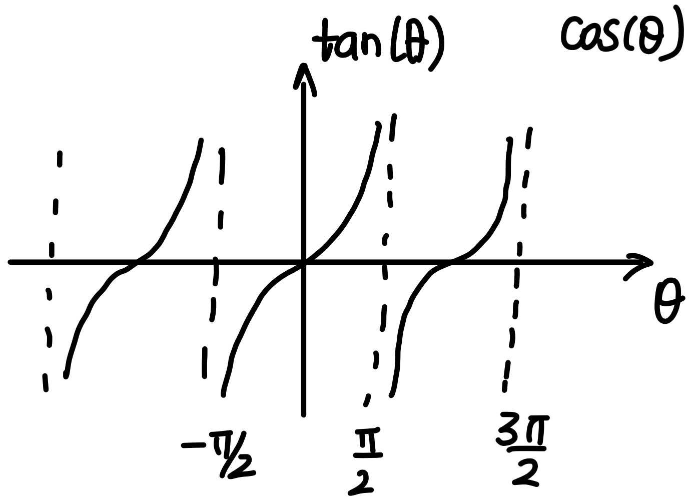


$\cos(\theta) = \frac{x}{r}$

$\sin(\theta) = \frac{y}{r}$

$\tan(\theta) = \frac{y}{x}$

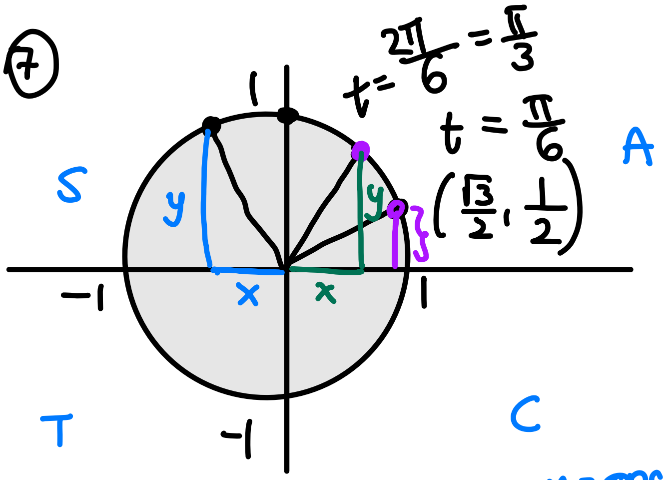
$= \frac{\sin(\theta)}{\cos(\theta)}$



Office hours today on Zoom at 3:30pm

WebAssign 5.1

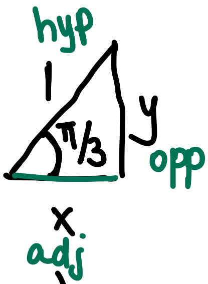
⑦



$\cos(\theta) = \frac{adj}{hyp}$

$\cos(\theta) = \frac{x}{1}$

$\Rightarrow x = \cos\theta$



$\cos\left(\frac{\pi}{3}\right) = \frac{x}{1}$

$x = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

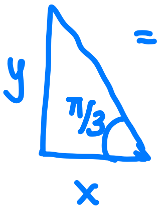
$\sin\left(\frac{\pi}{3}\right) = \frac{y}{1}$

$y = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$x = \cos\left(\frac{2\pi}{3}\right)$

$= -\cos\left(\frac{\pi}{3}\right)$

$= -\frac{1}{2}$



$y = \sin\left(\frac{2\pi}{3}\right)$

$= \sin\left(\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2}$

	t	Terminal point
	0	$(1,0)$
	$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (x, y)
	$\frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6}$ $= \frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
	$\frac{\pi}{2}$	$(0,1)$
	$\frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$+\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$+\frac{\pi}{6}$	π	$(-1,0)$
⋮	$\frac{7\pi}{6}$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
	$\frac{4\pi}{3}$	$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
	$\frac{3\pi}{2}$	$(0, -1)$
	$\frac{3\pi}{2} + \frac{\pi}{6}$	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
	$= \frac{9\pi}{6} + \frac{\pi}{6}$	
	$= \frac{10\pi}{6} = \frac{5\pi}{3}$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
	$\frac{11\pi}{6}$	
	2π	$(1,0)$

Trigonometric graphs

Recall $\sec(\theta) = \frac{1}{\cos(\theta)}$

$\operatorname{cosec}(\theta) = \frac{1}{\sin(\theta)}$

$\cot(\theta) = \frac{1}{\tan(\theta)}$

Definition

A function is periodic if there is a positive number p such that

$$f(t+p) = f(t)$$

for every value of t.

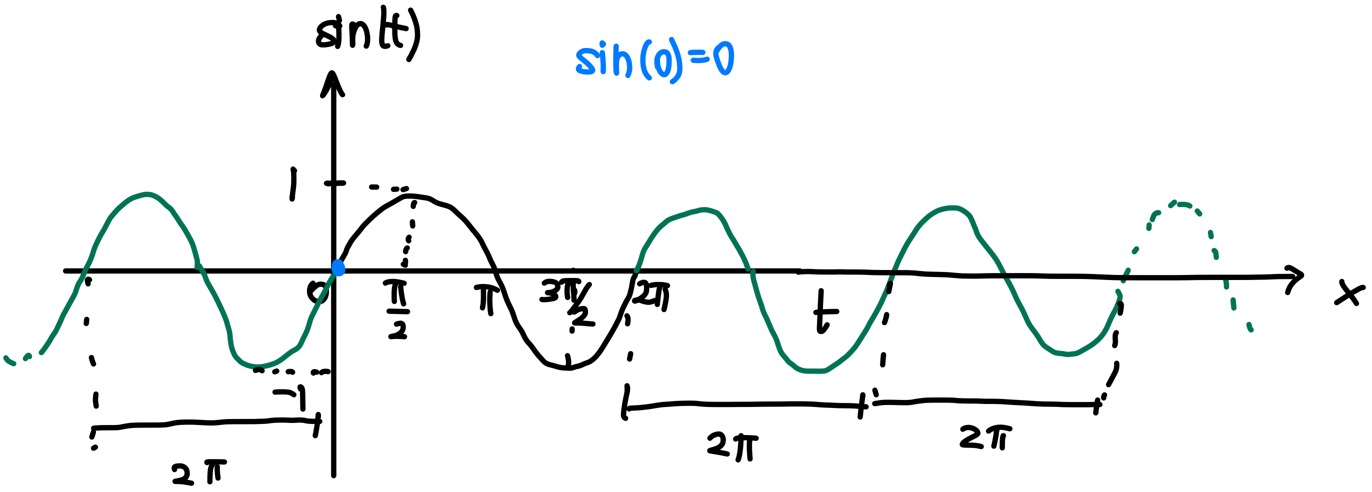
The smallest positive number p is called the period

Periodic properties of sine and cosine.

$\sin(t+2\pi) = \sin(t)$

$\cos(t+2\pi) = \cos(t)$

] sine and cosine have a period of 2π.



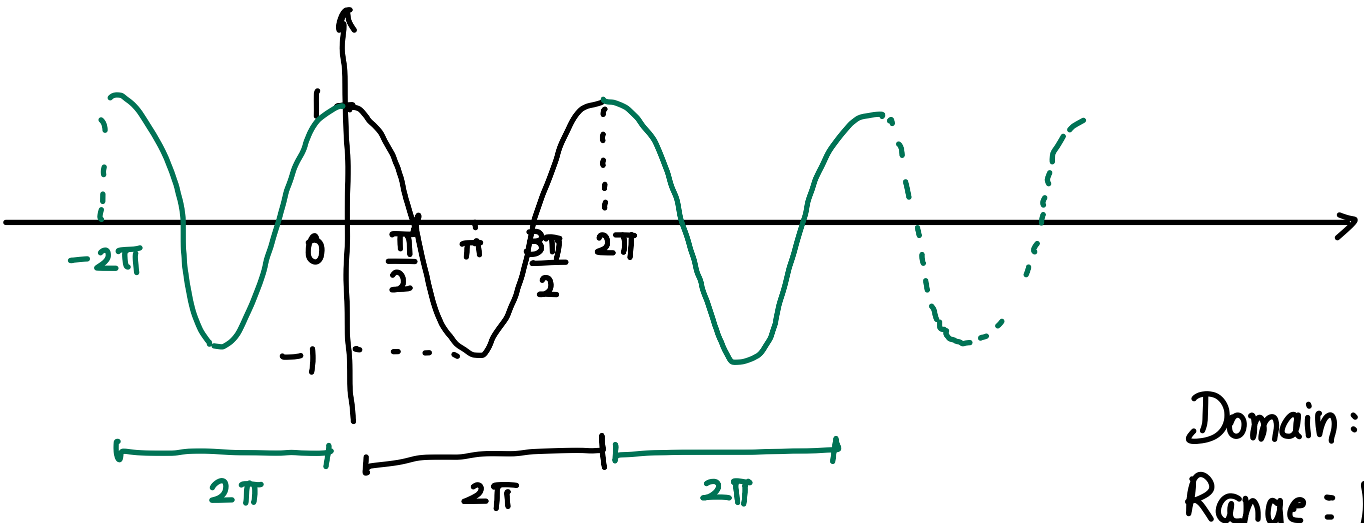
Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

cos(t)

$\cos(0) = 1$

$\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$



Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

Transformations of cosine and sine

The most general transformation is

$$y = -a\sin(k(x-b)) + h \quad , \quad y = -a\cos(k(x-b)) + h$$

Vertical transformations: 1. Reflection along x-axis

2. If $a > 1$ then there is a vertical stretch by a factor of a

If $0 < a < 1$ then there is a vertical compression by a factor of a.

3. If $h > 0$ there is a shift up by h.

Horizontal transformations: 1. If $k > 1$ this is a horizontal compression by a factor of $\frac{1}{k}$

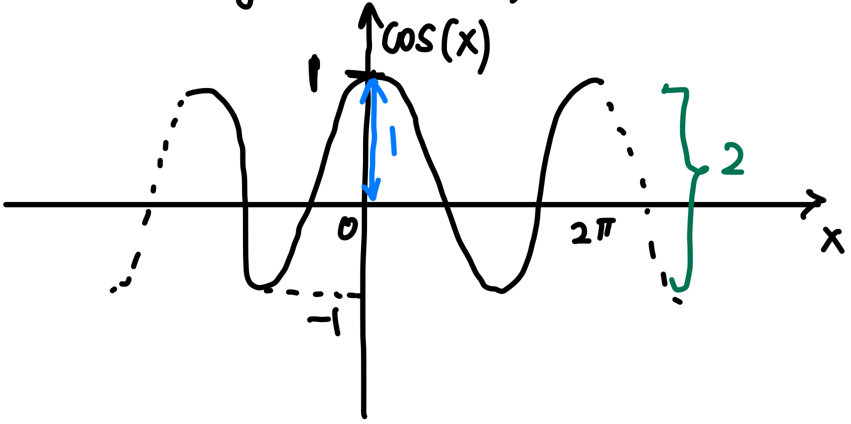
2. If $b > 0$ this is a shift to the right by b.

$$y = -a\sin(\overset{\downarrow}{k}(x-b))+h$$

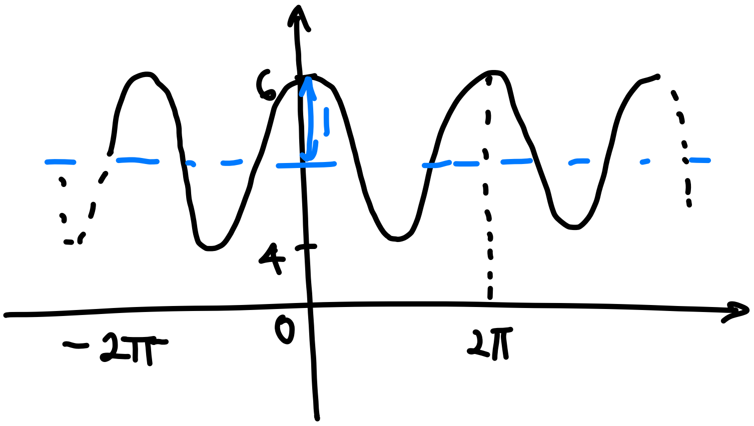
then the period is found by using $\boxed{\text{period} = \frac{2\pi}{k}}$

the amplitude is now $|a|$.

Examples 1. $y = 5 + \cos(x)$.



