Collinear interaction of vortex pairs with different strengths – criteria for leapfrogging

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Abstract

We formulate a system of equations that describes the motion of four vortices made up of two interacting vortex pairs, where the absolute strengths of the pairs are different. Each vortex pair moves along the same axis in the same sense. In much of the literature, the vortex pairs have equal strength. The vortex pairs can either escape to infinite separation or undergo a periodic leapfrogging motion. We determine an explicit criterion in terms of the initial horizontal separation of the vortex pairs given as a function of the ratio of their strengths, to describe a periodic leapfrogging motion when interacting along the line of symmetry. In an appendix we also contrast a special case of interaction of a vortex pair with a single vortex of the same strength in which a vortex exchange occurs.

1 Introduction

Vortex rings as coherent fluid structures that appear in numerous application fields have frequently been the subject of interest in vortex dynamics. These structures are well-studied using a variety of analytic, numerical, and experimental techniques — see Shariff and Leonard [29] for a review. Smoke rings are a familiar example and a concise description of the experiments devised by Tait [31, p. 291–293], using smoke rings in air, can be found in the review by Meleshko [24, p. 413–414]. The interaction of multiple vortex rings results in complex dynamical behaviour, such as the leapfrogging motion of a pair of vortex rings, which is the main motivation for the current paper.

In this paper we consider a simplified model of the leapfrogging of two vortex rings with a common axis. When the rings have the same strength, width, and sense of rotation they travel in the same direction. The rear ring shrinks and accelerates due to their mutual interaction and the leading ring widens and decelerates. The rear ring then passes through the leading ring, with this process of 'leapfrogging' then repeating again and again [23]. We note that interactions of such vortex rings — primarily with filament cores — have been considered in Hicks [20] and more recently in Borisov, Kilin, and Mamaev [13], in Caplan, Talley, Carretero-González, and Kevrekidis [14], in Cheng, Lou, and Lim [15] and in Aiki [4].

Of course in 'real' flows the filament cores have finite sizes and the effect of viscosity and three dimensionality is to induce diffusion of vorticity within the flow. At high Reynolds

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number and in a two-dimensional model of the interaction of N line vortices we have an idealised dynamical system, treatment of which has had a notable history — summarised for example in the book by Newton [27] and the paper by Aref [7].

The simplification considered here is to two line vortex pairs and a generalisation then to pairs of different strengths but with the same sense(s) of rotation. The behaviours then recorded should inform those to be expected for the generalised classical three-dimensional ring problem.

Of course the N-line vortex problem is well-known in fluid mechanics, with consideration there of the mutual motions of their intersection points with a transverse plane — see [6] for an earlier review. The cases of $N = 1$, $N = 2$, $N = 3$ are integrable, leading to regular dynamics. The $N = 3$ case was considered extensively by Gröbli in his doctoral thesis [18] — he showed explicitly the reduction of the problem to quadratures, and detailed various specific and non-trivial examples.

The interactions of $N = 4$ vortices without any imposed symmetry are generally chaotic, exhibiting the classic property of sensitivity to initial conditions. While the interactions of $N = 3$ vortices are not chaotic, they may still exhibit regimes of bounded (capture) and unbounded (scattering) motions.

The particular case we consider here does have symmetry in that we have two pairs — each consisting of equal and opposite strengths and the same axis. In any event, the $N = 4$ case with zero total circulation overall and zero impulse is known to be integrable — see e.g. Eckhardt [16], Aref and Stremler [9]. In contrast to various other treatments in the literature concerning two such vortex pairs — e.g. Love [23], Péntek, Tel, and Toroczkai [28], Newton [27], where the four vortices have equal absolute strengths — we allow the absolute strengths of the two pairs to be different. So, we consider a system of four point vortices with strengths $\pm K_1, \pm K_2$, that move in the same direction along a common symmetry axis perpendicular to the extension of both pairs. In this work we specify the conditions determining when two given vortex pairs leapfrog, and when they do not. The overall impulse is not zero in this case, but the imposed symmetry does restrict the motion strongly.

In a paper by Eckhardt and Aref [17] there is extensive treatment of the general collision dynamics of line vortex pairs, with the aim of quantifying scattering regimes, as well as those where there can be (maybe prolonged) periods of vortex partner exchange — with parallels in the theory of solitons and in chemical physics. In appendix B of that paper there is a treatment of coaxial vortex pairs (i) moving towards one another ('opposite polarity') and (ii) moving in the same direction ('same polarity'), as part of a more general dynamical systems approach to vortex scattering.

In later papers by Meleshko, Konstantinov, Gurzhi, and Konovaljuk [25], Meleshko and Van Heijst [26], the interaction between equal strength vortex pairs, without a common axis, is considered with the aim of addressing advection and stirring of the fluid initially confined within the atmospheres of the vortex pairs.

Published work on this specific asymmetric problem before [17] is (as they state) 'rather incomplete' and a search indicates that later work concentrates on matters of equal pair strength stability and on chaotic advection in a field of vortices. In [17] there is specific reference to Acton [3], which, in developing a treatment of large current eddies, refers in passing to the interaction of line vortex pairs with a criterion for the existence of bound-state motions which Eckhardt and Aref [17] find to be incorrect.

Our aims in this paper are to develop the following new results:

(I) To extend in particular some results of Péntek, Tel, and Toroczkai [28] and Eckhardt and Aref [17] for the 'same polarity' vortex pair interaction with pairs of different strengths;

(II) To give explicit criteria detailing when two given such vortex pairs and initial separation experience leapfrogging or move off to infinite relative separation;

(III) To illustrate the relative motion of the vortex pairs for a range of positive ratios of vortex strengths.

We also highlight a particular case of interaction of a vortex pair with a single vortex of the same strength that is not fixed. This example is mentioned in Gröbli [18], although not illustrated there, and in Aref, Rott, and Thomann [8], with only a brief sketch. In contrast to the interaction of vortex pairs, where the integrity of the pairs is maintained throughout, this case involves an exchange of vortices with some similarity to a perfectly elastic collision.

In [21] Koshel, Reinaud, Riccardi, and Ryzhov study the more general problem of a pair of point vortices impinging on a fixed point vortex with arbitrary strengths.

We do not consider here any conditions of stability of the solutions presented, although we note that there is some computational consideration of this matter contained in Acheson [2]. The dynamical systems approach and stability issues are also pursued analytically in Berger [12], in Behring and Goodman [11], and in a numerical investigation in Whitchurch, Kevrekidis, and Koukouloyannis [33].

The paper is organised as follows. In section 2 we detail the translational motion of a single vortex pair ($N = 2$) with equal and opposite circulations. In section 3 we discuss the leapfrogging mechanism for two pairs of equal and opposite vortices, together with criteria for its occurrence. The analysis is to some extent a generalisation of that presented for the case of equal pair strengths in [28]. In section 4, we illustrate various specific cases of leapfrogging for a range of different relative strengths of vortex pairs, incidentally showing how the behaviours relate to their different widths, and initial separations. Section 5 details a summary and conclusion, indicating when two such vortex pairs can experience zero, one, or an infinite number of passes, as indicated in section 4. In appendix A, we give an explicit analysis for the very special case of the three vortex problem referred to above.

Nomenclature

x_r, y_r	relative horizontal and vertical coordinates of the vortex pairs
$\overline{x}_r, \overline{y}_r$	scaled relative coordinates
x_{r_0}, y_{r_0}	initial relative horizontal and vertical coordinates of the vortex pairs
$\overline{x}_{r_0}, \overline{y}_{r_0}$	scaled initial relative coordinates
x_0, y_0	centre-of-vorticity coordinates
$\overline{x}_{r_m}$	maximal scaled relative horizontal separation for leapfrogging to occur
K	vortex strength
H	Hamiltonian of the system
μ	ratio of vortex pair strengths, $\mu = K_2/K_1$
D	pair width, $D = 2a$, where a is the half-width of a vortex pair
d	initial separation between vortex pairs
$\dfrac{1}{C_{\text{CRIT}}}$	critical parameter separating leapfrogging and non-leapfrogging evolution
ℓ_{ij}	distance between vortices i and j

2 A vortex pair

Suppose that two vortices have strengths K and $-K$ and that they are a distance D apart. The vortex pair moves together in a straight line and the relative position of the two vortices is maintained. These two vortices sit in an infinite region of fluid and move uniformly with speed $u = K/(2\pi D)$. Figure 1 depicts a schematic diagram of this situation, where we think of the frame of reference in which the two vortices are fixed as being equivalent to introducing a fluid flow of speed $u = K/(2\pi D)$ in the **opposite** direction to which the vortices are translated. This keeps the vortices at rest in this translating frame [1, p. 177–178].

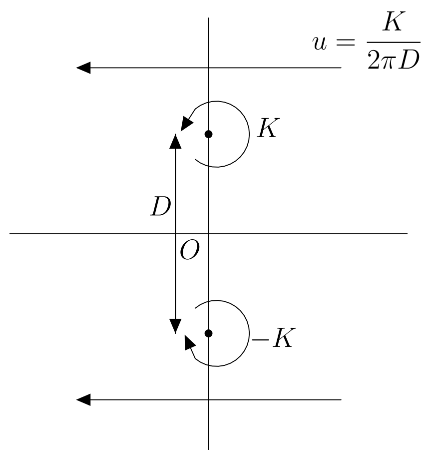

Figure 1: A vortex pair in a uniform flow. Here D denotes the distance between two vortices with opposite strength, K and $-K$, and u is the speed at which the vortices move.

The complex potential is given by $w(z) = \phi + i\psi$, where z is the complex number given by $x + iy$. Therefore, if we consider both the complex potential due to the fluid flowing through and due to the vortices, we obtain

$$\begin{aligned} w &= -\frac{iK}{2\pi} \ln \left| z - \frac{iD}{2} \right| - \frac{i(-K)}{2\pi} \ln \left| z + \frac{iD}{2} \right| - \frac{K}{2\pi D} z \\ &= -\frac{iK}{2\pi} \ln \left| x + i\left(y - \frac{D}{2} \right) \right| + \frac{iK}{2\pi} \ln \left| x + i\left(y + \frac{D}{2} \right) \right| - \frac{K}{2\pi D}(x + iy). \end{aligned} \quad (1)$$

We want to study the streamline topologies that arise from the two counter-rotating vortices that form the vortex pair. Streamlines are, by definition, lines with a constant value of the stream function, ψ. Thus, comparing the imaginary part of (1) gives the total stream function for the flow moving in a frame of reference with the vortex pair as follows:

$$\begin{aligned} \psi &= -\frac{K}{2\pi} \ln \left[x^2 + \left(y - \frac{D}{2} \right)^2 \right]^{1/2} + \frac{K}{2\pi} \ln \left[x^2 + \left(y + \frac{D}{2} \right)^2 \right]^{1/2} - \frac{K}{2\pi D} y \\ &= \frac{K}{4\pi} \ln \left[\frac{x^2 + (y + D/2)^2}{x^2 + (y - D/2)^2} \right] - \frac{K}{2\pi D} y. \end{aligned} \quad (2)$$

Then each streamline is defined through $\psi =$ constant. The streamlines produced from the interaction of the two counter-rotating vortices with equal absolute strength K are depicted in figure 2.

