

Sliding Glaciers and Cavitation



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1 Introduction

The objective of this article is to present glacier sliding laws and the effect that cavitation has on these. The results investigated here largely follow Schoof's papers [11] and [10].

Glaciers are blocks of ice that can occupy the surfaces of mountains. When they move over a flat surface, heat produced due to friction can maintain a water film which lubricates the ice and bedrock interface. The moving ice moves like ordinary slow viscous flow around an obstacle.

Cavitation occurs either when the pressure of the water film becomes equal to the pressure at which all three states (ice, water and vapour) co-exist or when the normal pressure reaches the drainage pressure. Since the pressure drops to a critical separation pressure p_c that is not sufficiently high to keep the ice in contact with the bedrock, the two separate and cavities form. This is analogous to cavitation in fluid flow. In this context, cavitation is defined as the process of formation of vapour or liquid water as a consequence of reduced pressure in some region of the flow.

Cavity formation depends on the size of the variations in normal stress and the effective pressure, N ,

$$N = p_i - p_c, \quad (1.1)$$

which is the difference between mean hydrostatic pressure p_i at the bed and the separation pressure p_c . The smaller the value of N , the more likely variations in normal stress are to cause cavitation, and the more extensive this will be. The glacier will thus experience reduced resistance to flow since cavities cause the base of the glacier to lose contact with the bed. It follows that both the basal stress τ_b and the effective pressure N determine the sliding velocity of a glacier. Therefore, the sliding law should be expressed as

$$\tau_b = f(u_b, N). \quad (1.2)$$

If cavity formation is taken into account, then the sliding law can be written as

$$\tau_b = C u_b^p N^q \quad (1.3)$$

where $C, p, q > 0$. This law shows that for a given sliding velocity u_b , the basal shear stress τ_b decreases when effective pressure N decreases and cavitation becomes more widespread. As we will see later in this article, Iken in [5] derived a bound for the basal shear stress, of the form $\tau_b \leq N \tan \beta$, which is solely determined by the bed geometry. However, Iken's bound and the sliding law (1.3) are contradictory, since the latter implies that τ_b can grow arbitrarily big despite the size of N . Iken's bound

was derived using the unrealistic bed geometry of rectangular steps. In an attempt to explain this contradiction, Fowler describes a model in [2] that takes into account a single cavity over a bedrock period. This allows the computation of sliding laws for more realistic bed surfaces. In [11], Schoof generalises this model to account for any finite number of cavities per period.

This article is organised as follows: In Section 2, we show how to derive bounds for $\frac{\tau_b}{N}$ for various bed geometries. In Section 3, we introduce different sliding laws and in Section 4, we discuss extensively how to determine the sliding law by taking into consideration cavitation. Finally, for the sake of self-containment, we include appendices at the end.

2 Bounds for $\frac{\tau_b}{N}$ for different bed shapes

Using a simple force balance of glacier sliding on a bed consisting of rectangular steps, Iken concluded that the quantity $\frac{\tau_b}{N}$ satisfies an upper bound which is determined only by the up-slope of the bed, $\frac{\tau_b}{N} \leq \tan \beta$. Schoof showed the bound for a general bed geometry which is only restricted to bounded slopes. We will investigate these in the subsections that follow.

2.1 Iken's proof for bed consisting of rectangular steps [5]

In this subsection, we show that the basal shear stress bound is $\tau_b \leq N \tan \beta$, as presented in [5]. Iken suggested that if the subglacial water pressure rises more than some value then a critical pressure p_{crit} will be reached above which the glacier will accelerate without bound.

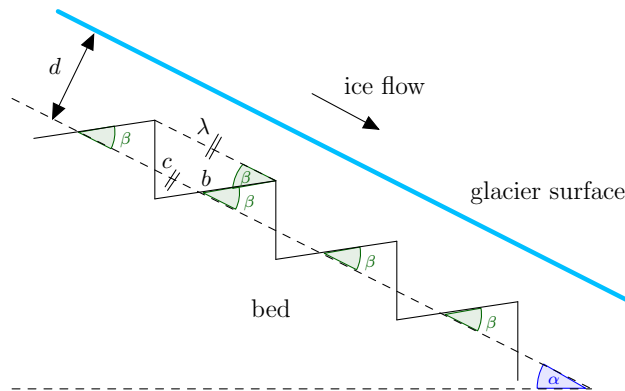


Figure 1: Bed consisting of rectangular steps to derive Iken's bound for τ_b .