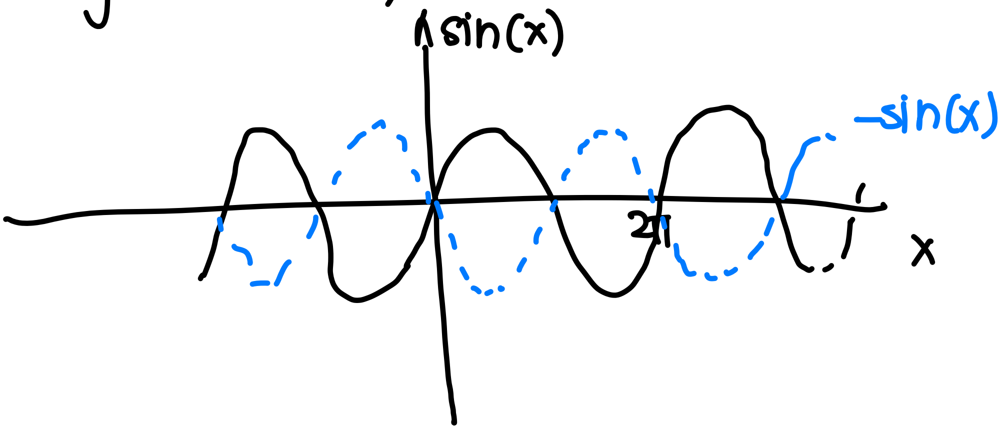
$$\boxed{\text{amplitude} = \frac{\max - \min}{2}}$$

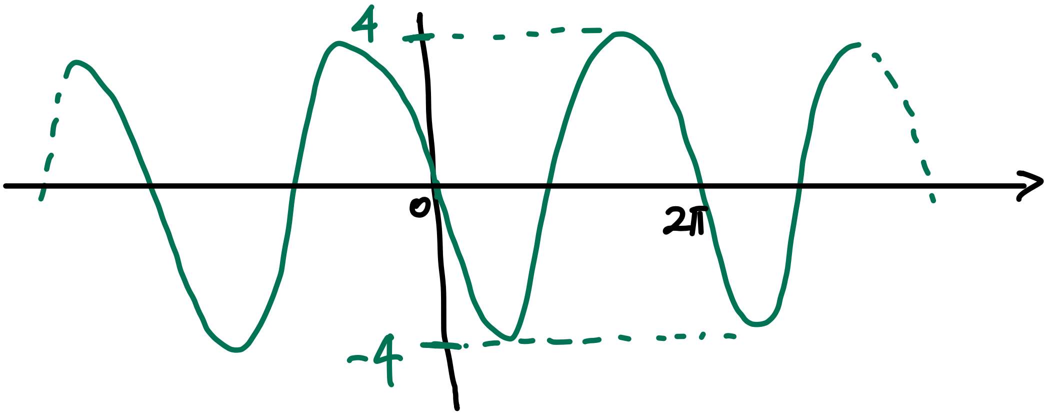


Domain: $(-\infty, \infty)$

Range: $[4, 6]$

2. Sketch $y = -4\sin(x)$





$y = -\boxed{4}\sin(x)$

$\text{amplitude} = \frac{\max - \min}{2}$
$= \frac{4-(-4)}{2}$
$= 4$

Domain: $(-\infty, \infty)$
Range: $[-4, 4]$

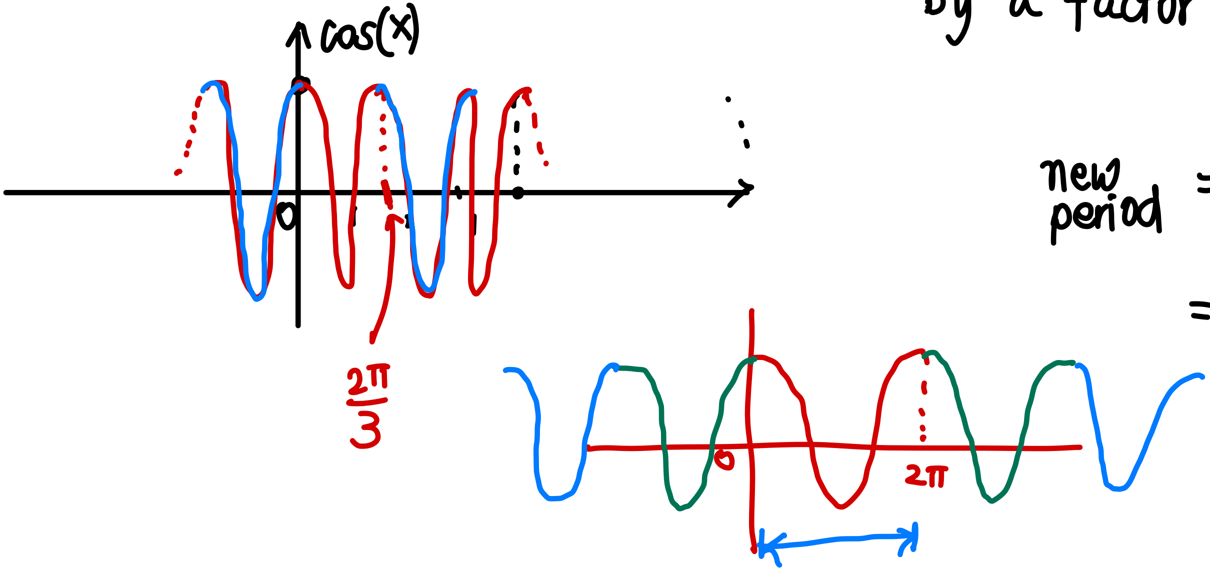
→ HW 10 posted on Gradescope

Office hours at 4:30 pm in Room 412 of WWH
(251 Mercer Street)

... continuing from transformations of trigonometric graphs.

Example ① Sketch $y = \cos(3x)$. Horizontal compression by a factor of $\frac{1}{3}$.

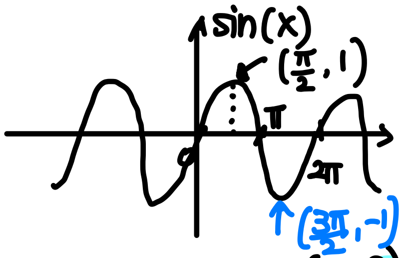
$\text{new period} = \frac{2\pi}{k}$
$= \boxed{\frac{2\pi}{3}}$

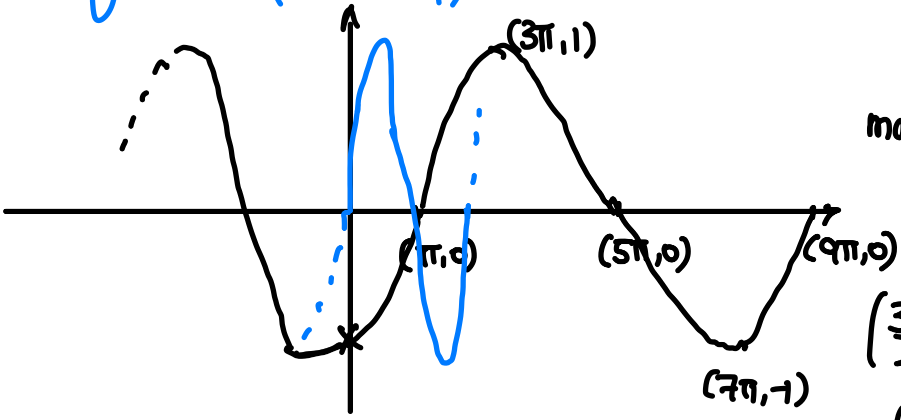


② Sketch $y = \sin\left(\frac{1}{4}(x-\pi)\right)$ in factored form *

$y = \sin\left(\frac{1}{4}x - \frac{\pi}{4}\right)$

- Hor. stretch by a factor of 4
- Hor. shift to the right π.





$(0,0) \rightarrow (\pi,0)$

max $(\frac{\pi}{2},1) \rightarrow (3\pi,1)$

$(\pi,0) \rightarrow (5\pi,0)$

$(\frac{3\pi}{2},-1) \rightarrow (7\pi,-1)$

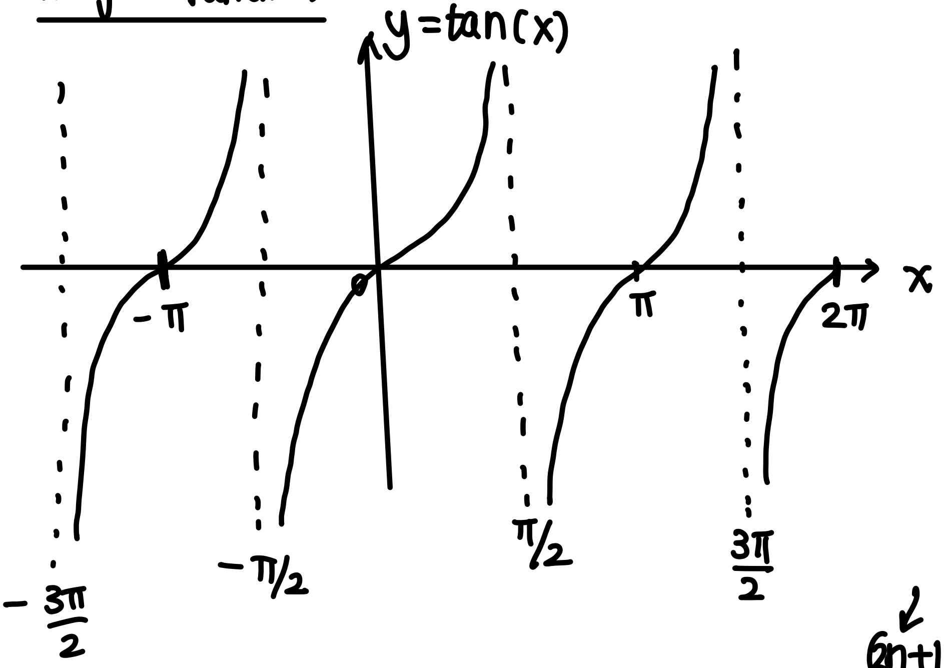
$(2\pi,0) \rightarrow (9\pi,0)$

new period $= \frac{2\pi}{k} = \frac{2\pi}{(\frac{1}{4})} = 8\pi$

$\longleftrightarrow$

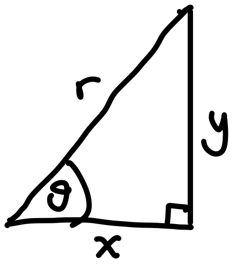
$9\pi - \pi$
$= 8\pi$

Tangent function



$\boxed{\text{period} = \pi}$

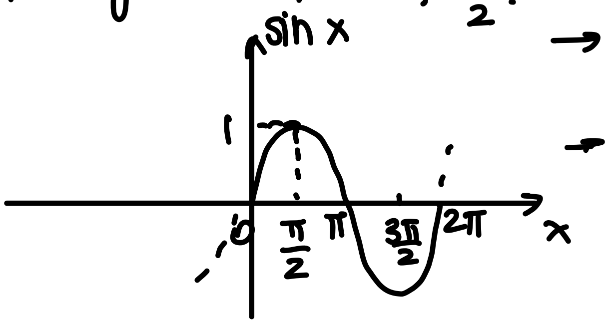
$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$



$\underbrace{(2n+1)}\frac{\pi}{2}$

Range: $(-\infty, \infty)$

Domain: All real numbers excluding odd integer multiples of $\frac{\pi}{2}$.



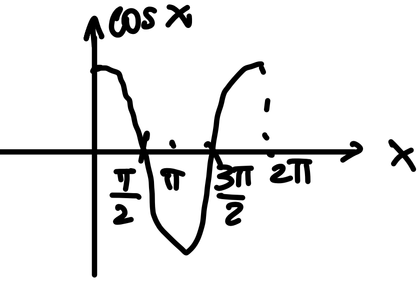
$\rightarrow \sin\frac{\pi}{2} = 1$

$\rightarrow \cos\frac{\pi}{2} = 0$

$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$



$\tan\frac{\pi}{2} = \frac{1}{0}$

undefined

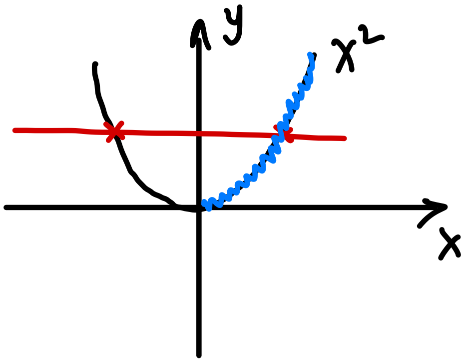
Inverse trigonometric functions

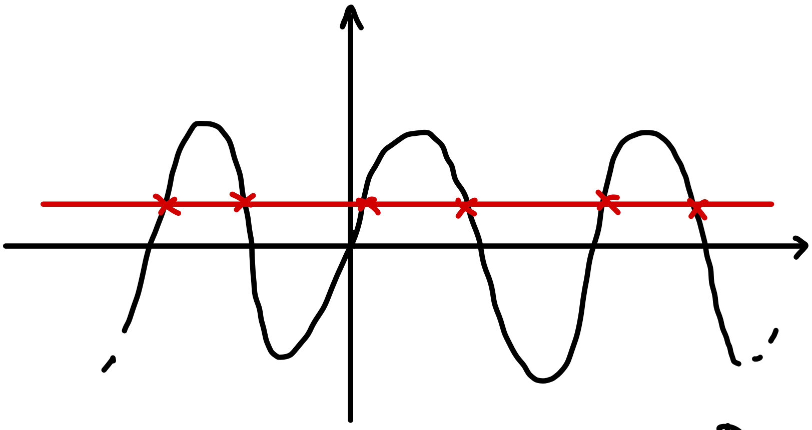
$f(f^{-1}(x)) = x$

$f^{-1}(f(x)) = x$

To check if a function is invertible use the horizontal line test.

e.g





$y = x^2$

$\sqrt{y} = x$

$\sqrt{x} = y$

$f^{-1}(x) = \sqrt{x}$

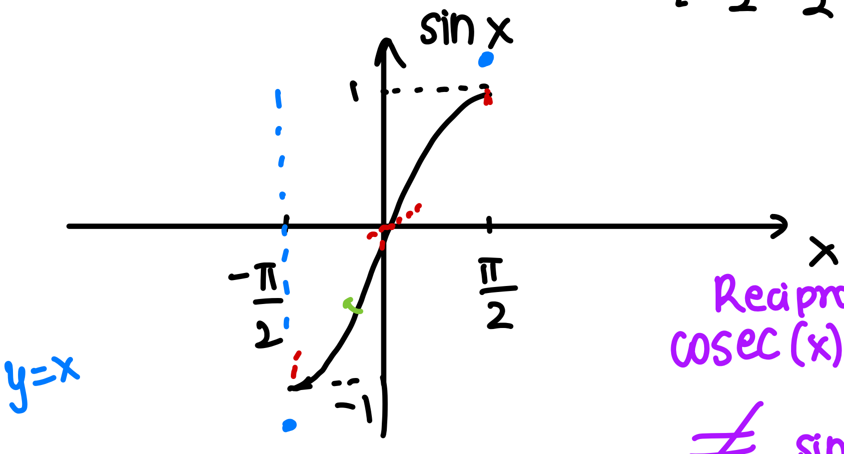
Domain of sin(x):

$(-\infty, \infty)$

There is no inverse if you do not restrict the domain of sin(x).

interval notation, inequality notation

Restricted domain of sin(x) : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\left(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right)$



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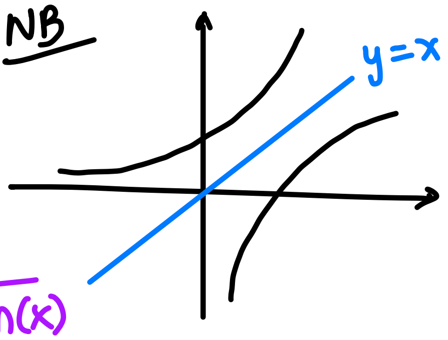