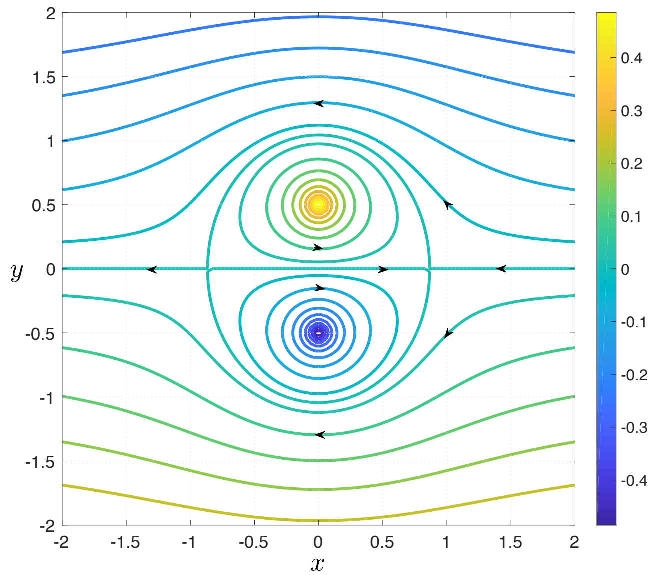

Figure 2: Flow of two vortices with equal and opposite circulations. The parameters used in this example are $K = \pm 1$. The color bar on the right has colors that correspond to values of ψ in (2).

Here we view a translating vortex pair in a frame that is moving with the two counter-rotating vortices. We note that a different choice for K would produce the same qualitative result, as long as the vortex strengths are equal and opposite. Now, it is useful to concentrate our attention on how to determine the separating streamline for the vortex pair. The result is a closed oval streamline surrounding the two vortices. The flow outside the oval is the same as it would be if the oval were replaced by a solid body, and the flow inside the oval is a circulation about the vortex lines that moves with the vortex.

First, the position of the relative stagnation points as shown in figure 2 (and schematically in figure 3) is found using the zero horizontal velocity component in this frame:

$$2\frac{Ka}{2\pi\left(a^2 + x_{S_1}^2\right)} - \frac{K}{2\pi(2a)} = 0, \tag{3}$$

where $a = D/2$ (i.e., half the distance between the two vortices). After a little algebra, we find that the relative stagnation points are

$$(x_{S_1}, y_{S_1}) = \left(\pm\sqrt{3}a, 0\right). \tag{4}$$

Finally, to determine the separating streamline, we evaluate the stream function (2) at (x_{S_1}, y_{S_1}), resulting in $\psi = 0$. Notice that the position of the two vortices and the stagnation points form an equilateral triangle, since $\tan\gamma = x_{S_1}/a = \sqrt{3}$, which implies that $\gamma = 60°$.

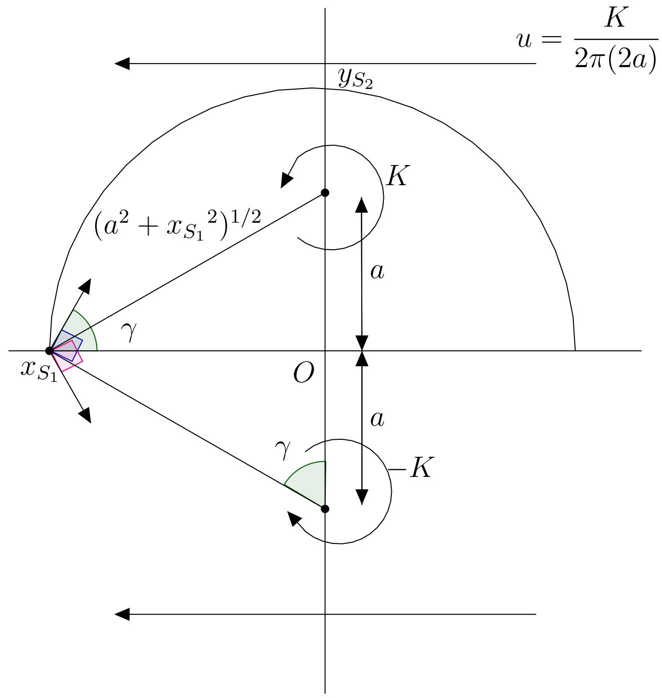

Figure 3: Schematic details of figure 2. Not drawn to scale.

Substituting $\psi = 0$ and $x = 0$ in (2) and solving for the position y_{S_2} yields

$$\frac{y_{S_2}}{a} = \coth\left(\frac{y_{S_2}}{4a}\right). \tag{5}$$

More precisely, the root of equation (5) is $y_{S_2} \approx 2.087a$. This is expected since $y_{S_2}/a = \coth\left(y_{S_2}/(4a)\right) > 1$, and we know that $y_{S_2} > a$. Note that in figure 2 we have $a = 1/2$, and therefore $(x_{S_1}, y_{S_1}) = \left(\sqrt{3}/2, 0\right)$ and $(x_{S_2}, y_{S_2}) \approx (0, 1.044)$.

We also remark that the doublet is at rest in the translational frame. To show this, we consider the complex potential function $w(z)$, and we calculate $\mathrm{d}w/\mathrm{d}z$, evaluating the derivative at $z = iD/2$ but not taking into account the contribution of the vortex at $z = iD/2$. Therefore,

$$\left.\frac{\mathrm{d}w(z)}{\mathrm{d}z}\right|_{z=iD/2} = \frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{iK}{2\pi}\ln\left|z + \frac{iD}{2}\right| - \frac{K}{2\pi D}z\right\}\Bigg|_{z=iD/2} = 0. \tag{6}$$

Since the translational velocity, $\underline{u}$, of the vortex is shown to be $\underline{u} = u + iv = \mathrm{d}w^{\star}/\mathrm{d}z = 0$, the vortex remains at rest. Similarly, repeating the above, we find that $\mathrm{d}w^{\star}/\mathrm{d}z = 0$ at $z = -iD/2$. Thus, we conclude that the relative positions of the vortices are maintained.

3 Leapfrogging of vortex pairs

Having studied the behaviour of a single vortex pair in the previous section, we investigate the possibility of two vortex pairs with a common axis to leapfrog. This was mentioned initially for vortex rings by Helmholtz in his paper on vortex dynamics [19]. We consider, here, a model of the so-called leapfrogging motion of two vortex pairs. In particular, we study the two-dimensional analogue of this process in an inviscid flow. In other words, the advection in the field of two pairs of ideal point vortices that have differing strengths, exhibiting leapfrogging motion.

In his paper [23], Love wrote that we are ignorant of the condition that determines whether the motion is periodic, and we can only guess the length of the period if the unknown condition is satisfied. In this paper, we provide a generalisation of the model first presented in [23] and then in [28], in each of which the vortex pairs are identical. In particular, we consider a special case of the four-vortex problem, still with zero overall circulation. The general case of four vortices is typically chaotic, but the special case with zero total circulation is integrable [9].

3.1 Governing equations

We consider a configuration of four vortices made up of two vortex pairs. The vortex pairs are arranged so that the x-axis is the perpendicular bisector of the line joining each vortex with the corresponding vortex having equal but opposite circulation [27, p. 89–90]. Instead of considering only the case where all the vortices have the same absolute strength, we let the strengths of the vortices be different (i.e., not necessarily equal in strength), but assume that K_1 and K_2 are both positive.

The leapfrogging motion of two vortex rings is modelled here using two vortex pairs. When corresponding vortices within the pairs have the same sense of rotation, so that K_1 and K_2 have the same sign, they travel in the same direction. Each vortex induces a transverse motion component in each of the other three vortices. The symmetry of the configuration is maintained thereby as the vortex pair widths and positions evolve. As we remarked in the Introduction there is here no consideration of the stability of this arrangement. The vortex pair that is behind will attempt to pass through the vortex pair in the front. The vertical separation between the point vortices of the pair that is originally in the front widens and the pair slows down. At the same time, the vertical separation between the point vortices of the pair that is initially at the back shrinks and the pair moves faster. It may then pass through the first one. This process may then be repeated exactly over and over again, and is conditional in that the vortex pair horizontal separation $|x_2 - x_1|$ can alternatively increase without limit.

In figure 4, we present a configuration of the system. It shows the motion of two vortex pairs with strengths K_1 and K_2 that move in the same direction along a common symmetry axis perpendicular to the extension of both pairs. In figure 4, the lower plane is a mirror image of the upper plane. This is the so-called *symmetry-reduced problem.* In the analysis below we take $K_1, K_2 > 0$ and $K_2/K_1 \leq 1$.

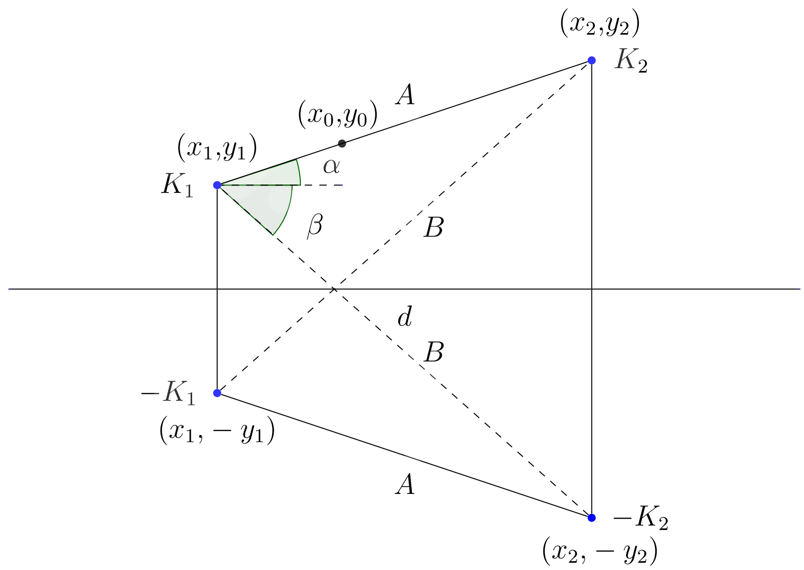

Figure 4: Configuration of the two vortex pairs.

The coordinates of the vortices at time t are $(x_1, y_1), (x_1, -y_1), (x_2, y_2)$, and $(x_2, -y_2)$. The two with suffix 1 form a pair and the two with suffix 2 form another pair. We assume initially that $x_2 > x_1$ and $y_2 > y_1$. Thus the wider pair is ahead of the narrower one. However, the case where the wider pair is behind the narrower pair is equivalent and gives rise to the same result.