Let us consider a glacier block on an inclined plane with slope α . Assume that the bed consists of rectangular steps for simplicity. For more irregular shapes like a sinusoidal bed, see § 2.2. Here we assume that the column of glacier has unit width, length λ and rests on faces b and c of these steps. Furthermore, the glacier block has weight equal to $F_{\text{weight}} = \rho g d \lambda$. In Figure 2, we split the weight of the glacier into the components perpendicular to face b and face c with angle $\beta - \alpha$.

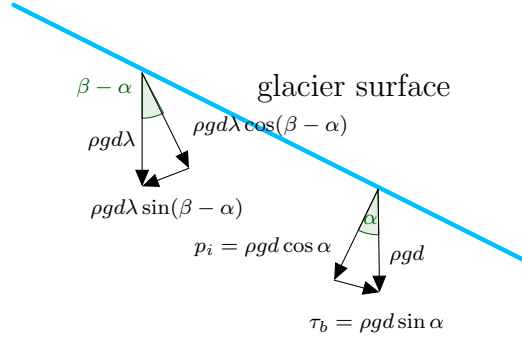


Figure 2: Force components resolved in different directions.

The component perpendicular to face b is

$$F_b = \rho g d \lambda \cos(\beta - \alpha) \quad (2.1)$$

and the component perpendicular to face c is

$$F_c = \rho g d \lambda \sin(\beta - \alpha), \quad (2.2)$$

where ρ is the density of the block of glacier, d is its thickness, g is the acceleration due to gravity and finally β is the angle that the stoss surface makes with the inclined plane.

Having determined the forces acting on faces b and c we can find the corresponding pressures acting on them. The pressure on each face can be calculated as the force divided by the length of the face

$$p_b = \frac{F_b}{b} = \rho g d \frac{\cos(\beta - \alpha)}{\cos \beta} \quad \text{and} \quad p_c = \frac{F_c}{c} = \rho g d \frac{\sin(\beta - \alpha)}{\sin \beta} \quad (2.3)$$

where we use that $b = \lambda \cos \beta$ and $c = \lambda \sin \beta$.

The basal water pressure, p_c , should remain below the critical pressure, or otherwise, if $p_c > p_{\text{crit}}$ there will be a net force on the ice which will move it upward along faces b and it will cause the glacier to accelerate without bound.

Next, using the overburden pressure $p_i = \rho g d \cos \alpha$ and the basal shear stress $\tau_b = \rho g d \sin \alpha$, we obtain the critical pressure

$$\begin{aligned} p_{\text{crit}} &= \rho g d \frac{\sin(\beta - \alpha)}{\sin \beta} = p_i \frac{\sin(\beta - \alpha)}{\sin \beta \cos \alpha} = p_i \frac{(\sin \beta \cos \alpha - \cos \beta \sin \alpha)}{\sin \beta \cos \alpha} \\ &= p_i \left(1 - \frac{\tan \alpha}{\tan \beta} \right) = p_i - \frac{\tau_b}{\tan \beta}. \end{aligned} \quad (2.4)$$

The basal water pressure, p_c , should remain below p_{crit} such that Iken's bound always holds. Recall that the effective pressure is $N = p_i - p_c$ and $p_c \leq p_{\text{crit}}$ such that

$$\frac{\tau_b}{N} = \frac{\tau_b}{p_i - p_c} \leq \frac{\tau_b}{p_i - \left(p_i - \frac{\tau_b}{\tan \beta} \right)} = \tan \beta. \quad (2.5)$$

To conclude, we see that for as τ_b increases, or equivalently, as the sliding speed, u_b , increases, p_{crit} decreases. If the water pressure, p_c , exceeds this critical value for the pressure, separation between the ice and the bedrock will occur and then cavities will form and will grow to a size which will be proportional to the amount by which p_c exceeds p_{crit} .

2.2 Schoof's proof for more irregular beds [11]

We consider a two-dimensional bedrock surface S as shown in Figure 3.