Reciprocal

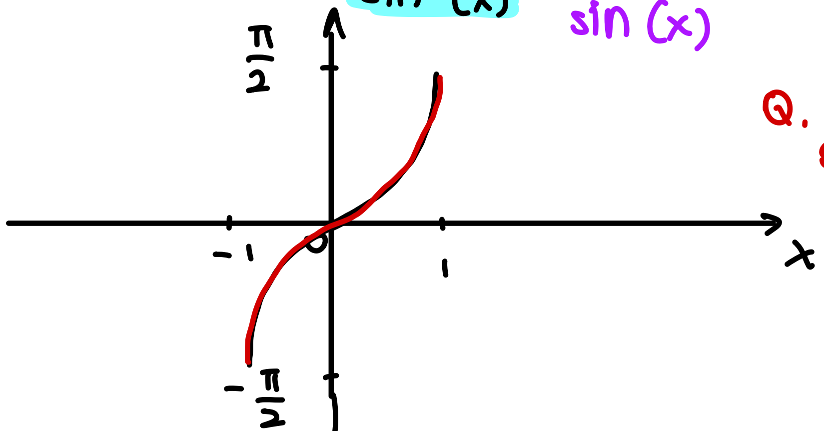
$\text{cosec}(x) = \frac{1}{\sin(x)}$

$\neq$ $\sin^{-1}(x)$ inverse

$\frac{\pi}{2} = \frac{3.14\ldots}{2} > 1$

Range: $[-1, 1]$

$\sin^{-1}(x)$ inverse of sin (x)



$f^{-1}(x)$

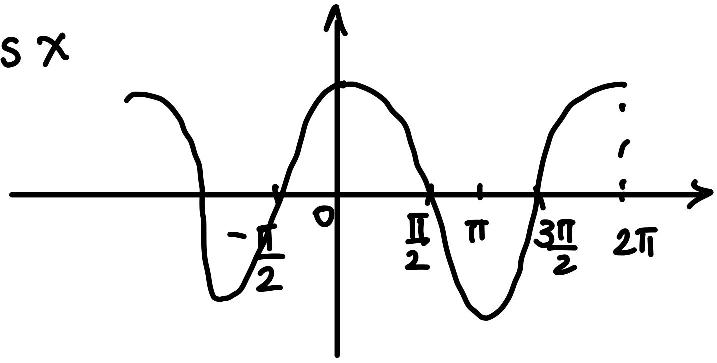
Q. Find $\sin^{-1}(-1) =$

$\sin^{-1}(x)$

Domain: $[-1, 1]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

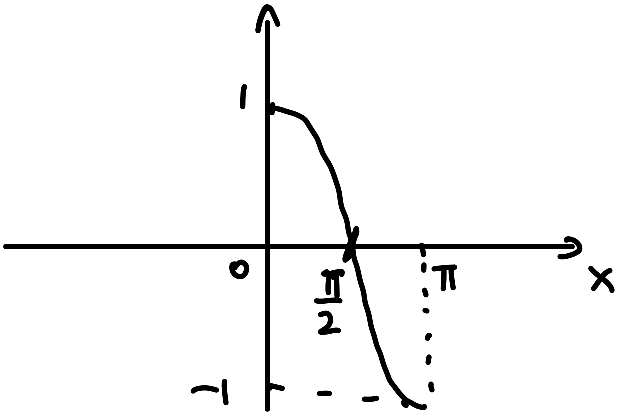
Looking at $\cos x$



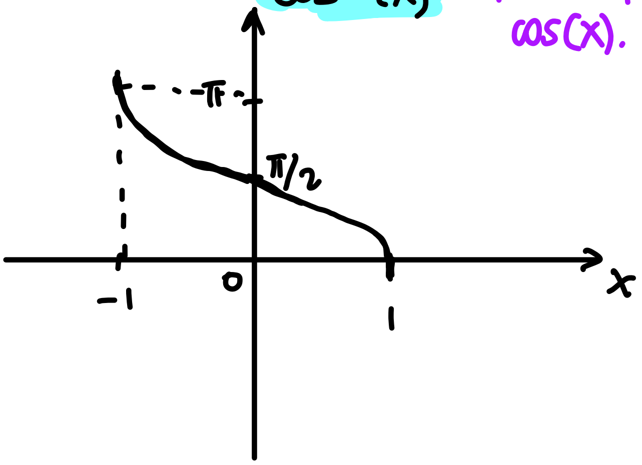
Restricted domain should be $[0, \pi]$

Range: $[-1, 1]$

Restricted $\cos(x)$



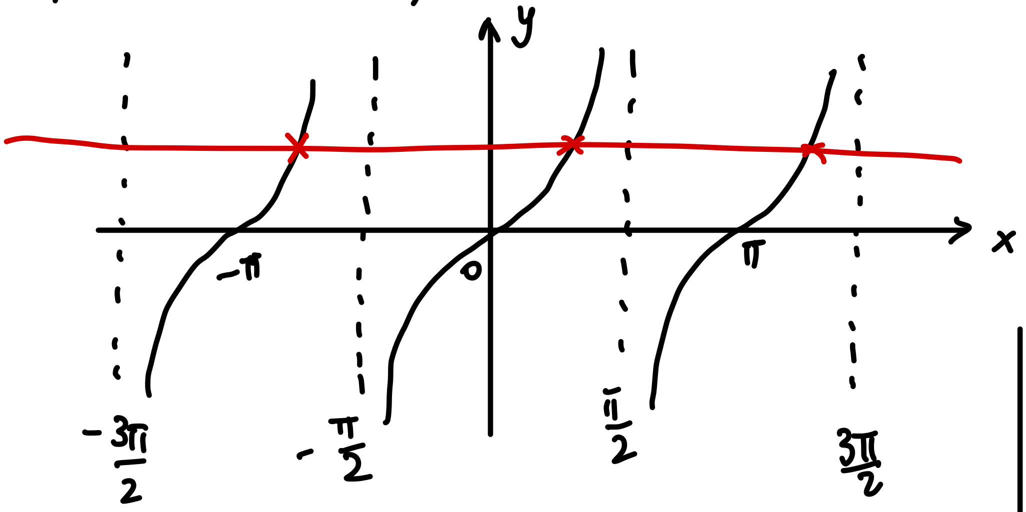
$\cos^{-1}(x)$ inverse of $\cos(x)$.



Domain: $[-1, 1]$

Range: $[0, \pi]$

Now, consider $\tan(x)$



Restricted domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Range: $(-\infty, \infty)$

reciprocal

$(f(x))^{-1} = \frac{1}{f(x)}$

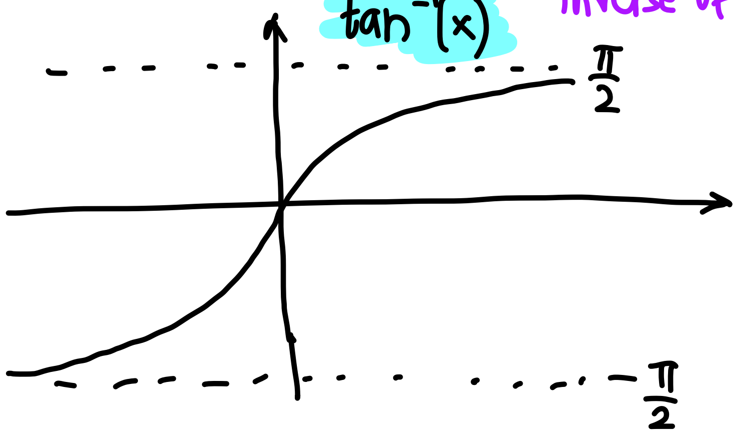
≠

$f^{-1}(x)$ inverse

$\cot(x) = \frac{1}{\tan x}$

$= (\tan x)^{-1}$

$\tan^{-1}(x)$ inverse of $\tan(x)$



Domain: $(-\infty, \infty)$

Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Aside: $\boxed{f(x) = x^2}$, $x > 0$

Write down: $(f(x))^{-1} = \frac{1}{f(x)} = \frac{1}{x^2}$

Write down $f^{-1}(x) = \sqrt{x}$

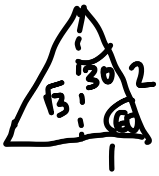
$$\boxed{(f(x))^{-1} \neq f^{-1}(x)}$$

$f(x) = \sin x$

Write down $(f(x))^{-1} = \frac{1}{f(x)} = \frac{1}{\sin x} = \operatorname{cosec}(x)$

$f^{-1}(x) = \sin^{-1}(x)$

$(f(x))^{-1} \neq f^{-1}(x) \Rightarrow \operatorname{cosec}(x) \neq \sin^{-1}(x$



Example. ① Find the exact values of

(a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$$\left[\begin{array}{l} \sin(x) = \frac{\sqrt{3}}{2} \\ x = \frac{\pi}{3} \end{array}\right.$$

If you see $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ you ask:
What x value would give $\sin(x) = \frac{\sqrt{3}}{2}$?

If you see $\sin^{-1}(a)$ you ask
What x value would give $\sin(x) = a$?

Q. What is

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

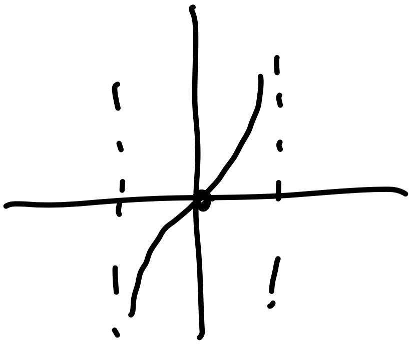
$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

(b) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

(c) $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

(d) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

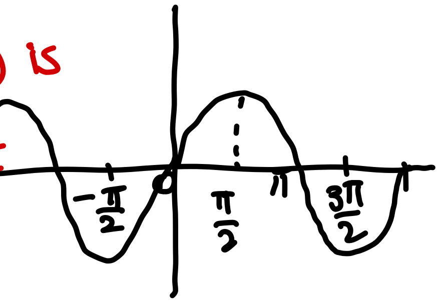


(e) $\tan^{-1}(1) = \frac{\pi}{4}$

(f) $\tan^{-1}(0) = 0$

(g) $\sin^{-1}(-1) = -\frac{\pi}{2}, \cancel{\frac{3\pi}{2}}$
↑ ↑

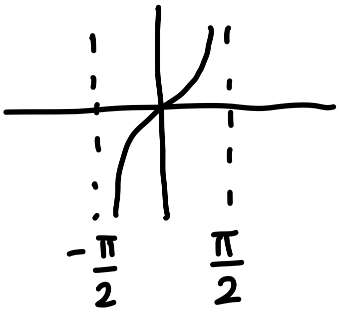
the range of $\sin^{-1}(x)$ is
$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

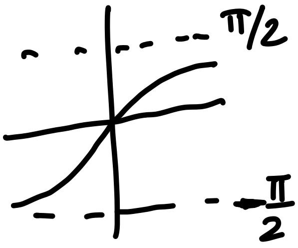


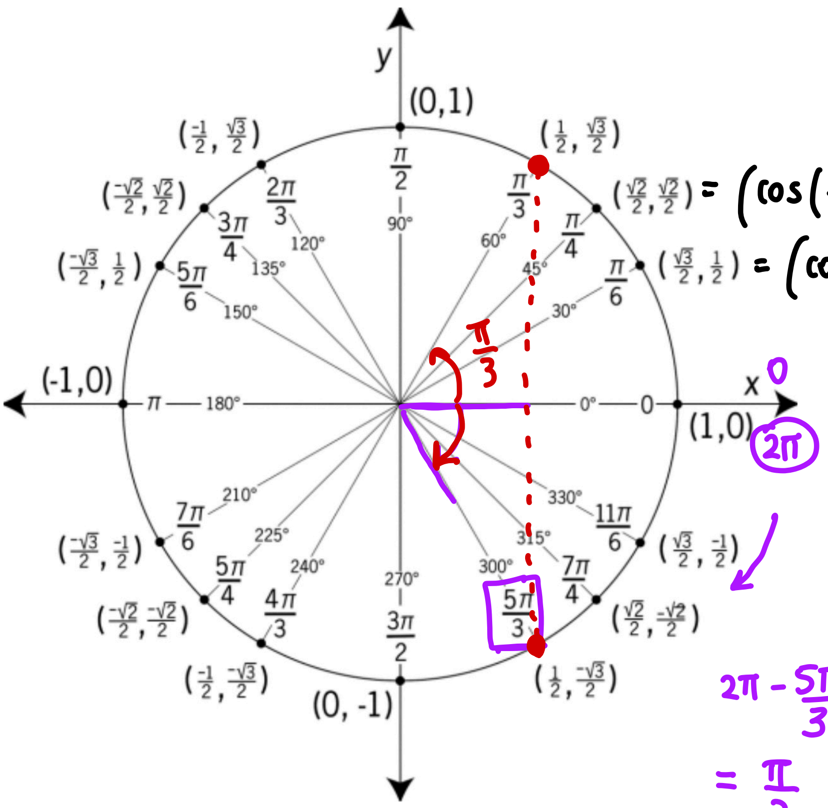
★ Office hours today on Zoom at 3:30pm - 4:30pm

Inverse trigonometric functions

Inverse function	Domain	Range
$\sin^{-1}(x)$	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$
$\cos^{-1}(x)$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}(x)$	$(-\infty,\infty)$	$(-\frac{\pi}{2},\frac{\pi}{2})$







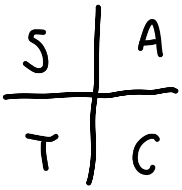
$= (\cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}))$

$= (\cos(\frac{\pi}{6}), \sin(\frac{\pi}{6}))$

$\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$
↑

$2\pi - \frac{5\pi}{3}$

$= \frac{\pi}{3}$



Reminder:

$\cos^{-1}(x)$ Range: $[0,\pi]$

$\sin^{-1}(x)$ Range: $[-\frac{\pi}{2},\frac{\pi}{2}]$

$\tan^{-1}(x)$ Range $(-\frac{\pi}{2},\frac{\pi}{2})$

Examples

① $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$

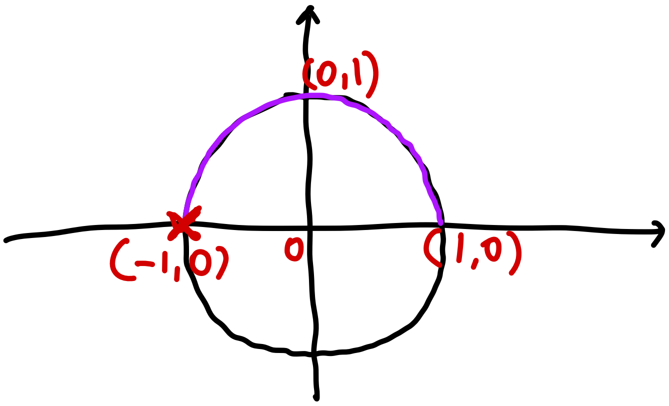
② $\sin^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$ lies in $[-\frac{\pi}{2},\frac{\pi}{2}]$

③ $\tan^{-1}(-1) = -\frac{\pi}{4}$ (Recall: $\tan(x) = \frac{\sin(x)}{\cos(x)}$).