Now, let us introduce the relative coordinates

$$x_r = x_2 - x_1, \quad y_r = y_2 - y_1, \tag{7}$$

as well as the centre-of-vorticity coordinates in the upper half plane $y \geq 0$, given by

$$x_0 = \frac{K_1 x_1 + K_2 x_2}{K_1 + K_2}, \quad y_0 = \frac{K_1 y_1 + K_2 y_2}{K_1 + K_2}. \tag{8}$$

We introduce $A = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]^{1/2}$ and $B = \left[(x_2 - x_1)^2 + (y_2 + y_1)^2\right]^{1/2}$, to simplify the notation. It is clear from figure 4 that we have:

$$\cos\alpha = \frac{x_2 - x_1}{A}, \sin\alpha = \frac{y_2 - y_1}{A},$$
$$\cos\beta = \frac{x_2 - x_1}{B}, \sin\beta = \frac{y_2 + y_1}{B}. \tag{9}$$

If we consider the vortex with strength K_1, then we find that the velocities induced by the other three vortices are

$$\dot{x}_1 = \frac{K_1}{2\pi(2y_1)} + \frac{K_2 \sin\alpha}{2\pi A} + \frac{K_2 \sin\beta}{2\pi B}, \tag{10}$$

$$\dot{y}_1 = \qquad - \frac{K_2 \cos\alpha}{2\pi A} + \frac{K_2 \cos\beta}{2\pi B}, \tag{11}$$

where the overdot represents differentiation with respect to time t. Now, let us consider the vortex with strength K_2. The velocity components of this vortex are

$$\dot{x}_2 = \frac{K_2}{2\pi(2y_2)} - \frac{K_1 \sin\alpha}{2\pi A} + \frac{K_1 \sin\beta}{2\pi B}, \tag{12}$$

$$\dot{y}_2 = \frac{K_1 \cos\alpha}{2\pi A} - \frac{K_1 \cos\beta}{2\pi B}. \tag{13}$$

After substituting the equations for A and B and equations (9), the preceding equations become

$$\dot{x}_1 = \frac{K_1}{4\pi y_1} + \frac{K_2(y_2 - y_1)}{2\pi\left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]} + \frac{K_2(y_2 + y_1)}{2\pi\left[(x_2 - x_1)^2 + (y_2 + y_1)^2\right]}, \tag{14}$$

$$\dot{y}_1 = -\frac{K_2(x_2 - x_1)}{2\pi\left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]} + \frac{K_2(x_2 - x_1)}{2\pi\left[(x_2 - x_1)^2 + (y_2 + y_1)^2\right]}, \tag{15}$$

and similarly,

$$\dot{x}_2 = \frac{K_2}{4\pi y_2} + \frac{K_1(y_1 - y_2)}{2\pi\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]} + \frac{K_1(y_1 + y_2)}{2\pi\left[(x_1 - x_2)^2 + (y_1 + y_2)^2\right]}, \tag{16}$$

$$\dot{y}_2 = -\frac{K_1(x_1 - x_2)}{2\pi\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]} + \frac{K_1(x_1 - x_2)}{2\pi\left[(x_1 - x_2)^2 + (y_1 + y_2)^2\right]}. \tag{17}$$

Note that $K_1\dot{y}_1 + K_2\dot{y}_2 = 0$ implies that $\dot{y}_0 = 0$, with y_0 as defined in (8).

The equations for $(\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2)$ given by (14)–(17) actually become Hamiltonian in structure if we take as canonical conjugate coordinate/momentum pairs:

$$(K_1 x_1, y_1) \equiv (q_1, p_1) \text{ and } (K_2 x_2, y_2) \equiv (q_2, p_2).$$

The Hamiltonian of the system in terms of x_1, x_2, y_1, and y_2 becomes

$$H = \frac{1}{4\pi} \ln\left((2y_1)^{K_1^2}(2y_2)^{K_2^2}\left[\frac{(x_2 - x_1)^2 + (y_2 + y_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]^{K_1 K_2}\right) =: E. \tag{18}$$

This is an autonomous system and H is conserved. Using (7) and (18), we find that

$$\left(\frac{x_r^2 + y_r^2 + 4y_1 y_2}{x_r^2 + y_r^2}\right)^{K_1 K_2} y_1^{K_1^2} y_2^{K_2^2} \equiv \text{constant}. \tag{19}$$

To arrive at the explicit form of the trajectories in relative coordinates, we first need to write the reciprocal of (19) and use the properties that follow. At this point, note that the centre-of-vorticity coordinate x_0 is *ignorable* since it does not appear in the Hamiltonian. Hamilton's equation gives $\dot{y}_0 = -\partial H/\partial x_0 = 0$, and therefore, the centre-of-vorticity coordinate y_0 is a conserved quantity as can be found explicitly above via (15), (17) and it appears as a constant in the Hamiltonian function

$$2y_0 = 2\frac{K_1 y_1 + K_2 y_2}{K_1 + K_2} = \text{constant}, \tag{20}$$

where $2y_0$ can be considered as the mean width of the vortex pairs. We can scale the problem according to

$$(\overline{x}_r, \overline{y}_r) \equiv \frac{1}{2y_0}(x_r, y_r) \quad \text{and} \quad (\overline{x}_i, \overline{y}_i) \equiv \frac{1}{2y_0}(x_i, y_i) \quad \text{for } i = 1, 2. \tag{21}$$

Also note that y_1 and y_2 can be written in terms of the centre-of-vorticity and relative coordinates as

$$y_1 = y_0 - \frac{K_2}{K_1+K_2} y_r \text{ and } y_2 = y_0 + \frac{K_1}{K_1+K_2} y_r. \tag{22}$$

The energy conservation (18) provides us with the explicit form of the trajectories in the relative coordinates as

$$(1-2k_2\overline{y}_r)^{-k_1/k_2}(1+2k_1\overline{y}_r)^{-k_2/k_1}\left[1-\frac{(1-2k_2\overline{y}_r)(1+2k_1\overline{y}_r)}{\left(\overline{x}_r^2+[1+(k_1-k_2)\overline{y}_r]^2\right)}\right] = C, \tag{23}$$

where we define $k_1 = K_1/(K_1+K_2)$, and similarly, $k_2 = K_2/(K_1+K_2)$.

As a check for our result, we may use $k_1 = k_2 = 1/2$, equivalent to $K_1 = K_2$ in the general form (23), which corresponds to

$$\frac{1}{1-\overline{y}_r^2} - \frac{1}{1+\overline{x}_r^2} = C,$$

the more specific form of the trajectories obtained in [28, p. 2194, Eq. 9]. Note that, C in our case and E in [28] are related through

$$C = \exp\left(-\frac{4\pi E}{K_1K_2} + \ln(2)\left(\frac{K_1}{K_2}+\frac{K_2}{K_1}\right)\right). \tag{24}$$

Furthermore, if we let $\overline{y}_r = 0$ then $\overline{x}_r^2/(\overline{x}_r^2+1) = C$, where C is a constant. Therefore, for real $\overline{x}_r$ we must have that $0 \le C \le 1$. We can draw the phase plane $(\overline{x}_r, \overline{y}_r)$ given the range of C values (see figures 8–11 in section 4). We note that our equations (14)–(17) are in accord with those displayed in the paper by Acton [3] which we cited in our Introduction section 1.

3.2 Leapfrogging criterion

Since vortex pairs do not always leapfrog, we need a criterion that determines when leapfrogging of vortex pairs occurs. The key point is to find the bounds on C that ensure the leapfrogging of vortex pairs. In other words, we must find C_{CRIT}. It is clear from (23) and the special case following, that $C = 1/C = 1$ is the bound for $\mu = K_2/K_1 = 1$, as in [28]. The interesting question to ask concerns how this bound changes with μ, when $\mu \in (0, 1]$. Therefore, given the vortex strengths K_1, K_2 and the initial positions of (x_1, y_1) and (x_2, y_2), we need to find C_{CRIT}. In figure 5, we illustrate the initial configuration of two vortex pairs.

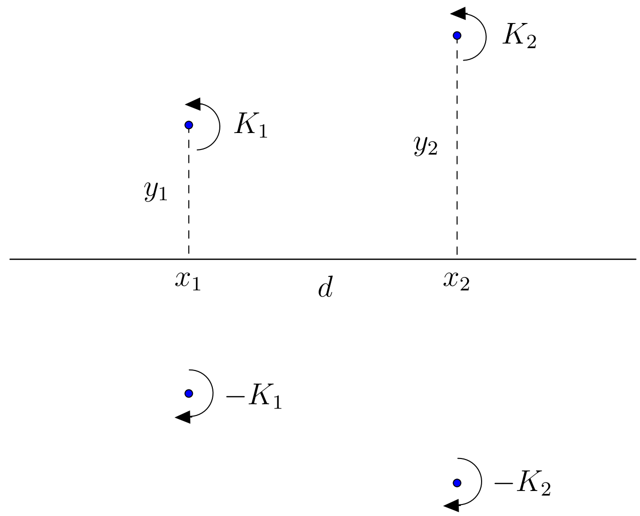

Figure 5: Two vortex pairs with different strengths K_1 and K_2. Here d denotes the horizontal distance between the two vortex pairs, and $(x_1, \pm y_1)$, $(x_2, \pm y_2)$ correspond to the vortex locations for pair 1 and 2, respectively.

The initial values $x_{r_0} = d$ and $y_{r_0} = y_2 - y_1$ are scaled by y_0 in (8) to give

$$\overline{x}_{r_0} = \frac{d(K_1 + K_2)}{2(K_1 y_1 + K_2 y_2)}, \ \overline{y}_{r_0} = \frac{(y_2 - y_1)(K_1 + K_2)}{2(K_1 y_1 + K_2 y_2)}. \tag{25}$$

When $\overline{y}_r$ has initial value $\overline{y}_{r_0}$ we can calculate the maximal scaled relative horizontal separation $\overline{x}_{r_m}$ for leapfrogging to occur as:

$$\overline{x}_{r_m}^2 = \frac{\left(1 - \frac{2\mu}{1+\mu}\overline{y}_{r_0}\right)\left(1 + \frac{2}{1+\mu}\overline{y}_{r_0}\right)}{\left[1 - C_{\text{CRIT}}\left(1 - \frac{2\mu}{1+\mu}\overline{y}_{r_0}\right)^{1/\mu}\left(1 + \frac{2}{1+\mu}\overline{y}_{r_0}\right)^{\mu}\right]} - \left(1 + \left(\frac{1-\mu}{1+\mu}\right)\overline{y}_{r_0}\right)^2. \tag{26}$$

After y_1, y_2 are scaled suitably, we obtain

$$\overline{y}_1 = \frac{1}{2} - \frac{K_2}{K_1 + K_2}\overline{y}_r, \quad \overline{y}_2 = \frac{1}{2} + \frac{K_1}{K_1 + K_2}\overline{y}_r. \tag{27}$$

We need y_1, y_2 and hence $\overline{y}_1$ and $\overline{y}_2$ to be non-negative, and we also know that $\mu > 0$ (recall the assumption that $K_1, K_2 > 0$). We find that there is an upper and a lower bound for $\overline{y}_r$, given by

$$\overline{y}_r \leq \left(\frac{1+\mu}{2\mu}\right) \text{ for } \overline{y}_1 \geq 0, \tag{28}$$

and similarly,

$$\overline{y}_r \geq -\left(\frac{1+\mu}{2}\right) \text{ for } \overline{y}_2 \geq 0. \tag{29}$$

Note that the existence of the upper and lower bound for $\overline{y}_r$ agrees with figures 8–11 in section 4. The $\overline{y}_r$ bounds are natural in that either $\overline{y}_1 \to 0$, or $\overline{y}_2 \to 0$ there. These conditions can be associated with unlimited speed of the respective vortex pair, as one or the other vortex pair approaches zero width about the x-axis.