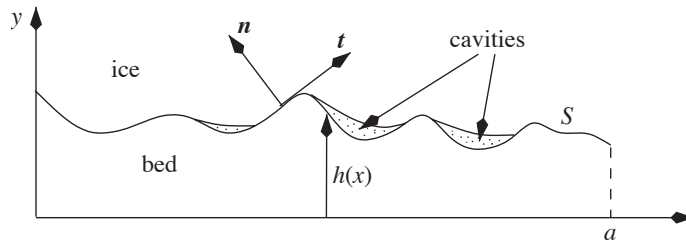


Figure 3: Geometry of bed under consideration taken from [11].

We focus on a section of the bedrock that lies in $0 \leq x \leq a$ and assume that the ice flow is parallel to the x -axis. The height of the bedrock depends upon the position along x and can thus be written as $y = h(x)$. The bed exerts a force on the glacier block lying above it and this force is denoted by \mathbf{f} . It can be calculated as

$$\mathbf{f} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = - \int_S \boldsymbol{\sigma} \mathbf{n} \, ds = - \int_S (\mathbf{n} \cdot \boldsymbol{\sigma} \mathbf{n}) \mathbf{n} \, ds - \int_S (\mathbf{t} \cdot \boldsymbol{\sigma} \mathbf{n}) \mathbf{t} \, ds = - \int_S \sigma_{nn} \mathbf{n} \, ds, \quad (2.6)$$

where we evaluate the stress tensor $\boldsymbol{\sigma}$ at the bed surface, $\sigma_{nn} = (\mathbf{n} \cdot \boldsymbol{\sigma} \mathbf{n})|_{y=h(x)}$ and we assume that due to the water film, the tangential shear stress is zero at the base.

The unit normal to the bedrock surface is $\mathbf{n} = \frac{1}{\sqrt{1+h'(x)^2}} \begin{pmatrix} -h'(x) \\ 1 \end{pmatrix}$ and an element of the arc S is $ds = \sqrt{1+h'(x)^2} dx$. Substituting these into Equation (2.6), yields

$$f_x = \int_0^a \sigma_{nn} h'(x) dx, \quad f_y = - \int_0^a \sigma_{nn} dx. \quad (2.7)$$

If a is regarded as the bedrock period then we can use it as the averaging distance. With this we can calculate the basal drag, τ_b , and the hydrostatic overburden pressure, p_i , in the following way

$$\tau_b = -\frac{f_x}{a} = -\frac{1}{a} \int_0^a \sigma_{nn} h'(x) dx, \quad p_i = \frac{f_y}{a} = -\frac{1}{a} \int_0^a \sigma_{nn} dx. \quad (2.8)$$

If the compressive normal stress $-\sigma_{nn}$ decreases below the critical pressure then cavities form. We define a local effective pressure as $N_{\text{loc}}(x) = -\sigma_{nn} - p_c$ and since we want the compressive normal stress to not drop below this critical pressure, we require

$$-\sigma_{nn} \geq p_c \quad \Rightarrow \quad N_{\text{loc}}(x) \geq 0. \quad (2.9)$$

Reconsider the basal drag force τ_b , this time as a function of the local effective pressure

$$\begin{aligned} \tau_b &= -\frac{1}{a} \int_0^a \sigma_{nn} h'(x) dx = \frac{1}{a} \int_0^a (N_{\text{loc}}(x) + p_c) h'(x) dx \\ &= \frac{1}{a} \int_0^a N_{\text{loc}}(x) h'(x) dx + \frac{1}{a} \int_0^a p_c h'(x) dx. \end{aligned} \quad (2.10)$$

Since $N_{\text{loc}}(x) \geq 0$, we can write $N_{\text{loc}}(x) h'(x) \leq N_{\text{loc}}(x) \sup(h'(x))$, and also evaluate the second integral in Equation (2.10) as

$$\frac{1}{a} \int_0^a p_c h'(x) dx = \frac{p_c}{a} \int_0^a h'(x) dx = \frac{p_c}{a} [h(a) - h(0)].$$