④ $\cos^{-1}(-1) = \theta$ unknown

$\cos(\theta) = -1$

What is θ such that $\underbrace{\cos(\theta) = -1}$



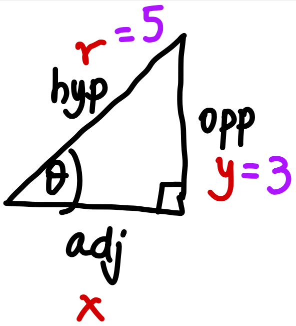
$\cos^{-1}(-1) = \pi$

$\cos^{-1}(x)$:
Range: $[0,\pi]$

Composition of trigonometric functions with their inverses.

1. Find $\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right) \overset{\text{given } (*)}{=} \cos(\theta)$

$= \frac{x}{r} = \boxed{\frac{4}{5}}$



(*) $\theta = \sin^{-1}\left(\frac{3}{5}\right)$

$\sin(\theta) = \frac{3}{5}$

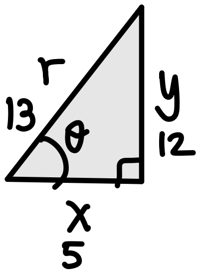
Find x: $x^2+y^2 = r^2$
$x^2+3^2=5^2$
$x^2+9=25$
$x^2=16$
$x=4$

$\sin(\theta) = \boxed{\frac{y}{r} = \frac{3}{5}}$

Note
$f^{-1}(x)=y$
$f(y)=x$

2. Find $\tan\left(\sin^{-1}\left(\frac{12}{13}\right)\right) = \tan(\theta)$

$= \frac{12}{5}$



$\theta = \sin^{-1}\left(\frac{12}{13}\right)$

$\sin(\theta) = \frac{12}{13}$

Find x:

$x^2+12^2=13^2$
$x^2+144=169$

$x^2 = 25$

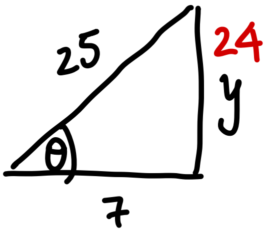
$x = 5$

3. Find $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \frac{12}{5}$

4. Find $\operatorname{cosec}\left(\cos^{-1}\left(\frac{7}{25}\right)\right) = \frac{1}{\sin\left(\cos^{-1}\left(\frac{7}{25}\right)\right)} = \frac{1}{\sin(\theta)} = \frac{1}{\left(\frac{24}{25}\right)} = \frac{25}{24}$

Recall

$\operatorname{cosec} x = \frac{1}{\sin x}$



Find $7^2 + y^2 = 25^2$

$49 + y^2 = 625$

$y^2 = 576$

$y = \sqrt{576}$

$y = 24$

$\cos^{-1}\left(\frac{7}{25}\right) = \theta$

$\cos(\theta) = \frac{7}{25}$

$$\begin{array}{r} 25\times \\ 25 \\ \hline 125 \\ 50 \\ \hline 625 \end{array}$$

5. $\cot\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \cot(\theta) = \frac{\sqrt{5}}{2}.$

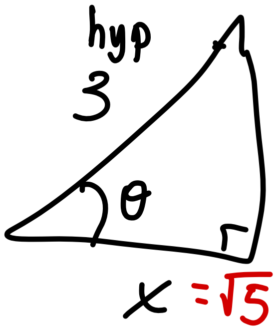
- Draw the triangle
- Use Pythagoras' theorem

$\cot(\quad \theta \quad)$

$$\frac{1}{\tan} = \frac{1}{\frac{opp}{adj}}$$

$$= \frac{1}{\frac{2}{\sqrt{5}}}$$

$$= \frac{1}{1} \div \frac{2}{\sqrt{5}} = \frac{\sqrt{5}}{2}$$



$\theta = \sin^{-1}\left(\frac{2}{3}\right)$

$\Rightarrow \boxed{\sin(\theta) = \frac{2}{3}}$

$2^2 + x^2 = 3^2$

$4 + x^2 = 9$

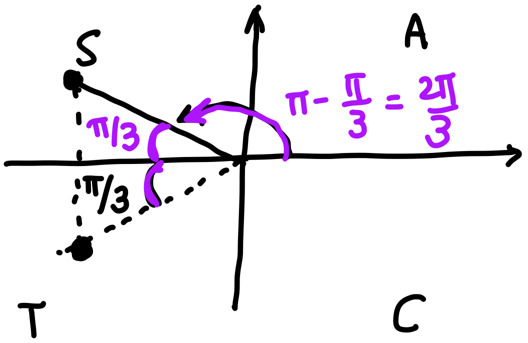
$x = \sqrt{5}$

6. Find $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$

$= \cos^{-1}\left(-\cos\left(\frac{\pi}{3}\right)\right)$

$= \boxed{\frac{2\pi}{3}}$

Do not use triangles

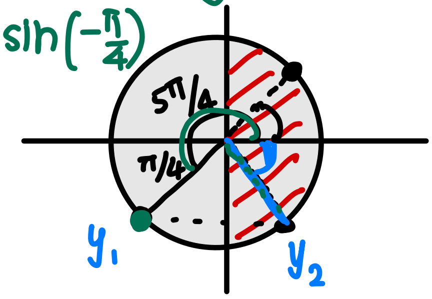


Range of $\cos^{-1}(x)$: $[0, \pi]$

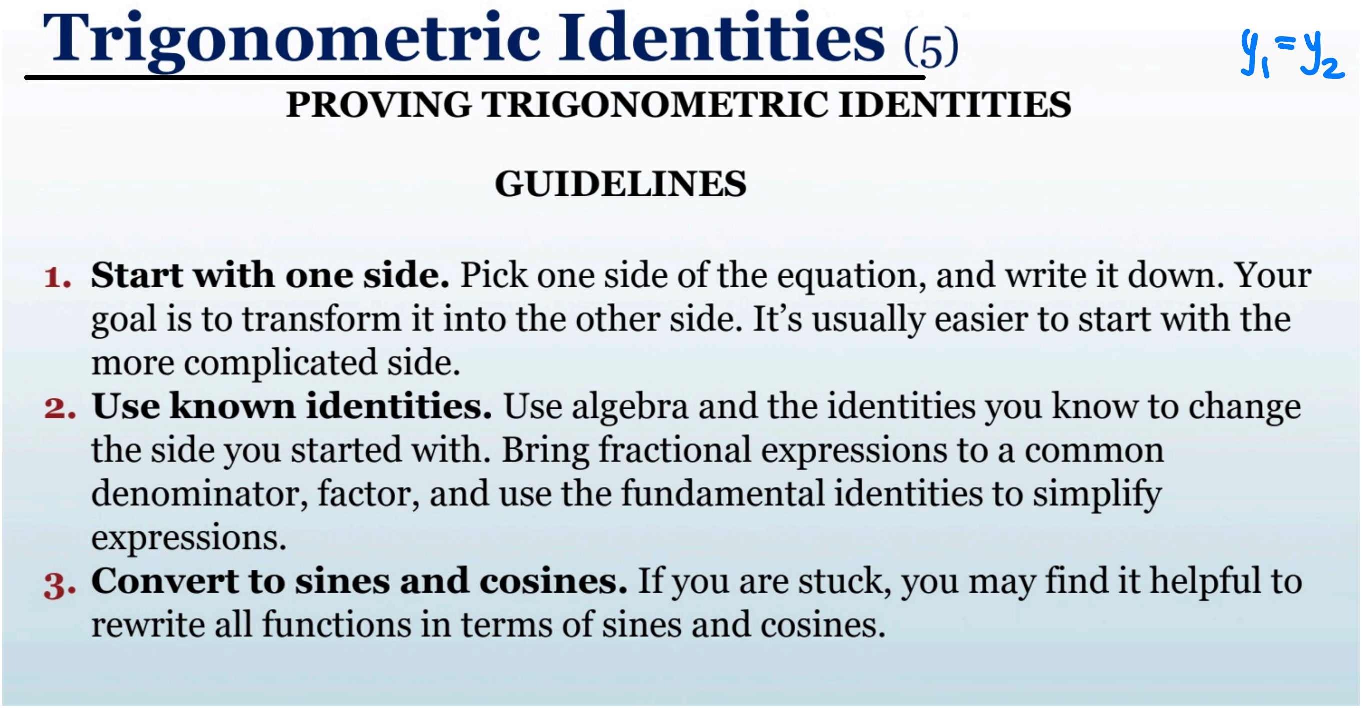
7. Find $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = -\frac{\pi}{4}$

Range of $\sin^{-1}(x)$: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Equivalent to checking $\sin\left(\frac{5\pi}{4}\right) = \sin\left(-\frac{\pi}{4}\right)$



$y_1 = y_2$



Trigonometric Identities (5)

PROVING TRIGONOMETRIC IDENTITIES

GUIDELINES

1. **Start with one side.** Pick one side of the equation, and write it down. Your goal is to transform it into the other side. It's usually easier to start with the more complicated side.
2. **Use known identities.** Use algebra and the identities you know to change the side you started with. Bring fractional expressions to a common denominator, factor, and use the fundamental identities to simplify expressions.
3. **Convert to sines and cosines.** If you are stuck, you may find it helpful to rewrite all functions in terms of sines and cosines.

⊛ Very important ⊛

Note $\sin^2 x = (\sin x)^2$

Identity: $\boxed{\sin^2 x + \cos^2 x = 1}$

Show that $\tan^2 x + 1 = \sec^2 x$.

LHS = $\tan^2 x + 1$ $= \left(\frac{\sin x}{\cos x}\right)^2 + 1$

left hand side

Use $\tan x = \frac{\sin x}{\cos x}$

$= \frac{\sin^2 x}{\cos^2 x} + 1$

$= \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}$

$\frac{1}{\cos x} = \sec x$

$= \frac{(\sin^2 x + \cos^2 x)}{\cos^2 x}$ =1 from the identity

$= \frac{1}{\cos^2 x}$

$= \frac{1}{(\cos x)^2}$

$= \sec^2 x$ ✓

$=$ RHS

$\boxed{\tan^2 x + 1 = \sec^2 x}$ ←

HW: $\boxed{1 + \cot^2 x = \operatorname{cosec}^2 x}$

Example Simplify $\cos(t) + \tan(t)\sin(t)$ ⇒

Use: $\tan(t) = \frac{\sin(t)}{\cos(t)}$

$= \cos(t) + \frac{\sin(t)}{\cos(t)} \cdot \sin(t)$

$= \cos(t) + \frac{\sin^2(t)}{\cos(t)}$

$= \frac{\cos^2(t)}{\cos(t)} + \frac{\sin^2(t)}{\cos(t)}$

$= \frac{\cos^2(t) + \sin^2(t)}{\cos(t)}$

$= \frac{1}{\cos(t)}$

$= \sec(t)$

2. Simplify $\dfrac{\cos(x)\cdot\sec(x)}{\cot(x)} = \dfrac{\cancel{\cos(x)}\ \dfrac{1}{\cancel{\cos(x)}}}{\left(\dfrac{1}{\tan(x)}\right)}$

$$= \frac{1}{\dfrac{1}{\left(\dfrac{\sin(x)}{\cos(x)}\right)}} = \frac{\sin(x)}{\cos(x)}.$$

$$= 1 \cdot \tan(x)$$

Verify that the identity holds.

1. $\dfrac{1}{1-\sin^2 z} = \boxed{1+\tan^2 z}$

$$\text{LHS} = \frac{1}{\underbrace{1-\sin^2 z}}$$

$$= \frac{1}{\cos^2 z}$$

$$= \sec^2 z$$

$$= 1+\tan^2 z$$

$$= \text{RHS} \quad \checkmark$$

$\dfrac{\cos x}{\sin x}$

$= \dfrac{1}{\tan x}$

$= \cot x$

Show that the LHS is equal to the RHS.

Reminder: $1 = \cos^2 x + \sin^2 x$

$1 - \sin^2 x = \cos^2 x$

1. $\boxed{\cos^2 x + \sin^2 x = 1}$

2. $\rightarrow 1 + \tan^2 x = \sec^2 x$

3. $\rightarrow 1 + \cot^2 x = \operatorname{cosec}^2 x$

Get 3. $\dfrac{\cos^2 x}{\sin^2 x} + \dfrac{\sin^2 x}{\sin^2 x} = \dfrac{1}{\sin^2 x}$

$$\cot^2 x + 1 = \operatorname{cosec}^2 x$$

Get 2. $\dfrac{\cos^2 x}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$

$$1 + \tan^2 x = \sec^2 x$$

2. Verify that $\dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x} = 2\tan x \sec x$

$$LHS = \frac{1}{1-\sin x} - \frac{1}{1+\sin x} = \frac{\cancel{1}+\sin x - (\cancel{1}-\sin x)}{(1-\sin x)(1+\sin x)}$$

difference of two squares

Use identity

$\cos^2 x + \sin^2 x = 1$

$1 - \sin^2 x = \cos^2 x$

$$= \frac{2\sin x}{1-\sin^2 x}$$

$$= \frac{2\sin x}{\cos^2 x}$$

$$= \underbrace{\frac{2\sin x}{\cos x}} \cdot \underbrace{\frac{1}{\cos x}}$$

$$= 2\tan x \sec x$$

$= RHS$ ✓

3. Verify $\frac{\sec t - \cos t}{\sec t} = \sin^2 t$

$$LHS = \frac{\sec t - \cos t}{\sec t} = \frac{\frac{1}{\cos t} - \cos t}{\left(\frac{1}{\cos t}\right)}$$

$$= \left(\frac{1}{\cos t} - \cos t\right) \cdot \cos t$$

use $\sin^2 t + \cos^2 t = 1$

$1 - \cos^2 t = \sin^2 t$

$$= \left(\frac{1-\cos^2 t}{\cancel{\cos t}}\right) \cancel{\cos t}$$

$1 - \cos^2 t$

$= \sin^2 t$

$$= \sin^2 t$$

$= RHS$ ✓

- Office hours today in WWH Room 412 4:30 pm
- Review session at WWH Room 101 at 4:30-6 pm. tomorrow (Dec 15).