In order to determine C_{CRIT} we must investigate what happens to the curves $C =$ constant as $\overline{x}_r \to \pm\infty$. Evidently from (23), at $\overline{x}_r \to \pm\infty$ we have

$$F \equiv \left(1 - \frac{2\mu}{1+\mu}\overline{y}_r\right)^{1/\mu} \left(1 + \frac{2}{1+\mu}\overline{y}_r\right)^{\mu} = \frac{1}{C}. \tag{30}$$

Considering $\mu \in (0, 1]$, we concentrate our attention at F when $-(1+\mu)/2 \le \overline{y}_r \le (1+\mu)/(2\mu)$. The next step is to find the maximum of this function, and so we find $\overline{y}_r$ for which $\mathrm{d}F/\mathrm{d}\overline{y}_r = 0$. We obtain then that

$$-\frac{2}{1+\mu}\left(1 - \frac{2\mu}{1+\mu}\overline{y}_r\right)^{-1} + \frac{2\mu}{1+\mu}\left(1 + \frac{2}{1+\mu}\overline{y}_r\right)^{-1} = 0,$$

which yields the separating $\overline{y}_r$ as $\overline{y}_r = -(1-\mu^2)/[2(1+\mu^2)] \le 0$, and thus (30) becomes

$$F(\overline{y}_r;\mu) = \left(\frac{1+\mu}{1+\mu^2}\right)^{1/\mu} \left(\frac{\mu(1+\mu)}{1+\mu^2}\right)^{\mu} \equiv \mu^{\mu}\left(\frac{1+\mu}{1+\mu^2}\right)^{\mu+1/\mu} =: F_{\max}. \tag{31}$$

Note that this is actually a maximum since the derivative of $F(\overline{y}_r;\mu)$ at $\overline{y}_r = 0$ satisfies $\mathrm{d}F(0;\mu)/\mathrm{d}\overline{y}_r = -2(1-\mu)/(1+\mu) \le 0$. A diagram of $F(\overline{y}_r;\mu)$ as given by (30) is shown in figure 6 for $\mu = 0.4, 0.7, 1$.

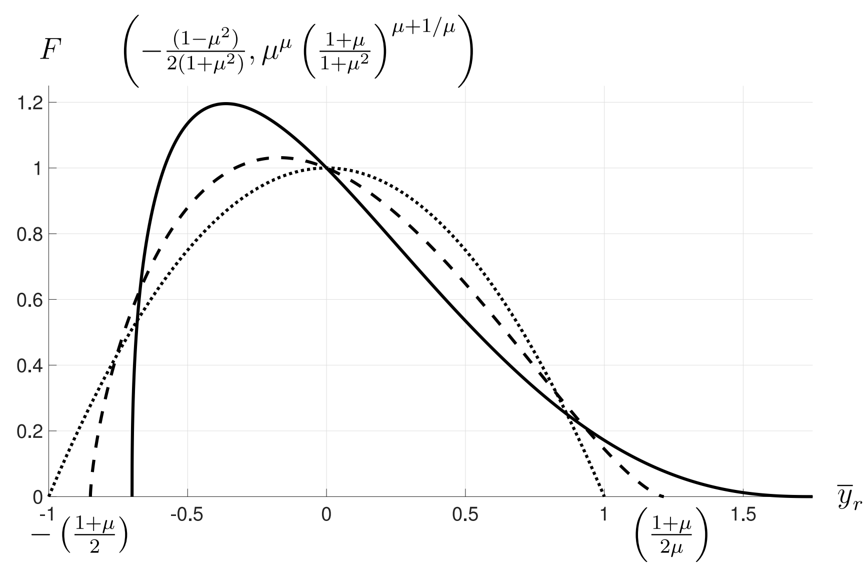

Figure 6: F vs. $\overline{y}_r$ graph for $\mu = 0.4$ (solid line), $\mu = 0.7$ (dashed line) and $\mu = 1$ (dotted line). Note that particular values are given for $\mu \in (0, 1]$ in Table 1 in section 4.

The three possible cases to be considered are given below:

- $\dfrac{1}{C} < F_{\max}$ that gives two distinct $\overline{y}_r$ values,

- $\dfrac{1}{C} = F_{\max}$ that gives two equal $\overline{y}_r$ values,
- $\dfrac{1}{C} > F_{\max}$ that gives no $\overline{y}_r$ values.

The curves C = constant are evidently closed, and with $\overline{x}_r$ bounded in (23), only when $1/C > F_{\max}$. Hence, leapfrogging occurs only in this last case, when

$$\frac{1}{C} \geq \mu^{\mu} \left(\frac{1+\mu}{1+\mu^2} \right)^{\mu+1/\mu} = \frac{1}{C_{\mathrm{CRIT}}}, \tag{32}$$

or, equivalently, when

$$C \leq C_{\mathrm{CRIT}} \equiv \left(\frac{K_1^2 + K_2^2}{K_1 + K_2} \right)^{\frac{K_2}{K_1} + \frac{K_1}{K_2}} K_1^{-\frac{K_1}{K_2}} K_2^{-\frac{K_2}{K_1}}. \tag{33}$$

In the case of equality, the period of the leapfrogging is infinite.

We claim that there is an upper bound for the distance, d, between two given vortex pairs for which leapfrogging occurs. Given $\overline{y}_{r_0}$, the $\overline{x}_{r_m}$ for leapfrogging must satisfy $\overline{x}_{r_0}^2 < \overline{x}_{r_m}^2$. That is, there exists a d such that

$$d^2 < \frac{4y_1y_2}{1 - \left(\dfrac{K_1^2 + K_2^2}{K_1y_1 + K_2y_2} \right)^{\frac{K_1}{K_2} + \frac{K_2}{K_1}} \left(\dfrac{y_1}{K_1} \right)^{\frac{K_1}{K_2}} \left(\dfrac{y_2}{K_2} \right)^{\frac{K_2}{K_1}}} - (y_1 + y_2)^2. \tag{34}$$

By putting $d = 0$ in (34) we may compute the critical size ratio at collinearity of all four vortices at $x = 0$, in agreement with figure B5 of [17].

We note that each vortex pair alone would translate with velocity $K_1/(4\pi y_1)$, $K_2/(4\pi y_2)$, respectively. Intuitively, (34) means that if the vortices start at a distance greater than d apart, then no leapfrogging takes place and the separation of vortices increases to infinity with or without a pass by.

Bounded trajectories (i.e., closed curves) are present for $C \leq C_{\mathrm{CRIT}}$ and there is a symmetry about the $\overline{y}_r$-axis. The C_{CRIT} curves correspond to the separatrix lying on the boundary between regions of open and closed trajectories. (See figures 8–11 in section 4.) The closed trajectories correspond to the leapfrogging motion of the vortex pairs.

For each value of C there exist four states with the same $|\overline{x}_r|$. Investigating the case $\mu \in (0, 1]$, we choose for instance $\mu = 0.7$ as illustrated in figure 7.

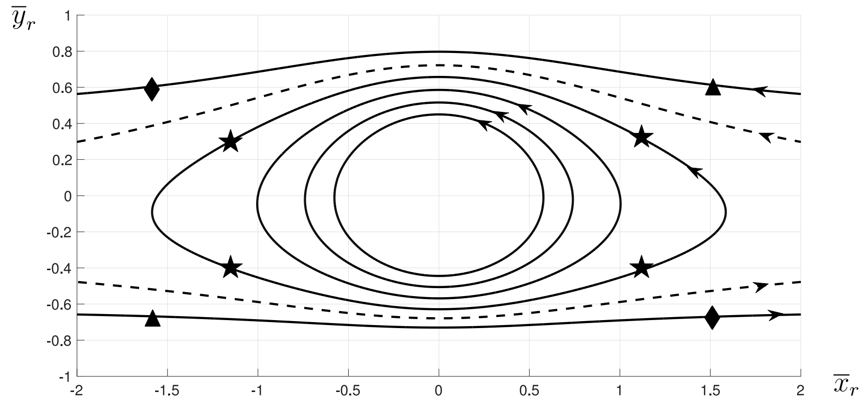

Figure 7: Vortex trajectories in $(\overline{x}_r, \overline{y}_r)$ for different C curves with $\mu = 0.7$ and lower bound limit $\overline{y}_r = -(1+\mu)/2$. Note that the motion is counterclockwise. There is successive leapfrogging when $C < C_{\mathrm{CRIT}}$ (★) and the period goes to infinity as C tends to C_{CRIT}^{-}. The dashed line represents the separatrix, and that corresponds to $C = C_{\mathrm{CRIT}}$. When $C > C_{\mathrm{CRIT}}$ there is no leapfrogging and $|\overline{x}_r| \to \infty$; for (▲) there is one pass ($|\overline{x}_r| = 0$); for (♦) there is no such pass.

Different types of motion are possible depending on the value of C. We may distinguish some cases for different values of μ — see section 4.

Note that for the case $\mu = 1$, the motions are symmetric about the $\overline{x}_r$-axis as well. It is straightforward to find in the $\mu = 1$ case that the separatrix is $E = 0$, or equivalently, that $C_{\mathrm{CRIT}} = 1$. This is true as, from (30), $1 - C(1 - \overline{y}_r^2) = 0$, which implies that $\overline{y}_r^2 = 1 - 1/C \to C_{\mathrm{CRIT}} = 1$.

Tophøj and Aref in [32] have studied the case where two coaxial vortex pairs leapfrog one another, when the pairs have the same absolute circulation. Both Gröbli and Love discovered that leapfrogging was possible only if the size ratio of the two pairs at the moment one slips through the other is not too large. In particular, Love [23] said "... the motion is periodic, if at the instant when one pair passes through the other, the ratio of the breadths of the pairs is less than $3+2\sqrt{2}$. When the ratio has this precise value the smaller pair shoots ahead of the larger and widens, while the larger contracts, so that each is ultimately of the same breadth..., and the distance between them is ultimately infinite. When the ratio in question is greater than $3+2\sqrt{2}$, the smaller shoots ahead and widens, and the latter falls behind and contracts... When the ratio is less than $3+2\sqrt{2}$, the motion is similar to the motion described by Helmholtz for two rings on the same axis, and it is probable that there is for this case also a critical condition in which the rings, after one has passed through the other, ultimately separate to an infinite distance, and attain equal diameters." We note that $3+2\sqrt{2} \approx 5.828$ and that its reciprocal is $(3+2\sqrt{2})^{-1} = 3 - 2\sqrt{2} \approx 0.172$. Here $3 - 2\sqrt{2} = \sigma^2$ where $\sigma = \sqrt{2} - 1$ is called the *silver ratio*. In our case we have a direct counterpart of this, in that our σ is $\left[\sqrt{F_{\max}+1} - \sqrt{F_{\max}}\right]$, where $C_{\min} = 1/F_{\max}$. We observe again that for the special case where the vortex pairs have the same absolute strength, we have $F_{\max} = 1$, and so we recover the silver ratio $\sigma = \sqrt{2} - 1$.

The centre of the closed leapfrogging curves in the gallery in section 4 is at $(0, 0)$ with $C \to 0$. Around this point, we do have in each case leapfrogging of two pairs that almost coincide even though they have different K_1 and K_2 strengths. Naturally this happens rapidly.

The criterion developed explicitly in (33) and (34) appears to be in general accord with the results obtained by a different analysis and given in Eckhardt and Aref appendix B [17].