Therefore, Equation (2.10) becomes

$$\tau_b \leq \frac{\sup(h'(x))}{a} \int_0^a N_{\text{loc}}(x) dx + \frac{p_c}{a} [h(a) - h(0)], \quad (2.11)$$

and we will see that the integral equation turns from a local effective pressure to a global effective pressure

$$\begin{aligned} \frac{\sup(h'(x))}{a} \int_0^a N_{\text{loc}}(x) dx &= \frac{\sup(h'(x))}{a} \int_0^a (-\sigma_{nn} - p_c) dx \\ &= \sup(h'(x)) \left[-\frac{1}{a} \int_0^a \sigma_{nn} dx - \frac{1}{a} \int_0^a p_c dx \right] \\ &= \sup(h'(x)) [p_i - p_c] \\ &= \sup(h'(x)) N, \end{aligned} \quad (2.12)$$

where we use in the third equality Equation (2.8) and the fact that the water pressure p_c is a constant, so $\frac{1}{a} \int_0^a p_c dx = p_c$.

At this point, let us consider what happens to the term $\frac{p_c}{a}[h(a) - h(0)]$ if the period a is large. Schoof assumes that $h(a)$ is bounded as $a \rightarrow \infty$ and therefore $\frac{p_c}{a}[h(a) - h(0)] \rightarrow 0$.

Taking everything into consideration, Equation (2.11) shows that

$$\tau_b \leq N \sup(h'(x)). \quad (2.13)$$

The bound is therefore determined by the bed topography. For instance, consider a bed with a sinusoidal shape that is given by $h(x) = a \sin\left(\frac{2\pi x}{\lambda}\right)$. Then the theoretical bound for $\frac{\tau_b}{N}$ is $\max(h'(x)) = \frac{2\pi a}{\lambda} \max(\cos\left(\frac{2\pi x}{\lambda}\right)) = \frac{2\pi a}{\lambda} = (\tan \beta)_{\max}$. Moreover, note that the bound (2.13) is equivalent to Iken's bound $\tau_b \leq N \tan \beta$ for rectangular steps.

3 Sliding Laws

Weertman in his paper [12] examined the theory of glacier sliding by considering a quite unrealistic bed consisting of uniformly spaced ice cubes on a flat surface as shown in Figure 4a. The simplified method he used is sometimes referred to as the tombstone model. Weertman concluded that the sliding law should be of the approximate form

$$\tau_b = C u_b^{\frac{2}{n+1}}, \quad (3.1)$$

where τ_b is the basal shear stress, u_b is the sliding velocity and n is the exponent as it appears in Glen's flow law ($\epsilon = A\tau^n$ for ϵ – shear stress, A and n constants, τ – basal shear stress). A glacier sliding over its bedrock may move past obstacles by either regelation¹ or plastic deformation.

Lliboutry was the first to introduce and examine the effects of cavitation on the sliding law and he presented his results in [6]. The sliding law suggested by Lliboutry is as illustrated in Figure 4b and we can observe that cavitation has the effect of reducing the effective roughness of the bed, or in other words, the friction for increasing sliding velocity.

¹Regelation is the process that results in melting of the ice on the upstream faces and freezing on the downstream faces. The flow of water through the lubricating film, through this process of melting and freezing, allows the ice to move through the obstacles.