Trigonometric identities

① Verify $(\tan x+\cot x)^2 = \sec^2 x + \overset{\text{cosec}^2 x}{\overbrace{\csc^2 x}}$

$$(x+y)^2 = x^2+2xy+y^2$$

Identity: $\boxed{\sin^2 x+\cos^2 x=1}$

$$\frac{\sin^2 x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}=\frac{1}{\sin^2 x} \Rightarrow ☺ \boxed{1+\cot^2 x = \text{cosec}^2 x}$$

$$\frac{\sin^2 x}{\cos^2 x}+\frac{\cos^2 x}{\cos^2 x}=\frac{1}{\cos^2 x} \Rightarrow \boxed{\tan^2 x+1=\sec^2 x} ⊛$$

$$\text{LHS} = (\tan x+\cot x)^2 = (\tan x+\cot x)(\tan x+\cot x)$$

$$= \tan^2 x + 2\underbrace{\tan x\cot x}+\cot^2 x$$

$$= \tan^2 x + 2\cancel{\tan}x\cdot\frac{1}{\cancel{\tan}x}+\cot^2 x$$

$$= \tan^2 x + 2 + \cot^2 x$$

$$= \underbrace{(\tan^2 x+1)}_{\sec^2 x} + \underbrace{(1+\cot^2 x)}_{\text{cosec}^2 x}$$

⊛ (☺)

$$= \sec^2 x + \text{cosec}^2 x$$

$$= \text{RHS}$$

② Verify $\dfrac{1+\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta} = 2\sec\theta$

$$LHS = \frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta}$$

$$= \frac{(1+\sin\theta)(1+\sin\theta) + \cos^2\theta}{\cos\theta\,(1+\sin\theta)}$$

$$= \frac{1 + 2\sin\theta + \left(\overbrace{\sin^2\theta + \cos^2\theta}^{=1 \text{ from the identity}}\right)}{\cos\theta\,(1+\sin\theta)}$$

$$= \frac{2 + 2\sin\theta}{\cos\theta\,(1+\sin\theta)}$$

$$= \frac{2\cancel{(1+\sin\theta)}}{\cos\theta\,\cancel{(1+\sin\theta)}}$$

$$= \frac{2}{\cos\theta} = 2\cdot\overset{=\sec\theta}{\boxed{\frac{1}{\cos\theta}}}$$

$$= 2\sec\theta$$

$$= RHS.$$

Algebra and Calculus
New York University
FINAL EXAM, Summer 2014
VERSION A

Name:____________________________________ ID:__________________

Read all of the following information before starting the exam:

- For multiple choice questions, only the answer is required. No work is required and no partial credit will be awarded. You must clearly circle your answer.
- For free response questions, you must show all work, clearly and in order, if you want to get full credit. We reserve the right to take off points if we cannot see how you arrived at your answer (even if your final answer is correct).
- The exam is closed book. You are not allowed to use a calculator or consult any notes while taking the exam.
- The exam time limit is 2 hours. **Good luck!**

SCORES

MC (45 points)	
1 (14 pts)	
2 (8 pts)	
3 (8 pts)	
4 (18 pts)	
5 (7 pts)	
TOTAL	

(45 points) This parts consists of 15 multiple choice problems. Nothing more than the answer is required; consequently no partial credit will be awarded.

1. If $f(x) = \dfrac{4}{4-x}$, find $f^{-1}(2)$
   - (a) 1/2
   - (b) 1
   - [✓ (c)] 2
   - (d) undefined
   - (e) none of the above

★ Inverses.

$f(x) = \frac{4}{4-x} \Rightarrow y = \frac{4}{4-x}$

$\Rightarrow (4-x)y = 4$

$4y - xy = 4$

$4y - 4 = xy$

$\frac{4y-4}{y} = x$

$\boxed{f^{-1}(x) = \frac{4x-4}{x}}$

$f^{-1}(2) = \frac{4(2)-4}{2}$

$= 2$

$\ln(x) = \log_e(x) = y$

$\ln(1) \Rightarrow e^y = 1$

$e^y = x$

2. Let $f(x) = \begin{cases} x-1 & \text{if } x \geq 0; \\ -x^2 & \text{if } x < 0. \end{cases}$ and let $g(x) = \ln(x+2)$ for $x > -2$.

   Find $f(g(-1))$
   - [✓ (a)] -1
   - (b) 0
   - (c) 1
   - (d) e
   - (e) Undefined

$g(-1) = \ln(-1+2)$

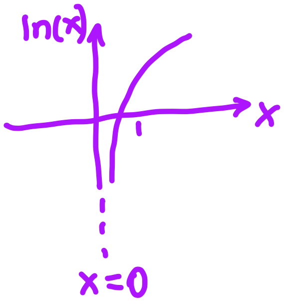
$= \ln(1)$

$= 0$

$f(\underbrace{g(-1)}_{0}) = f(0)$

$= 0-1$

$= -1$



Domain of $\ln(x)$: $(0,\infty)$

or $x>0$

3. The solution of the equation

$$2\sin^2 x + \sin x - 1 = 0 \text{ on } \left[0, \frac{\pi}{2}\right)$$

   is:
   - [✓ (a)] $\frac{\pi}{6}$
   - (b) $\frac{\pi}{2}$
   - (c) π
   - (d) $\frac{\pi}{3}$
   - (e) -1

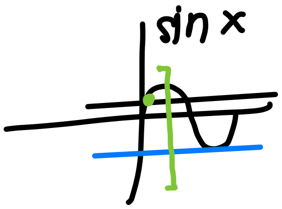
$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}$ or $\cancel{\sin x = -1}$

rejected since it's not going to be in $\left[0, \frac{\pi}{2}\right)$

$x = \sin^{-1}\left(\frac{1}{2}\right)$

$x = \frac{\pi}{6}$



$\sqrt{\cdots} \Rightarrow x-3 \geq 0 \Rightarrow x \geq 3$

4. The domain of the function

$$f(x) = \frac{\sqrt{x-3}}{x^2-5x-6}$$

$x \neq \cancel{-1}, 6$

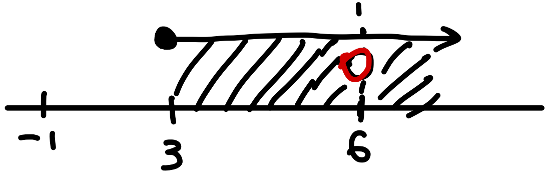
is:

$= \frac{\sqrt{x-3}}{(x+1)(x-6)}$

$x \geq 3,\ x \neq 6$

↳ $[3, 6) \cup (6, \infty)$

- (a) $[0, 3)$
- (b) $[3, \infty)$
- (c) $(-1, 6)$
- [✓ (d)] $[3, 6) \cup (6, \infty)$
- (e) $(-\infty, -1) \cup (-1, 6)$



5. Consider the following statements. In each case, P is a polynomial.

✓ I. The domain for $P(X)$ is $(-\infty, \infty)$.

× II. If the degree of P is odd, then we must have $P(x) \to -\infty$ as $x \to -\infty$.

✓ III. If $P(4) = 0$, then $x - 4$ is a factor of P.

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

Which of the above statements are true?

the exponents are non-negative integers.

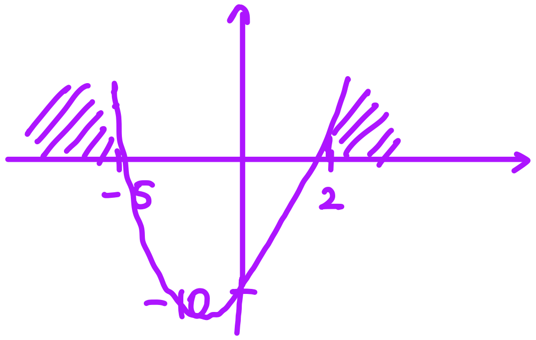
- (a) I and II
- [✓ (b)] I and III
- (c) II and III
- (d) I, II, and III
- (e) III only

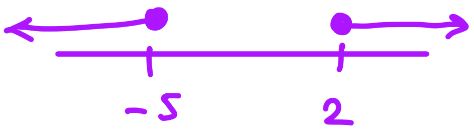
6. Find the domain of $f(x) = \sqrt{x^2 + 3x - 10}$. $= \sqrt{(x+5)(x-2)}$

- (a) $[0, \infty)$
- [✓ (b)] $(-\infty, -5] \cup [2, \infty)$
- (c) $(-\infty, -5) \cup (2, \infty)$
- (d) $(-5, 2)$
- (e) $[-5, 2]$

$\boxed{(x+5)(x-2) \geq 0}$

$y = \underset{\sim}{x^2} + \underset{\sim}{3x} \underset{\uparrow}{-10} \geq 0$





$\left.\begin{array}{l} x \geq 2 \\ x \leq -5 \end{array}\right\} (-\infty, -5] \cup [2, \infty)$

$$\boxed{\log_a(x) = b \iff a^b = x}$$

7. If $\log_2(2\log_3(x)) = 1$, find x.

   (a) $x = 0$
   (b) $x = 2$
   [✓ (c) $x = 3$]
   (d) $x = 4$
   (e) none of the above

$\log_2(2\log_3(x)) = 1$

$\iff \cancel{2}^{1} = \cancel{2}\log_3(x)$

$1 = \log_3(x)$

$3^1 = x \Rightarrow x = 3$

8. If $f(x) = \log_2(x-2)$, find $\cancel{f(3)} + f^{-1}(3)$. 0

   [✓ (a) 10]
   (b) 0
   (c) 1
   (d) 8
   (e) none of the above

$f(3) = \log_2(3-2) = \log_2(1) = 0$

$f^{-1}(3) = ?$

$f(x) = \log_2(x-2)$

$y = \log_2(x-2)$

$2^y = x-2$

$x = 2^y + 2$

$f^{-1}(x) = 2^x + 2$

$f^{-1}(3) = 2^3 + 2 = 8 + 2 = 10$

9. The amplitude, period and vertical shift of the trigonometric curve

$$y = 3\cos(2x - \pi) - 1$$

$= 3\cos\left(2\left(x - \frac{\pi}{2}\right)\right) - 1$

amplitude

shift vertically

$\frac{2\pi}{k} = \frac{2\pi}{2} = \pi$

are respectively:

   ~~(a)~~ 3, 2π and 1.
   ~~(b)~~ 2, 2π and -1
   ~~(c)~~ 3, π and 1
   [✓ (d) 3, π and -1]
   ~~(e)~~ 2, π and $\pi/2$

$y = a\cos(\underbrace{k(x-h)}) + b$

factored form

10. Let $f(x) = \begin{cases} \ln x & \text{if } x \geq 1; \\ -x+1 & \text{if } 0 < x < 1. \\ e^{-x} & \text{if } x \leq 0. \end{cases}$

Consider the following statements.

✓ I. $f(1) - f(0) = -1$ ✓ II. $f(x) \to \infty$ as $x \to -\infty$ × III. f has horizontal asymptote $y = 1$.

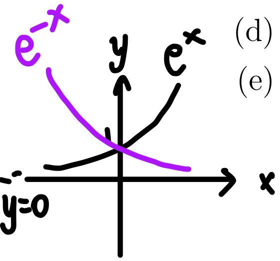
Which of the above statements are true?

   (a) I only
   (b) II only
   (c) I and II
   (d) I and III
   (e) II and III

I. $f(1) - f(0) = \cancel{\ln(1)} - \cancel{e^{-0}}$
$0 \qquad 1$

$= 0 - 1$

$= -1$



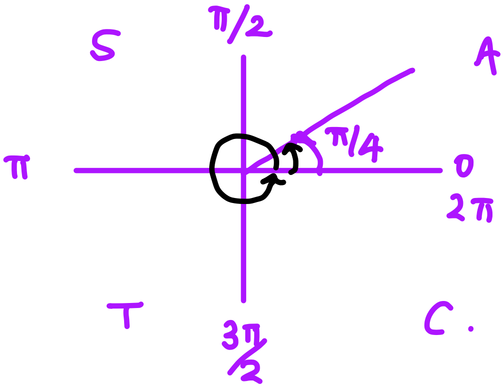
$e^{-x} \to \underset{\sim}{\infty}$ as $\underline{x \to -\infty}$

11. Find $\cos^{-1}\left(\cos\frac{9\pi}{4}\right)$. $= x$

- [✓ (a) $\pi/4$]
- (b) $3\pi/4$
- (c) $-\pi/4$
- (d) $9\pi/4$
- (e) $5\pi/4$

$\underbrace{\cos}\left(\frac{9\pi}{4}\right) = \cos(x)$

Range of the outer function $\cos^{-1}$ is $[0, \pi]$



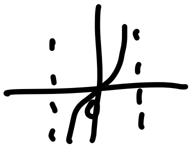
12. Find $\underbrace{\sin^{-1}}\left(\frac{-\sqrt{3}}{2}\right) + \underbrace{\tan^{-1}(1)}_{\frac{\pi}{4}}$.