4 A gallery of vortex leapfrogging

Here, we show the evolution of the leapfrogging motion for a small number of increasing μ values. Similar plots can be produced for any $\mu \in (0, 1]$. Different types of motion arise from different initial conditions.

In figures 8–11 the dashed curves represent the critical curve, C_{CRIT}. In other words, a dashed curve represents the separatrix between the open and closed trajectories. All the closed curves inside this dashed curve correspond to leapfrogging motion. In figures 8–11, the closed trajectories evolve in the anticlockwise sense, and these cases complement that for $\mu = 0.7$ in figure 7.

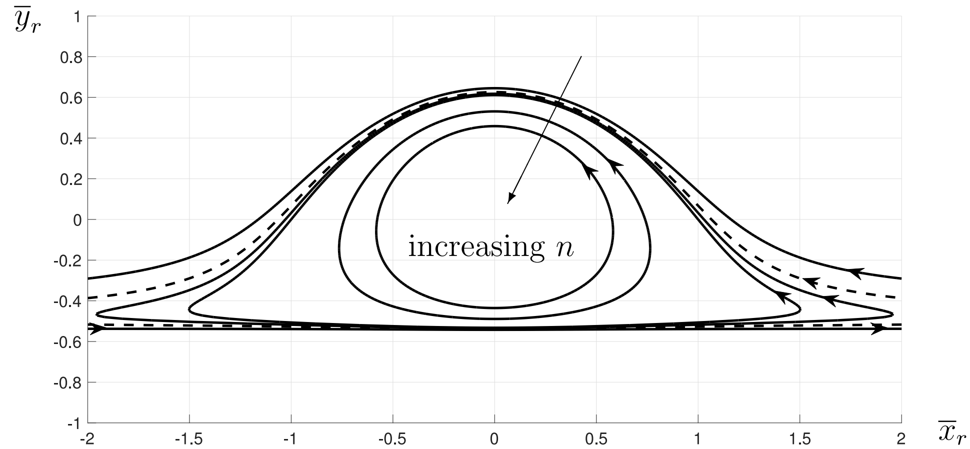

Figure 8: Vortex trajectories in $(\overline{x}_r, \overline{y}_r)$ for different C curves with $\mu = 0.1$ and lower bound limit $\overline{y}_r = -(1 + \mu)/2 = -0.55$. Note that the motion is counterclockwise. Here, $C = 1/2^n$, with $n = 0.8, 0.911599, 0.95, 1, 1.5, 2$.

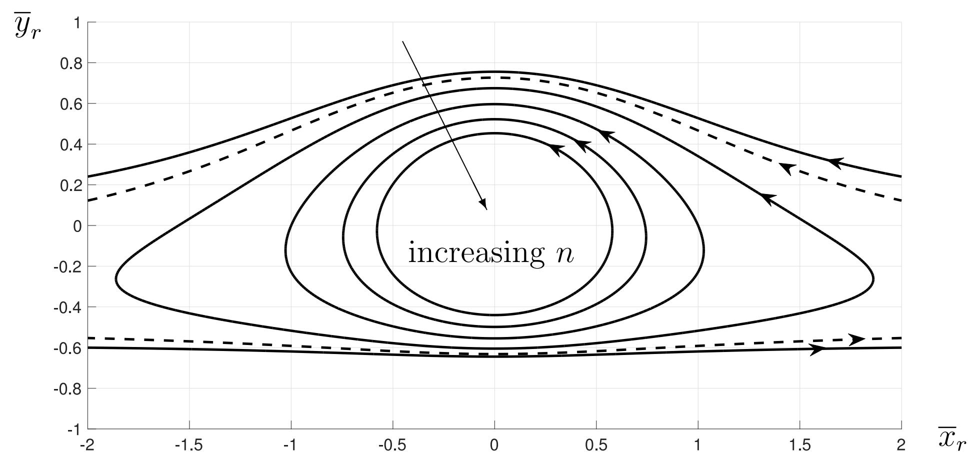

Figure 9: Vortex trajectories in $(\overline{x}_r, \overline{y}_r)$ for $\mu = 0.3$ and lower bound limit $\overline{y}_r = -0.65$. The different C curves are given by $C = 1/2^n$, with $n = 0, 0.402438, 0.5, 1, 1.5, 2$.

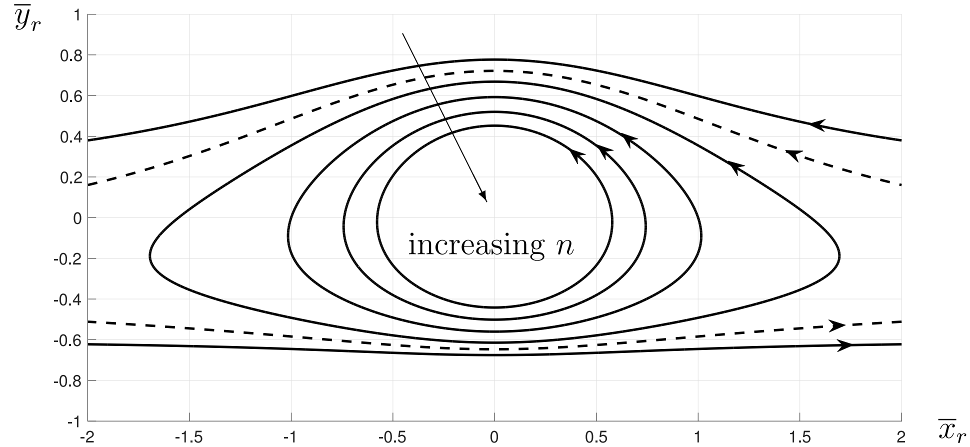

Figure 10: Vortex trajectories in $(\overline{x}_r, \overline{y}_r)$ for $\mu = 0.5$ and lower bound limit $\overline{y}_r = -0.75$. The different C curves are given by $C = 1/2^n$, with $n = 0, 0.157587, 0.5, 1, 1.5, 2$.

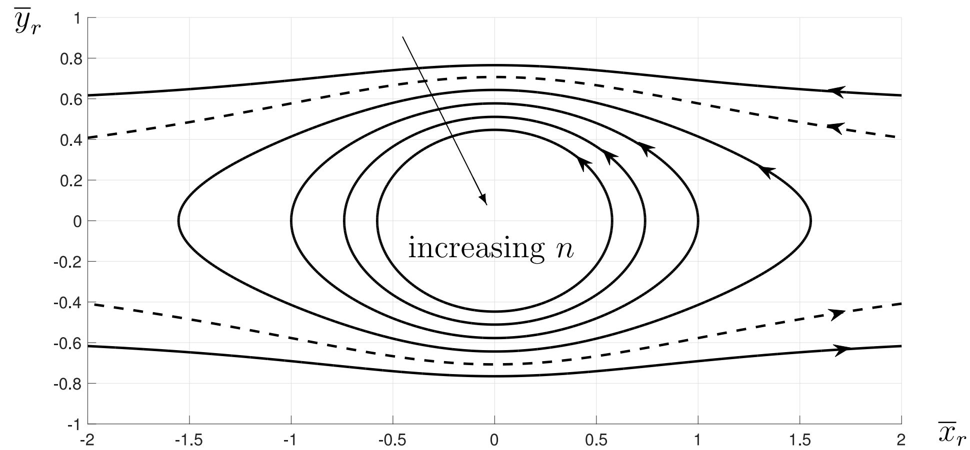

Figure 11: Vortex trajectories in $(\overline{x}_r, \overline{y}_r)$ for $\mu = 1$ and lower bound limit $\overline{y}_r = -1$. The different C curves are given by $C = 1/2^n$, with $n = -0.5, 0, 0.5, 1, 1.5, 2$. (This is the counterpart of figure 2 in [28].)

Figures 8–11 illustrate the relative motion of the vortex pairs leading to leapfrogging or overtaking motions. In figure 12 we plot C_{CRIT} as a function of vortex strength ratio μ and some specific values are given in Table 1.

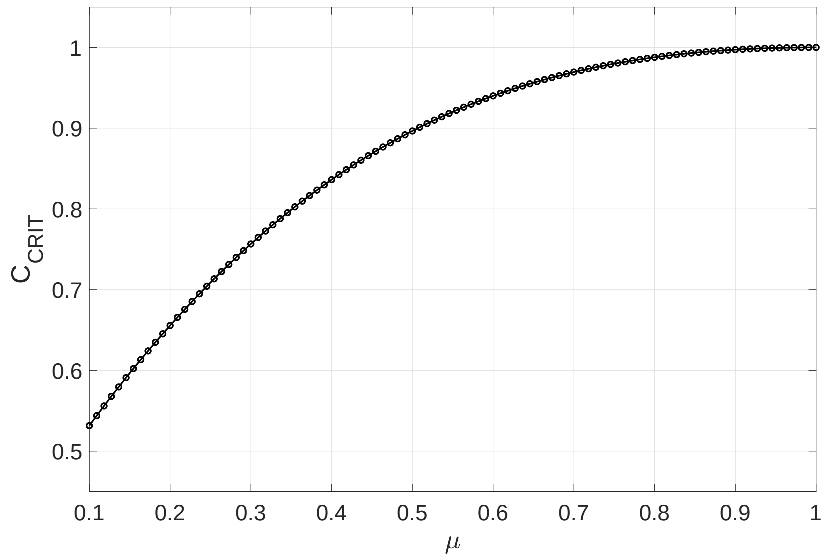

Figure 12: For a prescribed ratio of vortex strengths $\mu = K_2/K_1$, we plot the corresponding $C_{\mathrm{CRIT}} = (1/\mu^{\mu})[(1+\mu^2)/(1+\mu)]^{\mu+1/\mu}$ value.

$\mu = K_2/K_1$	$C_{\mathrm{CRIT}} = \frac{1}{\mu^{\mu}}\left(\frac{1+\mu^2}{1+\mu}\right)^{\mu+1/\mu}$	$1/C_{\mathrm{CRIT}}$	separating $\overline{y}_r = -\frac{(1-\mu^2)}{2(1+\mu^2)}$
0.1	0.53159	1.88114	−0.490099
0.2	0.65558	1.52537	−0.461538
0.3	0.75658	1.32174	−0.417431
0.4	0.83624	1.19582	−0.362069
0.5	0.89652	1.11542	−0.3
0.6	0.93999	1.06384	−0.235294
0.7	0.96949	1.03147	−0.171141
0.8	0.98776	1.01240	−0.109756
0.9	0.99724	1.00277	−0.052486
1.0	1.0	1.0	0.0

Table 1: Numerical results for the various μ values.

We may also look at the consequential motions of the vortex pairs in the (x, y) plane, by integrating our governing equations (14)–(17) using a fourth-order Runge-Kutta scheme.

Here we choose to illustrate as an example the case $K_1 = 2$, $K_2 = 1$, so that $\mu = 0.5$, for comparison with figure 10. Other cases may of course be treated similarly.