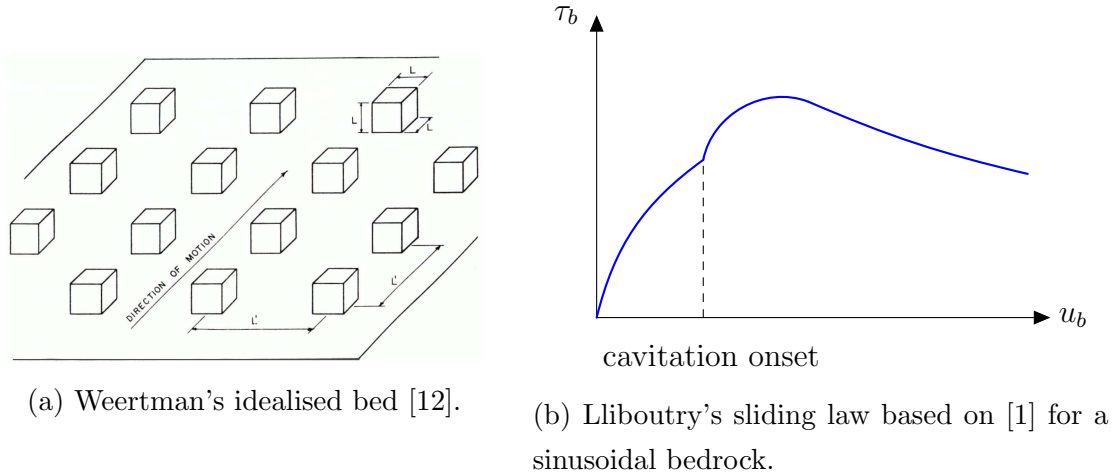


Figure 4: Sliding laws.

Fowler found an analogy between glacier sliding problems and the lubrication theory² of shallow ice problems. He introduced a boundary layer approach based on the assumption that the wavelengths of the basal obstacles are much smaller than the thickness of the ice lying above. Moreover, he determined the sliding law using matched asymptotic expansions and he presented the solution for the case of a periodic bedrock with one cavity per period. For details, see [2].

The model is restricted to the case of glaciers whose width is much greater than their depth and so the flow can be considered as two-dimensional. It is found that the basal stress decreases towards zero as the velocity tends to infinity, i.e. $\frac{\tau_b}{N}$ decreases as $\frac{u_b}{N}$ increases.

Since the glacier is modelled as an incompressible Newtonian fluid with low Reynolds number, the dimensionless Stokes flow equations take the form

$$\begin{aligned}\nabla^2 \mathbf{u} - \nabla p &= 0 \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}\tag{3.2}$$

where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ is the velocity of the glacier and ∇p is the pressure gradient.

After suitable non-dimensionalisation at leading order the following boundary

²Thin layers of fluid can prevent solid bodies from contact and the analysis of the fluids flow in thin layers is called lubrication theory.

conditions are obtained

$$\begin{aligned}
u_y + v_x &= 0, & v &= u_b h'(x) & \text{on } y = 0, x \in C' \\
u_y + v_x &= 0, & p - 2v_y &= -N, & v &= u_b h'_C(x) & \text{on } y = 0, x \in C \\
u, v, p &\rightarrow 0 & \text{as } y &\rightarrow \infty
\end{aligned} \tag{3.3}$$

where we require that u, v must have a period a in x . C' denotes the contact areas and C the regions that correspond to cavities. Furthermore, $h(x)$ is the height of the bedrock as shown in Figure 3 and $h_C(x)$ is the height of the cavity roof.

At the bedrock, zero shear stress is prescribed and this corresponds to $u_y + v_x = 0$ in Equation (3.3). This is the τ_{xy} component of the stress tensor as it appears in [9]. For an illustration of the components of the stress tensor, see Figure [3]. A vanishing normal velocity, $v = u_b h'(x)$ is also required. The condition $p - 2v_y = -N$ stems from the component of the stress tensor, τ_{yy} .

Equivalently, using the stream function, $u = \psi_y$ and $v = -\psi_x$, we have

$$\begin{aligned}
\psi_{yy} - \psi_{xx} &= 0, & \psi &= -u_b h(x) & \text{on } y = 0, x \in C' \\
\psi_{yy} - \psi_{xx} &= 0, & p + 2\psi_{xy} &= -N, & \psi &= -u_b h_C(x) & \text{on } y = 0, x \in C \\
\psi &\rightarrow 0 & \text{as } y &\rightarrow \infty.
\end{aligned} \tag{3.4}$$

4 Method of Solution using Complex Variables

A nice way of solving this problem is using the theory of complex variables and this is what we will do in this section.

4.1 Boundary conditions

Recall that we defined the velocity vector in terms of a stream function, ψ , as

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{4.1}$$

From the problem in (3.2) we use $\nabla^2 \mathbf{u} - \nabla p = 0$ and Equations (4.1) to write

$$\nabla p = \nabla^2 \mathbf{u} = \nabla^2 \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) \tag{4.2}$$

and thus

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \nabla^2 \psi \quad \text{and} \quad \frac{\partial p}{\partial y} = -\frac{\partial}{\partial x} \nabla^2 \psi \tag{4.3}$$

which are the Cauchy-Riemann equations for the function $p + i\nabla^2\psi$.