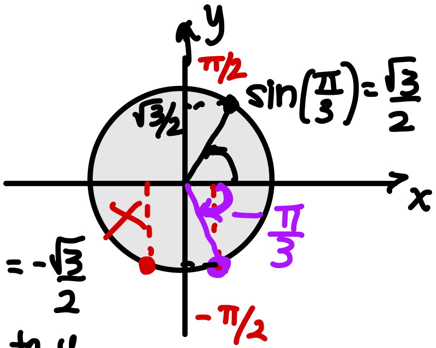
- (a) $\frac{\sqrt{3}}{2} - 1$
- (b) $\pi/6$
- (c) $\pi/12$
- (d) $\pi/4$
- [✓ (e) $-\pi/12$]

$\frac{\pi}{4} \Leftrightarrow \tan\left(\frac{\pi}{4}\right) = 1$

Range of $\sin^{-1}$:

$\boxed{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$





$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta \Rightarrow \sin\theta = -\frac{\sqrt{3}}{2}$

relates to y

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \tan^{-1}(1)$

$= -\frac{\pi}{3} + \frac{\pi}{4}$

$= -\frac{4\pi}{12} + \frac{3\pi}{12} = -\frac{\pi}{12}$

13. One solution for the equation

$$\frac{2}{x-2} + 3x = \underbrace{3(x-2)}_{1}$$

is:

- (a) $5/2$
- [✓ (b) $5/3$]
- (c) 2
- (d) $5/4$
- (e) no real solution

$2 + 3x(x-2) = 3(x-2)(x-2)$

$2 + 3x^2 - 6x = 3(x^2 - 4x + 4)$

$2 + 3\cancel{x^2} - 6x = 3\cancel{x^2} - 12x + 12$

$6x = 10 \Rightarrow x = \frac{10}{6}$

$= 5/3$

$\frac{2}{x-2} = \cancel{3x} - 6 - \cancel{3x}$

$2 = -6(x-2)$

$2 = -6x + 12$

$6x = 10 \Rightarrow \boxed{x = \frac{5}{3}}$

14. A point in the 1st quadrant satisfies $\cos(2t) = \frac{1}{2}$. Find $\sin(t)$.

- [✓ (a) $1/2$]
- (b) 1
- (c) $\sqrt{3}/2$
- (d) $\pi/6$
- (e) $\pi/3$

$2t = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$t = \frac{\pi}{6}$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

15. Define $f(x)$ for $x \geq 1$ by $f(x) = x^2 + 4$. What is the range of f^{-1}?

- (a) $[0, \infty)$
- (b) $[1, 4]$
- [✓ (c) $[1, \infty)$]
- (d) $(-\infty, \infty)$
- (e) $[5, \infty)$

Range of $f^{-1}(x)$

$=$ Domain of $f(x)$ $= [1, \infty)$

(55 points) Problems 1-5 are free response problems. Put your work/explanations in the space below the problem.

- **Read and follow the instructions of every problem.**
- **Show all your work for purposes of partial credit. Full credit may not be given for an answer alone.**
- **Justify your answers.**

1. (a) (6 pts) Let $f(x) = \frac{3x}{x-2}$. Find $f^{-1}(x)$.

$$y = \frac{3x}{x-2}$$

$$(x-2)y = 3x$$

$$xy - 2y = 3x$$

$$xy - 3x = 2y \Rightarrow x(y-3) = 2y \Rightarrow x = \frac{2y}{y-3}$$

$$f^{-1}(x) = \frac{2x}{x-3}$$

(b) What is the domain and range of both f and f^{-1} in part (a)?

Domain of f : $x \neq 2$ or $(-\infty, 2) \cup (2, \infty)$

$=$ Range of f^{-1}

Domain of f^{-1} : $x \neq 3$ or $(-\infty, 3) \cup (3, \infty)$

$=$ Range of f

(c) (8 pts) Find the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

for $f(x) = x^2 + 3x - 1$

$$f(x+h) = (x+h)^2 + 3(x+h) - 1$$

$$= \underbrace{x^2 + 2xh + h^2 + 3x + 3h - 1}$$

be careful to place parenthesis

$$\frac{f(x+h) - f(x)}{h} = \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h \cancel{-1} - (\cancel{x^2} + \cancel{3x} \cancel{-1})}{h}$$

$$= \frac{2xh + h^2 + 3h}{h} = \frac{\cancel{h}(2x + h + 3)}{\cancel{h}} = 2x + h + 3$$

2. The function $f(x) = 2x^2 - 12x + 14$ represents the number of mosqutios (in thousands) that are flying about in Texas in June where x is the number of days past May 31.

   ← output

   (a) (4 pts) In context to this problem, what does the point (5,4) mean?

   ↑ input

Input = # of days past May 31

Output = # of mosquitos in 1000s

→ On June 5 there are 4000 mosquitos flying about in Texas

(b) Write $f(x)$ in vertex form.

COMPLETING THE SQUARE

$$
\begin{aligned}
f(x) &= 2x^2 - 12x + 14 \\
&= 2(x^2 - 6x) + 14 \\
&= 2\left[(x-3)^2 - 3^2\right] + 14 \\
&= 2\left[(x-3)^2 - 9\right] + 14 \\
&= 2(x-3)^2 - 18 + 14 \\
&= 2(x-3)^2 - 4
\end{aligned}
$$

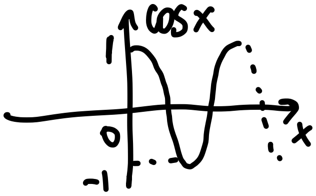
$$
\begin{aligned}
& ax^2 + bx + c \\
&= a\left[x^2 + \frac{b}{a}x\right] + c \\
&= a\left[\left(x + \frac{b}{2a}\right) - \left(\frac{b}{2a}\right)^2\right] + c
\end{aligned}
$$

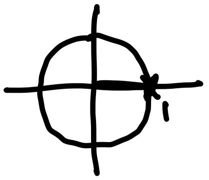
(c) (2 pts) What is the smallest number of mosquitos in June flying about Texas?

Vertex $(3, -4)$

but we cannot have negative # of mosquitos

⇒ zero mosquitos





3. (a) (12 points) Let $f(x) = -1 + 2\cos(2x)$.

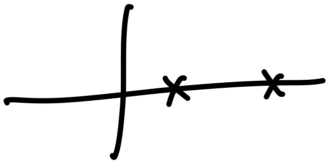
i. (2 pt) Find the y-intercept of f.

$x=0 \Rightarrow f(0) = -1+2\cos(2\cdot 0) = -1+2\cos(0) = -1+2(1)$
$= 1$

y-int. $(0, 1)$

ii. (4 pts) Find all x-intercepts of f on $[0, 2\pi]$.

$\underline{y=0}$ $\quad 0 = -1+2\cos(2x)$



$1 = 2\cos(2x)$

$\frac{1}{2} = \cos(2x)$

$\cos^{-1}\left(\frac{1}{2}\right) = 2x$

$\cos(\theta) = \frac{1}{2}$
↑
?

$\frac{\pi}{3} = 2x$

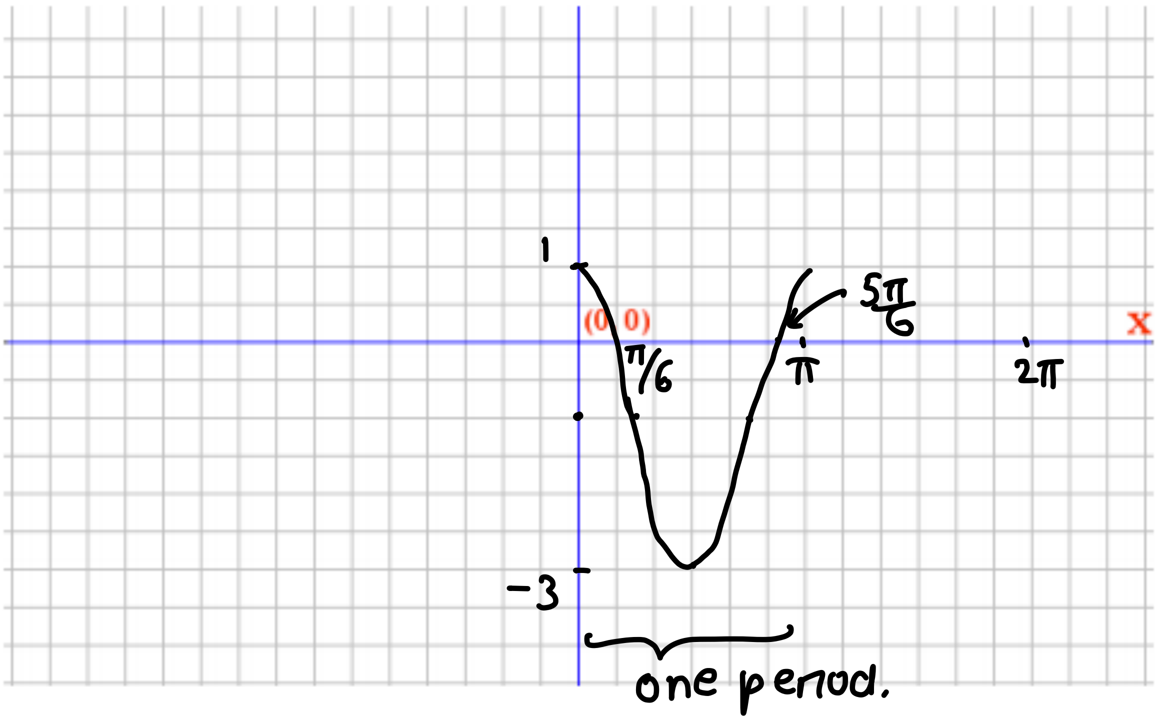
$x = \frac{\pi}{6}$.

Solving trig. equations.

iii. (6 pts) Graph one period of f. Clearly label the points at the beginning and end of the period, and label all intercepts. Clearly indicate the range of f.

$f(x) = 2\cos(2x) - 1$

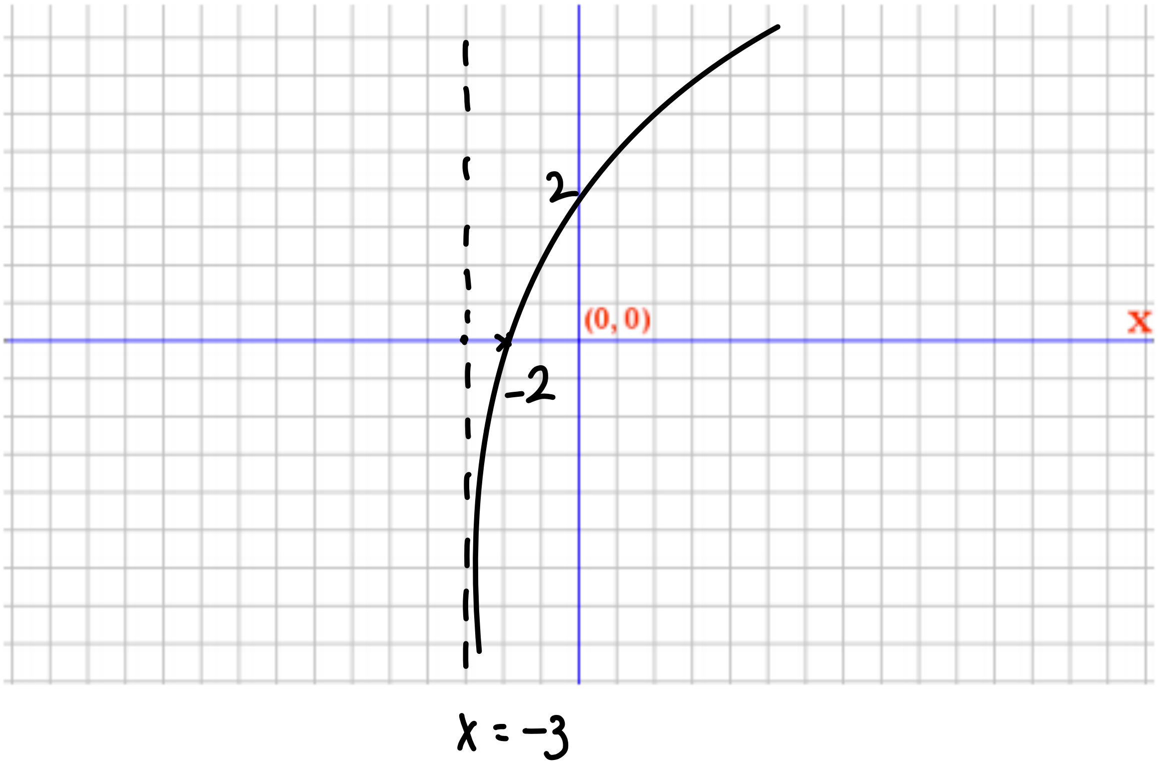
period $= \frac{2\pi}{k}$
$= \frac{2\pi}{2}$
$= \pi$



(Free-response problem 4, continued)

(b) (6 pts) Sketch the graph of $g(x) = 2\log_3(x+3)$ below, *not by merely plotting points*, but instead by applying transformations to the graph of $y = \log_3(x)$.