(a)

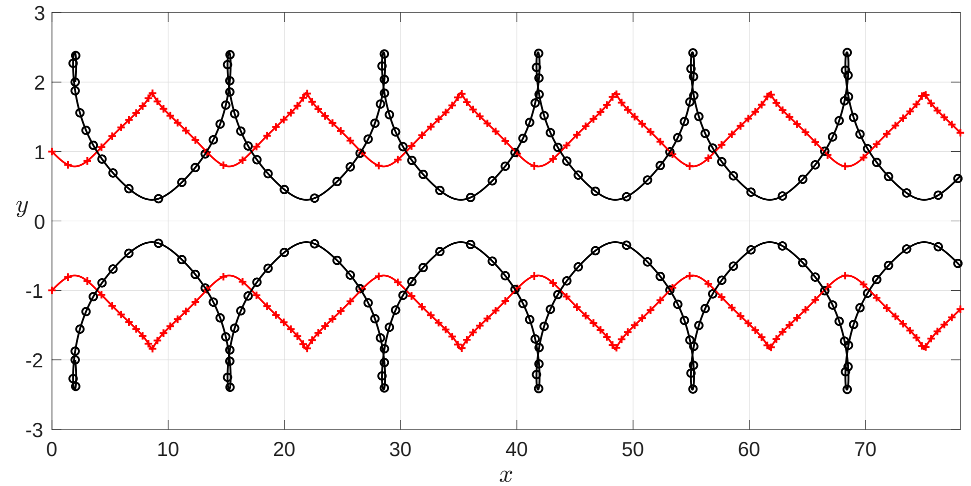

Figure 13: Vortices $+K_1$, $-K_1$ initially at $(0,1)$, $(0,-1)$ respectively [red line with crosses], and $+K_2$, $-K_2$ initially at $(2,2)$, $(2,-2)$ respectively [black line with circles], leading to a periodic (leapfrogging) motion.

The corresponding temporal plot in figure 14 shows the moment when the passing occurs.

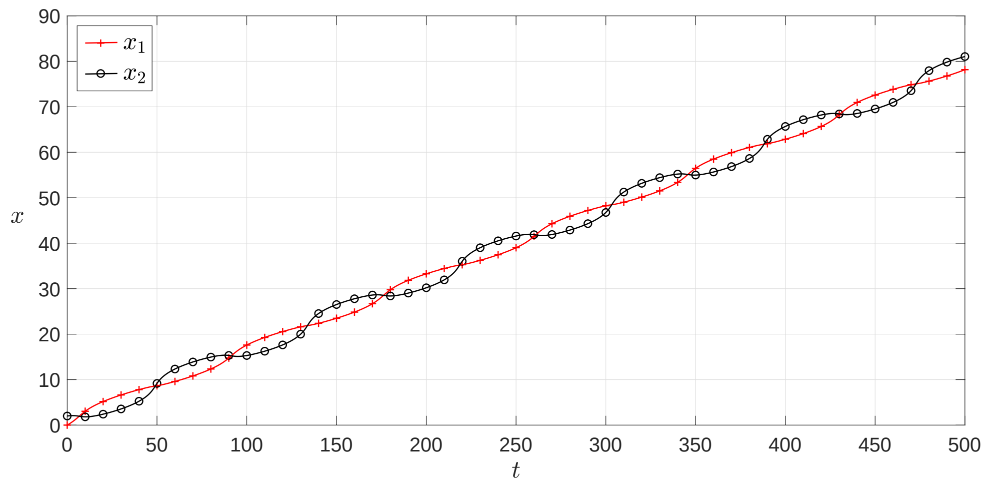

Figure 14: The leapfrogging motion on the t-x plane. The red line with crosses shows the time evolution of the x-coordinate for the vortex with strength $+K_1$, initially at $x=0$, and the black line with circles shows the time evolution of the x-coordinate for the vortex with strength $+K_2$, initially at $x=2$. The points of intersection of the x_1 and x_2 lines show the moment in time when the passing occurs.

(b)

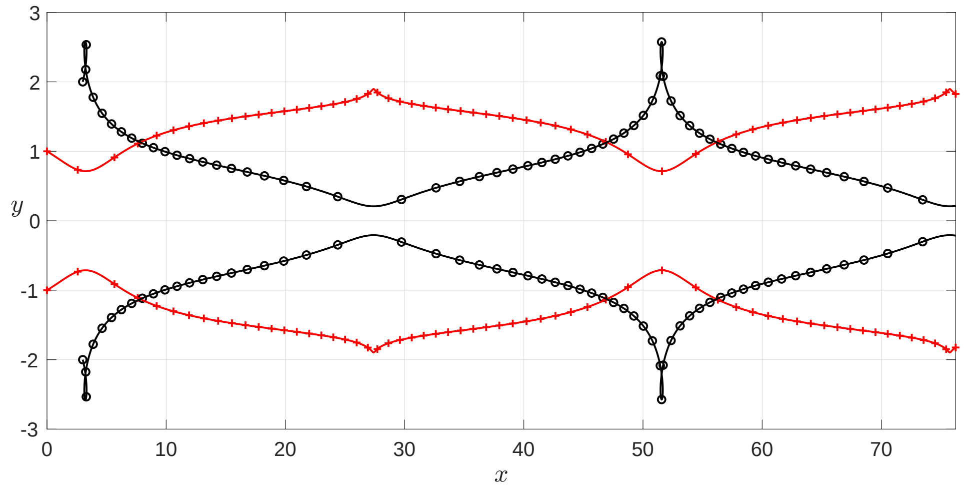

Figure 15: Vortices $+K_1$, $-K_1$ initially at $(0, 1)$, $(0, -1)$ respectively [red line with crosses], and $+K_2$, $-K_2$ initially at $(3, 2)$, $(3, -2)$ respectively [black line with circles]. The greater initial horizontal separation leads to a periodic (leapfrogging) motion with a longer period than in the case shown in (a) above.

(c) Our criterion for leapfrogging (34) gives the critical initial separation d to be 3.459. So we have:

(i) for separation just less than the critical value — $d = 3.43$ in figure 16 — leading to leapfrogging;

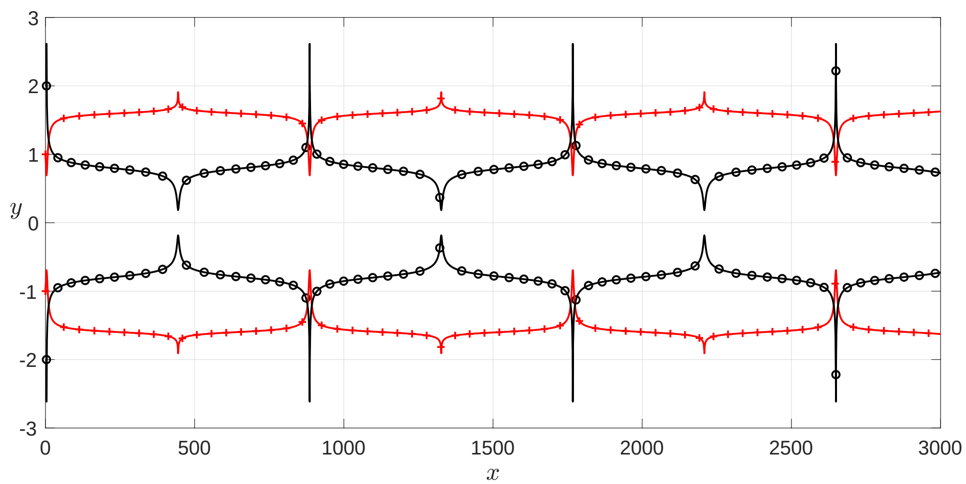

Figure 16: Vortices $+K_1$, $-K_1$ initially at $(0, 1)$, $(0, -1)$ respectively [red line with crosses], and $+K_2$, $-K_2$ initially at $(3.43, 2)$, $(3.43, -2)$ respectively [black line with circles]. The initial separation is just below the critical value, so there is periodic (leapfrogging) motion, but the period is now very large.

(ii) for separation just greater than the critical value — $d = 3.46$ in figures 17, 18 — leading to separation.

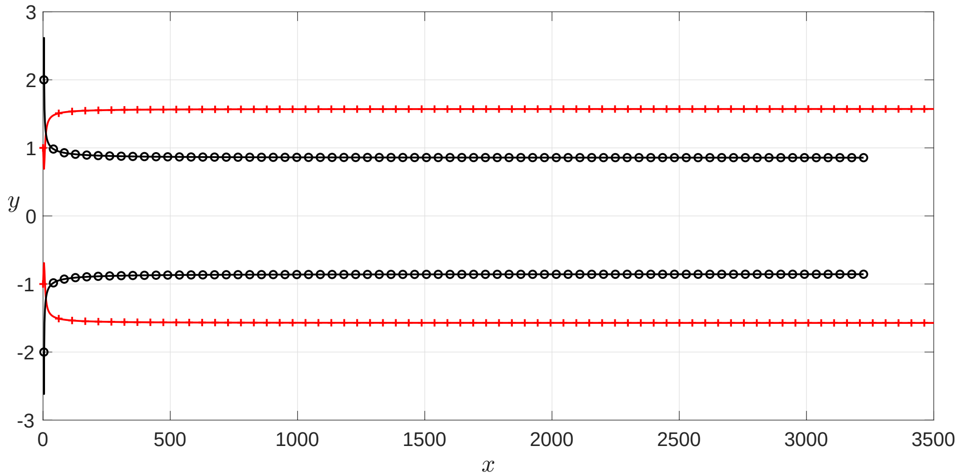

Figure 17: Vortices $+K_1$, $-K_1$ initially at $(0, 1)$, $(0, -1)$ respectively [red line with crosses], and $+K_2$, $-K_2$ initially at $(3.46, 2)$, $(3.46, -2)$ respectively [black line with circles]. The initial separation is now just above the critical value; there is now no periodic (leapfrogging) motion, but there is a single overtake by the first vortex pair (which then expands in width) as the second vortex pair then falls further and further behind (decreasing in width). The widths of the pairs asymptote to constant values as their separation $\to \infty$.

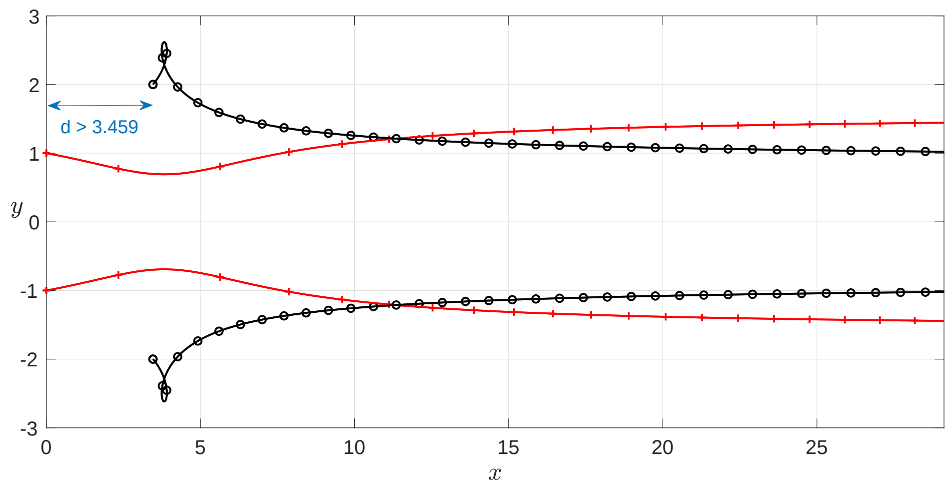

Figure 18: Zoomed-in version of figure 17 that focuses on the single overtake by the first vortex pair as the second pair falls further behind.