Clearly label all asymptotes and the x and y intercepts.



y-intercept: $x = 0$

$g(0) = 2\log_3(3)$

$= 2(1)$

$= 2$

$(0, 2)$

x-intercept : $y = 0 \Rightarrow 2\log_3(x+3) = 0$

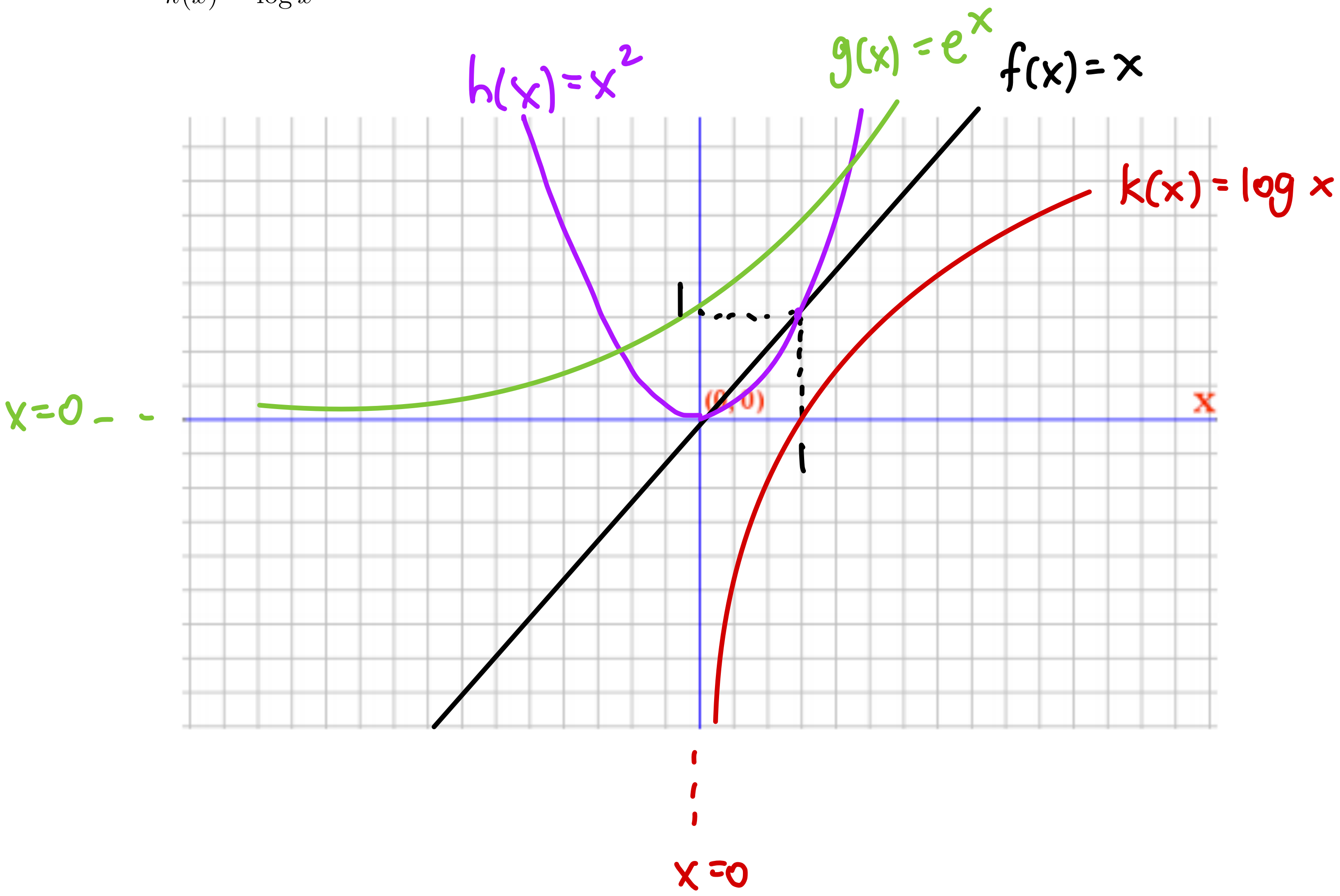
$3^0 = x+3$

$1 = x+3$

$x = -2$

$[-2, 0)$

4. (7 pts) On the same set of axes, sketch the graphs of $f(x) = x, g(x) = e^x, h(x) = x^2$ and $k(x) = \log x$



MATH-UA 009: Algebra and Calculus

Final Exam

Wednesday, December 20, 2017

Name: ______________________________

This exam is scheduled for 110 minutes, to be done individually, without calculators, notes, textbooks, and other outside materials. You can detach the last sheet for scratch work; do not detach any other sheets from this exam.

Show all work to receive full credit, except in multiple choice probelms.

Mark an "X" next to your lecture section

X	Section	Instructor	Lecture Time & Location
	001	Ruojun Huang	MW, 9:30-10:45AM, GCASL C95
	006	Mutiara Sondjaja	TTh, 12:30-1:45PM, 5WP 101
	011	Madhura Joglekar	TTh, 2:00-3:15PM, CANT 101

I pledge that I have observed the NYU honor code, and that I have neither given nor received unauthorized assistance during this examination.

Signature:

Problem	Points
MC	/36
FR 1	/10
FR 2	/14
FR 3	/10
FR 4	/10
Total	/80

Multiple Choice

(2 points each) **Please clearly write your answer in the box next to the question.** You need not explain your answer. No partial credit will be given.

1. Find the equation of a line passing through point $(7, 7)$ and perpendicular to the line $7x + 3y = -1$.

   (A) $y = \frac{3}{7}x + 7$
   (B) $y = \frac{7}{3}x + 7$
   [✓ (C)] $y = \frac{3}{7}x + 4$
   (D) $y = -\frac{7}{3}x + 4$
   (E) None of the above

$7x+3y=-1$

$3y=-7x-1$

$y=-\frac{7}{3}x-\frac{1}{3}$

$m=-\frac{7}{3}$, the perpendicular line will have slope $\frac{3}{7}$.

Problem 1

$(7,7)$ with slope $\frac{3}{7}$ $\Rightarrow$ $y-7=\frac{3}{7}(x-7)$

$y=\frac{3}{7}x-3+7$

$y=\frac{3}{7}x+4$

2. Solve the inequality for x

$$|3x + 2| < 4$$

   (A) $[-2, \frac{2}{3}]$
   (B) $x = -2, x = \frac{2}{3}$
   (C) $(-\infty, -2) \cup (\frac{2}{3}, \infty)$
   [✓ (D)] $(-2, \frac{2}{3})$
   (E) None of the above

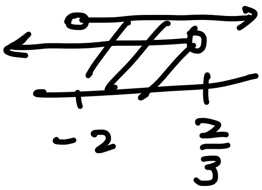
$3x+2<4$ $\qquad$ $(3x+2)>-4$

$3x<2$

$x<\frac{2}{3}$

$3x>-6$

$x>-2$



Problem 2

3. Which of the following is true about the rational function

$$q(x) = \frac{2x^3+2x}{x^2-1}\ ? \quad = \frac{2x(x^2+1)}{x^2-1} = \frac{2x(x^2+1)}{(x-1)(x+1)}$$

[✓ (A)] The graph of $q(x)$ has vertical asymptotes at $x = 1, x = -1$ and no horizontal asymptote.

(B) The graph of $q(x)$ has one vertical asymptote given by $x = -1$ and no horizontal asymptote.

(C) The graph of $q(x)$ has vertical asymptotes at $x = 1, x = -1$ and horizontal asymptotes given by $y = 2$ and $y = -2$.

(D) The graph of $q(x)$ has vertical asymptotes at $x = 1, x = -1$ and a horizontal asymptote given by $y = 2$.

(E) None of the above

Problem 3

☐

4. Solve for x.

$$\boxed{e^{2x} - e^x - 6 = 0} \longrightarrow v^2 - v - 6 = 0$$

let $\boxed{v = e^x}$

$v^2 = e^{2x}$

$(v+2)(v-3) = 0$

$v = -2 \quad v = 3$

$e^x \cancel{=} -2 \quad \boxed{e^x = 3}$

rej

$x = \ln(3)$

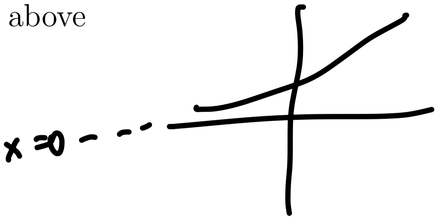
(A) $x = \frac{6}{e^2+1}$

[✓ (B)] $x = \ln(3)$

(C) $x = \ln(3), \ln(-2)$

(D) $x = \log_2(3), \log_2(-2)$

(E) None of the above



Problem 4

☐

5. Find $\cos^{-1}(\cos(\frac{7\pi}{6})) = y$ $\Rightarrow \cos(y) = \cos\left(\frac{7\pi}{6}\right)$

Range of $\cos^{-1}$ is $[0, \pi]$ so y should be in $[0, \pi]$

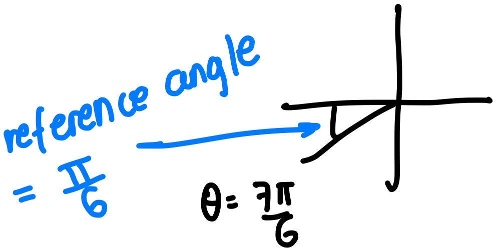
(A) $\frac{7\pi}{6}$

[✓ (B) $\frac{5\pi}{6}$]

(C) $\frac{\pi}{6}$

(D) $\frac{\cos(\frac{7\pi}{6})}{\cos(1)}$

(E) None of the above



so actually

$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Problem 5

6. Simplify the following expression using trigonometric identities:

$$\frac{1+\sin(x)}{\cos(x)} + \frac{\cos(x)}{1+\sin(x)}.$$

$$= \frac{(1+\sin x)(1+\sin x) + \cos x \cdot \cos x}{\cos x\,(1+\sin x)}$$

$$= \frac{1+2\sin x + \overset{=1}{\sin^2 x + \cos^2 x}}{\cos x(1+\sin x)}$$

$$= \frac{1+2\sin x+1}{\cos x(1+\sin x)}$$

$$= \frac{2+2\sin x}{\cos x(1+\sin x)}$$

$$= \frac{2\cancel{(1+\sin x)}}{\cos x\,\cancel{(1+\sin x)}}$$

$$= \frac{2}{\cos x}$$

(A) $\frac{1+2\sin(x)\cos(x)}{\cos(x)(1+\sin(x))}$

(B) $2\cos(x)$

[✓ (C) $\frac{2}{\cos(x)}$]

(D) $2\tan(x)$

(E) None of the above

Problem 6

7. Let $f(x) = \dfrac{1}{\sqrt{x}}$ and $g(x) = \ln(x-1)$. Find the domain of $f \circ g$.

(A) $[0, \infty)$
[✓ (B) $(1, \infty)$]
(C) $[2, \infty)$
(D) $(2, \infty)$
(E) None of the above

Problem 7 ☐

$$f(g(x)) = f(\ln(x-1))$$
$$= \frac{1}{\sqrt{\ln(x-1)}}$$

Domain of $\ln(x-1)$
is $\boxed{x > 1}$
Under the radical we must have >0

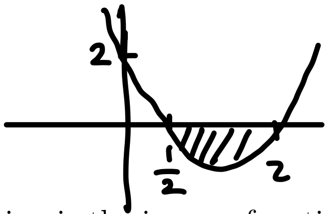
8. Which of the following is the solution of the inequality

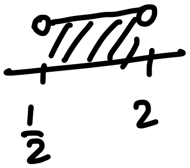
$$2x^2 - 5x + 2 < 0 \ ?$$

(A) $(1, 1.5)$
[✓ (B) $(0.5, 2)$]
(C) $(-\infty, 0.5) \cup (2, \infty)$
(D) $(-\infty, 1) \cup (1.5, \infty)$
(E) None of the above

Problem 8 ☐

$$2x^2 - 5x + 2 = (2x-1)(x-2) < 0$$





9. Which of the following is the inverse function of

$$f(x) = 2^x - 1 \ ?.$$

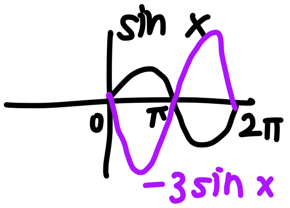
[✓ (A) $g(x) = \log_2(1 + x)$]
(B) $g(x) = \log_2(1 - x)$
(C) $g(x) = \log_2(x) + 1$
(D) $g(x) = e^{x+1}$
(E) None of the above

Problem 9 ☐

$$y = 2^x - 1$$
$$y + 1 = 2^x$$
$$\log_2(y+1) = x$$
$$f^{-1}(x) = \log_2(x+1)$$

10. Simplify the following expressions using trigonometric identities.

$$f(x) = -3\sin(\pi + x).$$



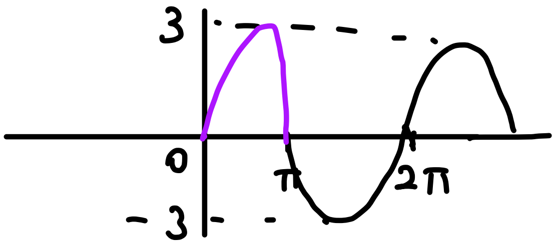
(A) $f(x) = 3\cos(x)$

[✓ (B)] $f(x) = 3\sin(x)$

(C) $f(x) = -3\sin(x)$

(D) $f(x) = -3\cos(x)$

(E) None of the above



Problem 10

11. Which one of the following functions does NOT go to ∞ as $x \to \infty$?

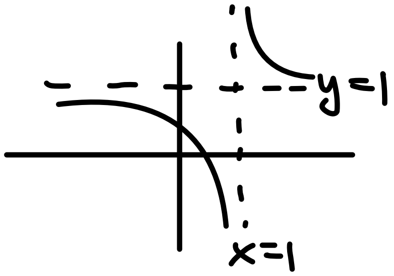
(A) $f(x) = 10 + \log_2(x) \rightarrow \infty$

(B) $f(x) = x^3 + x^2 \rightarrow \infty$

(C) $f(x) = xe^x \rightarrow \infty$

[✓ (D)] $f(x) = \frac{x+1}{x-1} = \frac{x-1+2}{x-1} = 1 + \frac{2}{x-1}$

(E) None of the above



Problem 11

12. Which of the following is a factor of the polynomial

$$P(x) = x^4 + 3x^3 - 2x^2 - 8x - 4 \ ?$$

Hint: Use the remainder theorem or long division.

[✓ (A)] $x + 1$ is $P(-1)=0$?

(B) $x - 1$ is $P(1)=0$?

(C) $x - 2$ is $P(2)=0$?

(D) $x^2 - 4 = (x-2)(x+2)$ are both $P(2)=0$ & $P(-2)=0$?

(E) None of the above

Problem 12

$$P(-1) = (-1)^4 + 3(-1)^3 - 2(-1)^2 - 8(-1) - 4$$

$$= 1 - 3 - 2 + 8 - 4$$

$$= 0$$

13. Which of the following is equal to

$$\left(\frac{8a^{-3}b^2}{a^6b^{-1}}\right)^{1/3}\ ?$$

(A) $\frac{a^3}{2b}$

[✓ (B)] $\frac{2b}{a^3}$

(C) $\frac{a^9}{8b}$

(D) $\frac{2b^{1/3}}{a}$

(E) None of the above

Problem 13 ☐

$$\left(\frac{8a^{-3}b^2}{a^6b^{-1}}\right)^{1/3} = \frac{8^{1/3}(a^{-3})^{1/3}\,b^{2/3}}{a^{6/3}\,b^{-1/3}}$$

$$b^{\frac{m}{n}} = \sqrt[n]{b^m}$$

$$= \frac{\sqrt[3]{8}\,a^{-1}\,b^{2/3}}{a^2\,b^{-1/3}}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$= \frac{2b^{2/3+1/3}}{a^3} = \frac{2b}{a^3}$$

14. Rationalize the numerator of the following expression:

$$\frac{1+\sqrt{x}}{2}.$$

(A) $\frac{1+x}{2(1-\sqrt{x})}$

(B) $\frac{1-x}{2(1+\sqrt{x})}$

[✓ (C)] $\frac{1-x}{2(1-\sqrt{x})}$

(D) $\frac{1+x}{2(1+\sqrt{x})}$

(E) None of the above

Problem 14 ☐

$$\frac{1+\sqrt{x}}{2}\cdot\left(\frac{1-\sqrt{x}}{1-\sqrt{x}}\right) = \frac{(1+\sqrt{x})(1-\sqrt{x})}{2(1-\sqrt{x})}$$

difference of two squares

$$(a-b)(a+b) = a^2-b^2$$

$$= \frac{1^2-(\sqrt{x})^2}{2(1-\sqrt{x})}$$

$$= \frac{1-x}{2(1-\sqrt{x})}$$

15. Below is the graph of a function $f(x)$.

    Find the average rate of change of $f(x)$ from $x = -1$ to $x = 3$.