5 Summary and conclusion

In sections 3 and 4 we have considered the motion of two line vortex pairs, with a common axis, but with, in general, different strengths K_1 and K_2. From their relative strengths we derived a criterion (34) for leapfrogging to occur. The gallery of pictures in section 4 and the qualitative sketch (figure 7) lead to a set of criteria that we observe when we are given strengths, widths, and separation, i.e., K_1, K_2, y_1, y_2, d.

Then, with $\overline{x}_{r_0}$, $\overline{y}_{r_0}$ given by (25) with the choice $K_2 \leq K_1$ we may distinguish some cases of the motion of two vortex pairs and briefly describe each of them:

(a) *Leapfrogging.*

As described in section 3 this type of motion exists when the criterion in subsection 3.2 is satisfied. Recall that leapfrogging motion corresponds to a periodic motion in which the vortex pairs alternately pass through each other. More specifically, for the case $0 < \mu = K_1/K_2 \leq 1$, leapfrogging occurs when $1/C \geq F_{\max}$, and there are no solutions for $\overline{y}_r$ at $|\overline{x}_r| \to \infty$. Leapfrogging occurs anticlockwise for the K_2 pair.

(b) *Single overtake of one vortex pair by the other.*

This type of motion occurs in two instances. Referring to figures 8–11, consider the case in which initially $\overline{x}_r > 0$ and $\overline{y}_r > 0$. Vortex pair 2 (as labelled in figure 4) is ahead of vortex pair 1 and it is wider. It will pass over vortex pair 1 and then go relatively to negative infinity. An equivalent kind of motion occurs when the initial configuration of the vortex pairs satisfies $\overline{x}_r < 0$ and $\overline{y}_r < 0$. The curve must satisfy $C > C_{\text{CRIT}}$ and $\overline{y}_r > -(1+\mu)/2$. This implies that the two vortices in vortex pair 2 are very close together. The speed acquired by vortex pair 2 is large. Thus, vortex pair 2 will pass through the wider vortex pair ahead and go relatively to plus infinity. In the single overtake case, we can validate our description by analysing figures 8–11. There is a transition stage where $\overline{x}_r$ changes from positive to negative and vice versa — the $\overline{x}_r = 0$ stage is when the single overtake of one vortex pair actually takes place. So, if $1/C < F_{\max}$, the two situations that can arise are:

1.
$$\overline{y}_{r_0} > -\frac{K_1^2 - K_2^2}{2(K_1^2 + K_2^2)} \text{ and } \overline{x}_{r_0} > 0, \tag{35}$$
with $\overline{x}_{r_0}$ and $\overline{y}_{r_0}$ as defined in (25). In this situation, vortex pair 1 passes through vortex pair 2 and $\overline{x}_r \to -\infty$.

2.
$$\overline{y}_{r_0} < -\frac{K_1^2 - K_2^2}{2(K_1^2 + K_2^2)} \text{ and } \overline{x}_{r_0} < 0. \tag{36}$$
Now vortex pair 2 passes through vortex pair 1 and $\overline{x}_r \to +\infty$.

(c) *No overtake of vortex pairs.*

Again, this type of motion is characteristic to two initial conditions. Consider figures 8–11, and in particular the case in which initially $\overline{x}_r > 0$ and $\overline{y}_r < 0$. Vortex pair 2 is ahead at this instant and it will just move off to plus infinity. The influence of vortex pair 1 on it is insufficient to lead to periodic leapfrogging motion. Similarly, if the initial configuration of the vortex pairs is $\overline{x}_r < 0$ and $\overline{y}_r > 0$, then the narrower vortex pair 2 behind will just move to negative infinity. In both cases there is no stage at which the $\overline{x}_r$ changes sign and therefore there is no overtake of vortex pairs at all. To summarise, if $1/C < F_{\max}$ then the two situations that can arise are:

1.
$$\overline{y}_{r_0} > -\frac{K_1^2 - K_2^2}{2(K_1^2 + K_2^2)} \text{ and } \overline{x}_{r_0} < 0. \tag{37}$$

This case implies that vortex pair 2 never catches vortex pair 1 and $\overline{x}_r \to -\infty$.

2.
$$\overline{y}_{r_0} < -\frac{K_1^2 - K_2^2}{2(K_1^2 + K_2^2)} \text{ and } \overline{x}_{r_0} > 0. \tag{38}$$

If these conditions hold, then vortex pair 1 never catches vortex pair 2 and $\overline{x}_r \to +\infty$.

We note that in Eckhardt and Aref appendix B [17] results are developed in terms of dimensionless impulse, rather than our dimensionless vertical vortex separation $\overline{y}_{r_0}$ – we contend that our approach is much easier to interpret physically and is therefore advantageous.

The criteria we have presented can be expected to be in only qualitative accord with those presented in Borisov, Kilin, and Mamaev [13] and in Aiki [4].

We indicated in the Introduction (section 1) our aims in this paper. These have been addressed and achieved as follows:

(I) To extend and enhance results by Péntek, Tel, and Toroczkai [28] and Eckhardt and Aref appendix B [17] explicitly in terms of physical variables for the relative motion of two given vortex pairs (of the 'same polarity') with specific initial separation — to address the question 'How does the system evolve?';

(II) To develop specific criteria — summarised in (a), (b), (c) above for when leapfrogging occurs. In particular (b) and (c) give details of the relative strength, separation, and initial conditions that lead to a non-periodic motion of the pairs — with eventual infinite separation, and either one or no occurrence of all four vortices being instantaneously in a straight line;

(III) To illustrate the motions with a gallery of vortex leapfrogging (section 4) for a range of relative pair strengths.

Acknowledgements

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Appendix A A Pythagorean configuration of three vortices of equal absolute strength

We have considered in the main body of this paper a particular interaction of vortex pairs where the integrity of each pair is maintained throughout. As we indicated in the Introduction, such an interaction is special, and that quite typically periodic and/or non-periodic motion can occur. In contrast to this situation we now give an analysis of another special case — that of the $N = 3$ vortex problem mentioned in Gröbli [18] and referred to in Aref, Rott, and Thomann [8]. The result for this case is an exchange of partners between a vortex pair and a single vortex of the same strength. The three vortices (1) $+K$, (2) $+K$, (3) $-K$ are in Pythagorean configuration at the vertices of a right angled triangle (see figure 19). This property of the triangle formed by the vortices is maintained and the vortices travel on straight line paths along and parallel to the x-axis. When the separation ℓ_{12} is very large then so is ℓ_{23} also and the configuration then models the approach of a vortex pair (1) (3) from afar to interact with vortex (2).

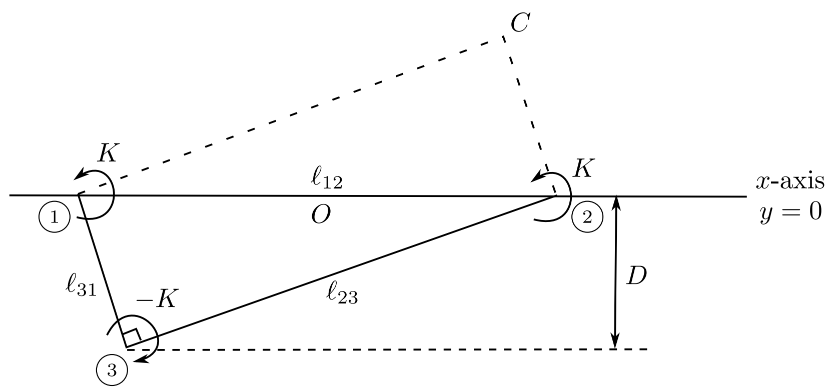

Figure 19: Schematic diagram of a vortex pair approaching a single vortex. Here D is the vertical distance between the vortex pair and the single vortex, ℓ_{12}, ℓ_{23}, ℓ_{31} are corresponding distances between the point vortices.

We specialise the analysis from the general three vortex results contained in e.g. Aref [5], Newton [27].

It follows from similarity of triangles that the vertical distance between the vortex pair and the point vortex is given by $D = \ell_{31}\ell_{23}/\ell_{12}$ and it is a constant. For the remainder of this section we will also use that $K_1 = K_2 = K$, $K_3 = -K$.

The equations of motion of point vortices can be written in the form of Hamilton's canonical equations [5]. If we let K_i be the strength and (x_i, y_i) be the position of vortex $i = 1, 2, 3$, then we have

$$K_i \dot{x}_i = \frac{\partial H}{\partial y_i}, \quad K_i \dot{y}_i = -\frac{\partial H}{\partial x_i}. \tag{A1}$$

The Hamiltonian function for this system of 3 vortices is defined as

$$\begin{aligned} H &= -\frac{1}{4\pi}\sum_{i\neq j}^{3} K_i K_j \ln(\ell_{ij}) \\ &= -\frac{2K^2}{4\pi}[\ln(\ell_{12}) - \ln(\ell_{31}) - \ln(\ell_{23})] \\ &= \frac{K^2}{2\pi}\ln\left(\frac{\ell_{31}\ell_{23}}{\ell_{12}}\right) \propto \ln(D), \end{aligned} \tag{A2}$$

where $i, j \in \{1, 2, 3\}$, ℓ_{ij} is the distance between vortices i and j, and the summation excludes the case $i = j$ [27, 5]. The Hamiltonian does not depend on time explicitly and so it is a constant of the motion, so that of course D is constant. Additionally, the Hamiltonian is invariant under translations and rotation of the coordinates. Via initial conditions we can determine the rest of the constants of the motion [10, 22, 30] which are (Q, P, I, M):

$$
\begin{aligned}
Q &= \sum_{i=1}^{3} K_i x_i = 0, \\
P &= \sum_{i=1}^{3} K_i y_i = KD, \\
I &= \sum_{i=1}^{3} K_i |z_i|^2 = KD^2, \\
M &= \sum_{i,j=1}^{3} K_i K_j \ell_{ij}^2 \equiv KI - Q^2 - P^2 = 0 \equiv K^2[\ell_{12}^2 - \ell_{31}^2 - \ell_{23}^2] = 0,
\end{aligned}
$$

where by convention we use that $i < j$ in M. For determining that M is a constant of the motion we use the fact that $\sum_{i=1}^{3} K_i = K \neq 0$ and so the center of vorticity C, given by

$$\frac{1}{\sum\limits_{i=1}^{3} K_i}\left(\sum_{i=1}^{3} K_i x_i, \sum_{i=1}^{3} K_i y_i\right), \tag{A3}$$

provides a fixed point of reference in the flow. Note that $\sum_{i=1}^{3} K_i = K$ here.

In this configuration, C is stationary, whereas O moves at speed $K/(4\pi D)$, and point vortex ③ moves with speed $K/(2\pi D)$. In figure 20 we show the $(+K, +K, -K)$ configuration of the vortex pair with the point vortex inscribed in a circle with centre O, with the induced velocities being, as usual, perpendicular to the line connecting the point vortices. The net induced velocities in the horizontal direction can be found using a geometrical argument (see figure 20), and they are given by:

- Point vortex ①: $\dfrac{K}{2\pi}\dfrac{\ell_{23}}{\ell_{31}\ell_{12}}$
- Point vortex ②: $\dfrac{K}{2\pi}\dfrac{\ell_{31}}{\ell_{23}\ell_{12}}$,
- Point vortex ③: $\dfrac{K}{2\pi}\left(\dfrac{\ell_{23}}{\ell_{31}\ell_{12}} + \dfrac{\ell_{31}}{\ell_{23}\ell_{12}}\right)$.