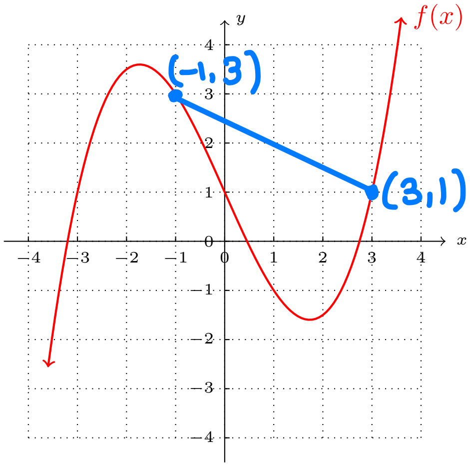
    (A) 2

    [✓ (B)] -0.5

    (C) 1

    (D) 0.5

    (E) None of the above



Problem 15

$$\frac{f(3)-f(-1)}{3-(-1)} = \frac{1-3}{3+1} = -\frac{2}{4} = -\frac{1}{2}$$

16. The graphs of the functions $f(x)$ and $g(x)$ are given below. Using the graph, solve the inequality

$$f(x) \geq g(x).$$

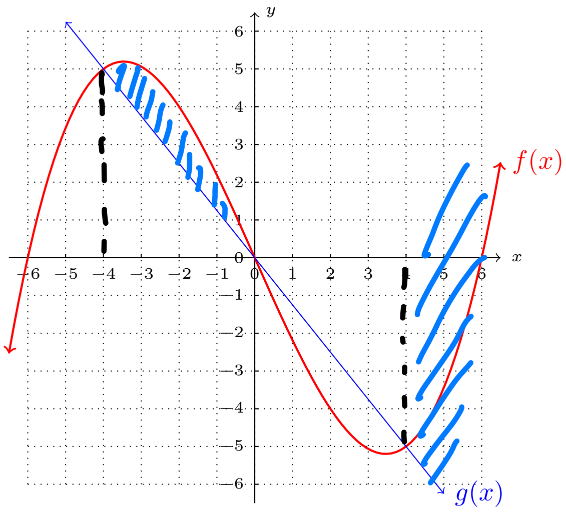
(A) $[-4, 4]$

(B) $(-\infty, -4] \cup [4, \infty)$

[✓ (C)] $[-4, 0] \cup [4, \infty)$

(D) $(\infty, -4] \cup [0, 4]$

(E) None of the above



Problem 16

$-4 \leq x \leq 0$

or $x \geq 4$

17. Given the table below, determine $(g \circ f)(2)$.

x	$f(x)$	$g(x)$
0	2	3
1	1	5
2	3	1
3	4	0
4	1	3
5	0	4

[✓ (A) 0]

(B) 1

(C) 3

(D) 4

(E) 5

$g(f(2)) = g(3)$

$= 0$

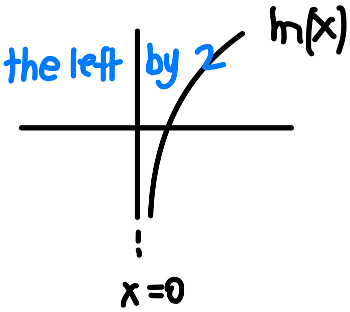
Problem 17

☐

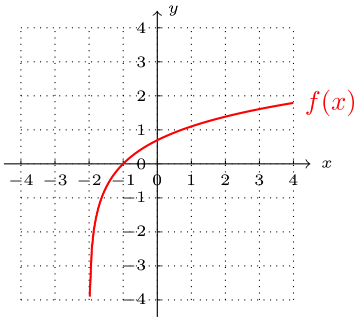
reflection about the x-axis

18. Which of the following is the graph of the function $f(x) = -\ln(x+2)$?

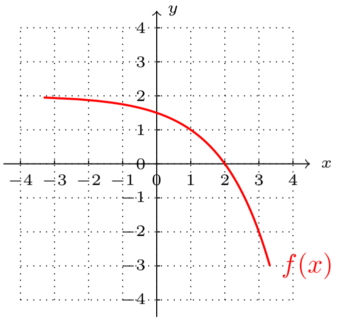
shift to the left by 2

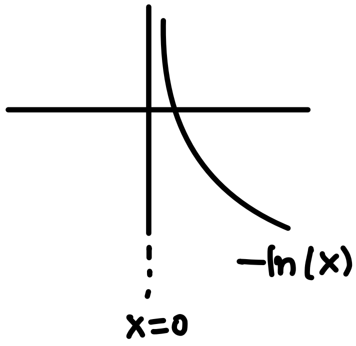


(A)

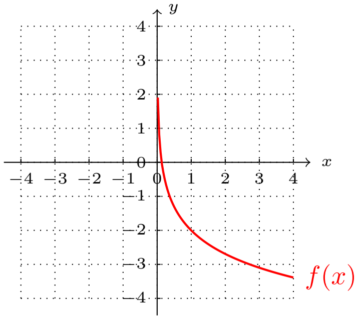


(C)

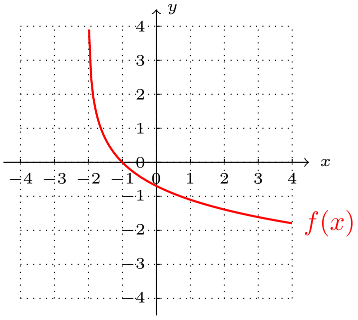


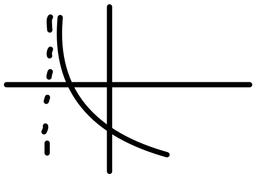


(B)



[✓ (D)]





(E) None of the above

Problem 18

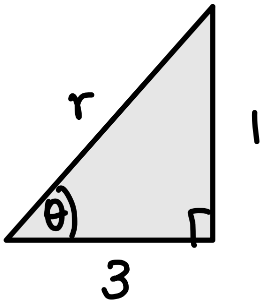
☐

Free Response

Please show all work and justification.

1. (10 points) Suppose that $\tan(\theta) = \frac{1}{3}$ and θ is an angle in Quadrant III. Find $\cos(\theta)$ and $\sin(\theta)$. Show all work.

$\tan(\theta) = \frac{1}{3}$



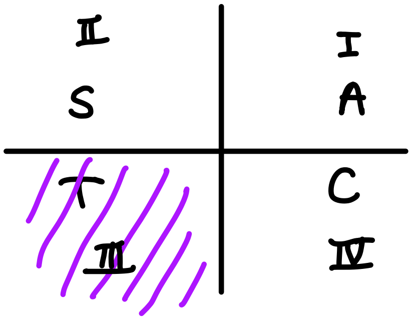
$r^2 = 1^2 + 3^2 = 1 + 9 = 10$

$r = \sqrt{10}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{\sqrt{10}}$

but we have to adjust the sign given we are in Quadrant III

$\Rightarrow \boxed{\cos\theta = -\frac{3}{\sqrt{10}}}$



here cos & sin are both negative

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{10}}$

but we have to adjust the sign given we are in Quadrant III

$\Rightarrow \boxed{\sin\theta = -\frac{1}{\sqrt{10}}}$

2. (14 points) Suppose that $f(x) = 7 - 6x - x^2$.

   (a) (3 points) Find the x and y intercepts of f. Show all work.

x-int. when $y=0$: $0 = 7-6x-x^2$

$= -(x^2+6x-7)$

$= -(x+7)(x-1)$

$\Rightarrow x = -7, 1$

Thus x-interepts are $(-7,0)$ and $(1,0)$

y-int. when $x=0 \Rightarrow f(0) = 7$

Thus the y-intercept is $(0,7)$

   (b) (5 points) Express f in standard form. (That is, in the form $f(x) = a(x-h)^2+k$.) Show all work.

$f(x) = 7-6x-x^2$

$= -x^2-6x+7$

$= -(x^2+6x)+7$

$= -[(x+3)^2-3^2]+7$

$= -(x+3)^2+9+7$

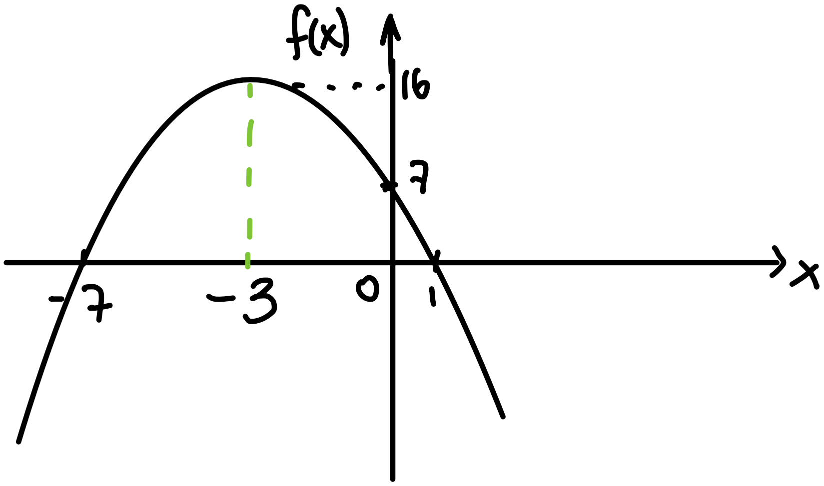
$= -(x+3)^2+16$

(c) (2 points) Based on your solution in part (b): (1) find the x and y coordinates of the vertex of f and (2) determine whether the vertex corresponds to the maximum or minimum value of f. Give a brief, 1 sentence, justification.

(1) Vertex: $(-3, 16)$

(2) max since the leading coefficient is negative

(d) (4 points) Sketch a graph of f. Clearly label the graph.



3. (10 points) Solve the following equation. Show all work.

$$\log_9(x+2) - \log_9(x) = 0.5 - \log_9(x-2) \qquad (*)$$

$$\log_9(x+2) - \log_9(x) + \log_9(x-2) = 0.5$$

$$\log_9\left(\frac{(x+2)(x-2)}{x}\right) = 0.5$$

$$\log_9\left(\frac{x^2-4}{x}\right) = 0.5$$

Recall

$\log_a(x) = b$

$\Leftrightarrow a^b = x$

$$9^{0.5} = \frac{x^2-4}{x}$$

$9^{0.5} = 9^{1/2} = \sqrt{9}$

$$\sqrt{9} = \frac{x^2-4}{x}$$

$$3 = \frac{x^2-4}{x}$$

$$3x = x^2-4$$

$$x^2 - 3x - 4 = 0$$

$$(x-4)(x+1) = 0$$

$$\boxed{x=4} \qquad x = -1$$

rejected. Since if we try to substitute into the original equation $(*)$ we have $\log_9(-1)$ which is <u>not</u> possible

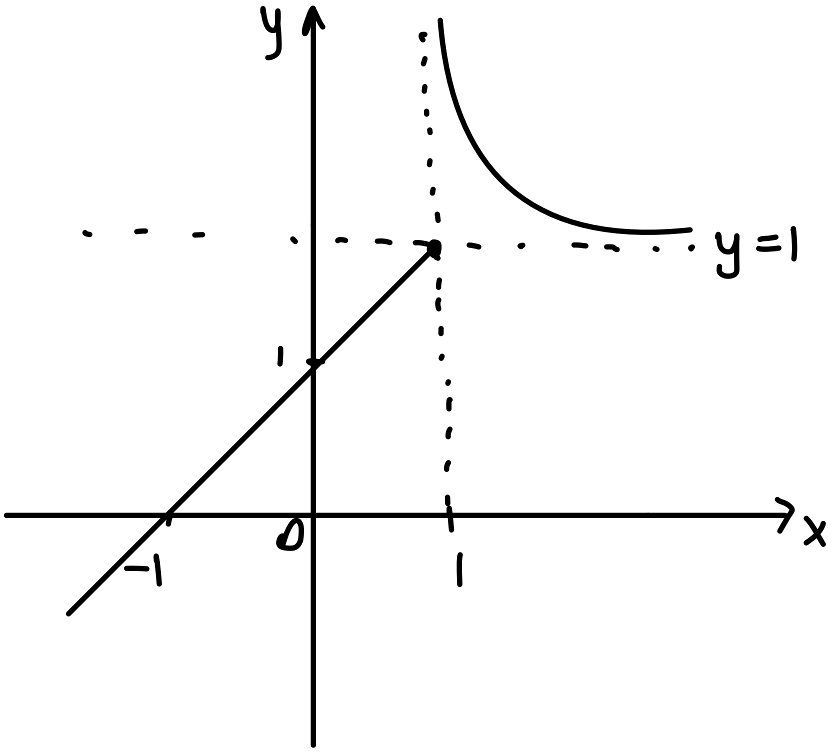
4. *Note: The two parts below are independent of one another.*

   (a) (6 points) Consider the piecewise defined function

$$f(x) = \begin{cases} \dfrac{x}{x-1} & \text{if } x > 1, \\ x+1 & \text{if } x \le 1. \end{cases}$$

$$\rightarrow \frac{x-1+1}{x-1} = 1 + \frac{1}{x-1}$$

   Sketch the graph of $f(x)$. Clearly label your graph (including important features such as intercepts and asymptotes) and show all work/reasoning.



   (b) (4 points) Find the inverse of the function $g(x) = \dfrac{x}{x-1}$. Show all work.

$$y = \frac{x}{x-1} \Rightarrow y(x-1) = x$$

$$xy - y = x$$

$$xy - x = y$$

$$x(y-1) = y \Rightarrow x = \frac{y}{y-1} \Rightarrow \boxed{g^{-1}(x) = \frac{x}{x-1}}$$

THE END

Thanks for a wonderful semester ☺