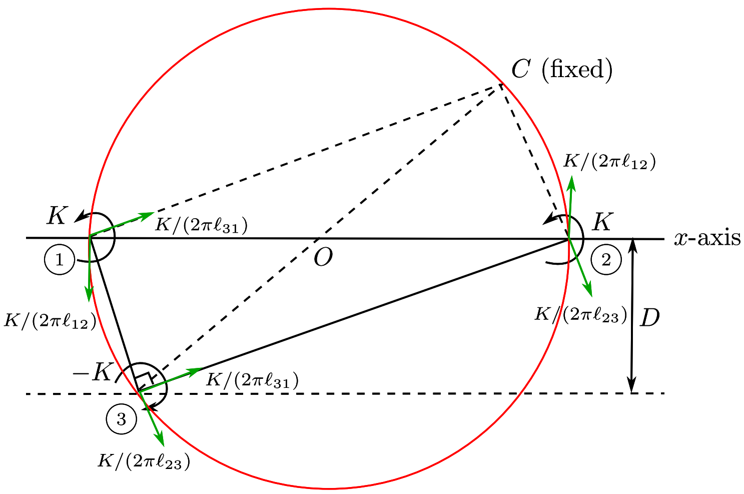

Figure 20: Schematic diagram of a vortex pair approaching a single vortex, inscribed in a circle with centre O. The induced velocities are labeled in the diagram, resulting in motions for each vortex along and parallel to the x-axis.

Now, we let the position of the circle centre O be at $x = s(t)$. Then the speed of O is given by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{K}{4\pi}\left(\frac{\ell_{23}}{\ell_{13}\ell_{12}} + \frac{\ell_{13}}{\ell_{23}\ell_{12}}\right) \equiv \frac{K}{4\pi}\frac{\ell_{12}}{\ell_{13}\ell_{23}} \equiv \frac{K}{4\pi D}, \tag{A4}$$

where we use that $\ell_{13}^2 + \ell_{23}^2 = \ell_{12}^2$, from Pythagoras' theorem. Therefore, the position $s(t)$ is a linear function of time, given by,

$$s(t) = \frac{K}{4\pi D}t + s_0, \tag{A5}$$

where s_0 is the initial position of the circle centre. Again using Pythagoras' theorem, we obtain that the radius of the circle can be expressed as $(s^2 + D^2)^{1/2}$ as shown in figure 21.

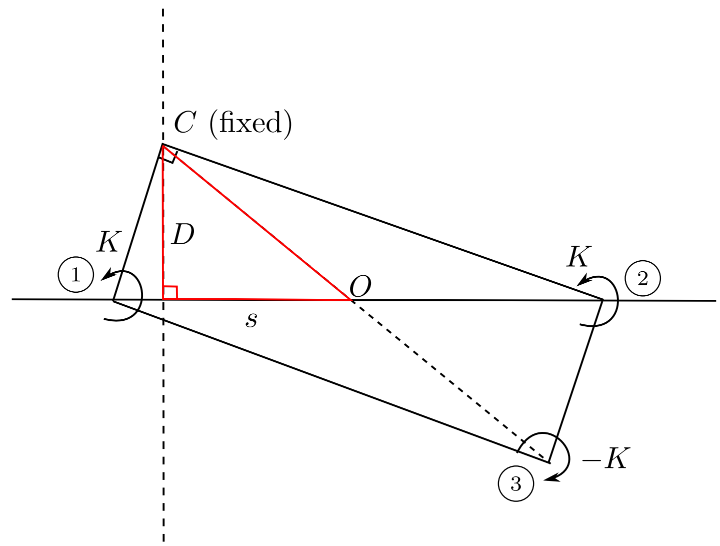

Figure 21: Schematic diagram of a vortex pair approaching a single vortex, showing the right-angled triangle used to find the radius of the circle.

The x-coordinate of point vortex (1) can then be obtained as

$$x_1 = s - (s^2 + D^2)^{1/2} \equiv \left(\frac{K}{4\pi D}t + s_0\right) - \left[\left(\frac{K}{4\pi D}t + s_0\right)^2 + D^2\right]^{1/2}, \tag{A6}$$

and similarly the x-coordinate of point vortex (2) is

$$x_2 = s + (s^2 + D^2)^{1/2} \equiv \left(\frac{K}{4\pi D}t + s_0\right) + \left[\left(\frac{K}{4\pi D}t + s_0\right)^2 + D^2\right]^{1/2}. \tag{A7}$$

From figure 21, it is evident that the x-coordinate of point vortex (3) is

$$x_3 = 2s = 2\left(\frac{K}{4\pi D}t + s_0\right). \tag{A8}$$

For convenience, we use the following transformation

$$\frac{K}{4\pi D}t + s_0 = D\sinh\phi. \tag{A9}$$

Now we can rewrite (A6)–(A8) as $x_1 = -De^{-\phi}$, $x_2 = De^{\phi}$, and finally $x_3 = D(e^{\phi} - e^{-\phi}) = 2D\sinh\phi$. So there is the following progression: we start with $\phi \to -\infty$ which implies that point vortices (1) and (3) start out at large negative x whereas point vortex (2) starts out at small positive x. Then at an intermediate point in time, $\phi = 0$, so $x_1 = -D$, $x_2 = D$, and $x_3 = 0$. Finally, at large time, $\phi \to \infty$ and so point vortex (1) goes to small negative x and point vortices (2) and (3) end at large positive x. The y coordinates are constant throughout. This transformation of variables is monotonic and $t : -\infty \to +\infty$ corresponds to $\phi : -\infty \to +\infty$. The evolution of the three-vortex configuration can be seen schematically in figure 22.

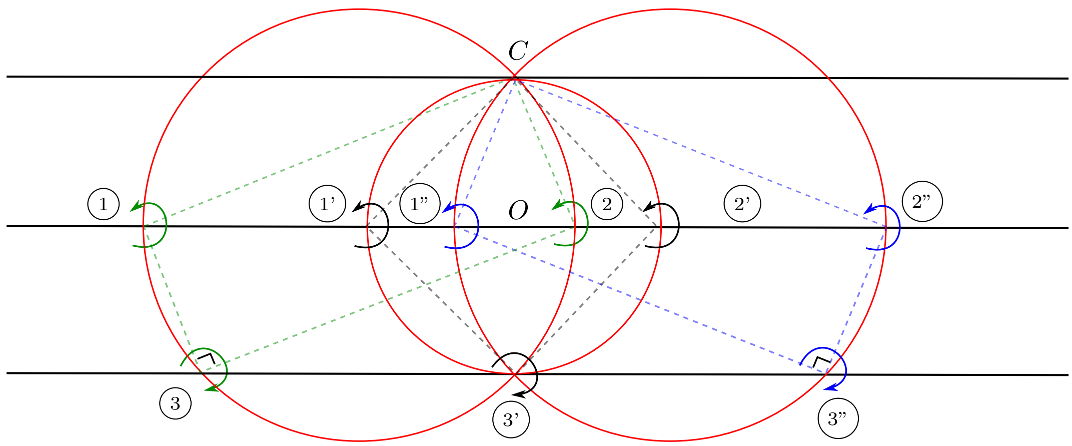

Figure 22: Schematic diagram of the evolution of the three-vortex configuration.

This special solution appears to be the only simple non-trivial explicit time-dependent solution for a three vortex problem. The effect is one of a vortex exchange of partners in that the vortex pair (1) (3) and the single vortex (2) evolve to a pair (2) (3) leaving behind the single vortex (1). A noteworthy feature is that the vortex (3) moves throughout at constant velocity, so that it experiences no time delay – that is to say it is at all times at the position it would have been if the interaction with the single vortex had not occurred.

This feature is in direct contrast with the pair-pair interactions considered in sections 3, 4, 5 of this paper, where indeed the integrity of the vortex pairs was maintained throughout.

In figure 23 we show the time evolution of the three-vortex Pythagorean configuration as a result of a numerical integration of the governing equations:

$$\dot{x}_i(t) = -\frac{1}{2\pi}\sum_{i\neq j}^{3} K_j \frac{y_i - y_j}{(x_i - x_j)^2 + (y_i - y_j)^2}, \tag{A10}$$

$$\dot{y}_i(t) = \frac{1}{2\pi}\sum_{i\neq j}^{3} K_j \frac{x_i - x_j}{(x_i - x_j)^2 + (y_i - y_j)^2}, \tag{A11}$$

where $i, j \in \{1, 2, 3\}$ and $K_1 = K_2 = 1 = -K_3$. A fourth-order Runge-Kutta scheme is used for time integration of the governing equations.

We note that the initial conditions need to be set in a particular way ('Pythagorean') since this case is very special and certainly not stable.

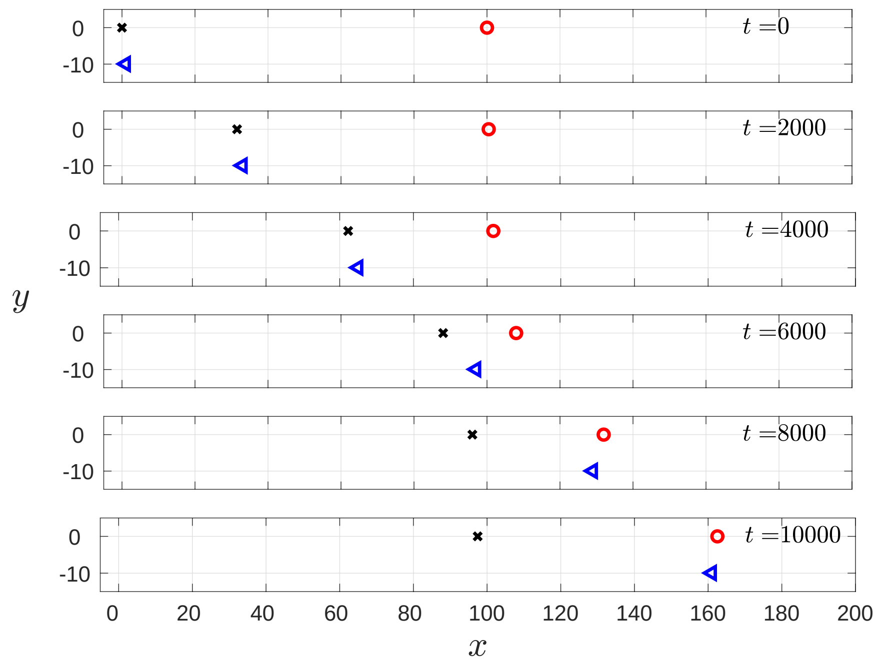

Figure 23: Time evolution of the three-vortex Pythagorean configuration with vortex strengths: $K_1 = K_2 = 1$ and $K_3 = -1$. The initial positions are given by $(x_1, y_1) = (0, 0)$ (black cross), $(x_2, y_2) = (100, 0)$ (red circle) and $(x_3, y_3) = (1, -9.9499)$ (blue triangle).

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