Thus, ψ satisfies the biharmonic equation

$$\nabla^4\psi = \frac{\partial^4\psi}{\partial z^2\partial\bar{z}^2} = 0. \quad (4.4)$$

The solution of the biharmonic equation has the Goursat representation [4]

$$\psi = \Re\{\bar{z}f(z) + g(z)\} = \bar{z}\theta(z) + \theta(z) + z\overline{\phi(z)} + \overline{\phi(z)} \quad (4.5)$$

where θ and ϕ are both holomorphic functions in the upper half plane, S^+ and ψ is a real function as required.

The Laplacian is $\nabla^2\psi = 4\frac{\partial^2\psi}{\partial z\partial\bar{z}}$. Therefore, if we differentiate the general solution (4.5) with respect to \bar{z} and with respect to z , we obtain $\frac{\partial^2\psi}{\partial z\partial\bar{z}} = \theta'(z) + \overline{\theta'(z)}$, which gives $\nabla^2\psi = 4(\theta'(z) + \overline{\theta'(z)})$. Therefore, we can write

$$p + i\nabla^2\psi = p + 4i(\theta'(z) + \overline{\theta'(z)}). \quad (4.6)$$

Schoof in [10] defines a new holomorphic function $\rho(z)$ as

$$\rho(z) = (p + i\nabla^2\psi) - 8i\theta'(z) \quad (4.7)$$

yielding

$$\rho(z) = p - 4i(\theta'(z) - \overline{\theta'(z)}). \quad (4.8)$$

The difference between $\theta'(z)$ and its complex conjugate is $2i\Im(\theta'(z))$ and thus when multiplied by i gives a real number. Since p is also real we can rearrange (4.8) and write p as

$$p = \kappa + 4i(\theta'(z) - \overline{\theta'(z)}) \quad (4.9)$$

where $\rho(z) = \kappa$ is a constant. For a proof of this, see Appendix A. This constant has to be zero since at infinity the velocity field components should vanish. In stream function notation this is equivalent to

$$\psi \rightarrow 0 \quad \text{as} \quad \Im(z) \rightarrow \infty. \quad (4.10)$$

Equation (4.5) implies that this is equivalent to

$$\theta \rightarrow 0 \quad \text{and} \quad \phi \rightarrow 0 \quad \text{as} \quad \Im(z) \rightarrow \infty, \quad (4.11)$$

and since p must be zero at infinity as well, we choose $\kappa = 0$. This yields

$$p = 4i(\theta'(z) - \overline{\theta'(z)}). \quad (4.12)$$

Now let us consider what happens at the base of the bedrock. We look separately at the boundary conditions prescribed at $y = 0$ for the contact areas (C') and for cavity regions (C).

The vertical velocity at $y = 0$ and $x \in C'$ is $v = -\psi_x = u_b h'(x)$. Therefore

$$v_x = -\psi_{xx} = u_b h''(x). \quad (4.13)$$

Using Equation (A.1) and Equation (4.5) we get

$$\begin{aligned} v_x = -\psi_{xx} &= -\frac{\partial^2 \psi}{\partial z^2} - 2\frac{\partial^2 \psi}{\partial z \partial \bar{z}} - \frac{\partial^2 \psi}{\partial \bar{z}^2} \\ &= -\bar{z}\theta''(z) - \phi''(z) - 2\theta'(z) - 2\overline{\theta'(z)} - z\overline{\theta''(z)} - \overline{\phi''(z)} \\ &= -[x\theta''(x) + \phi''(x) + 2\theta'(x) + 2\overline{\theta'(x)} + x\overline{\theta''(x)} + \overline{\phi''(x)}]^+ \\ &= u_b h''(x), \end{aligned} \quad (4.14)$$

where in the third equality we use the fact that at the bed, $y = 0$, we have $z = x + i0 = x$ and thus $\bar{z} = x$. Note that $^+$ indicates that we approach the boundary from S^+ .

However, on the cavity roof we have that the normal stress is given by

$$p - 2v_y = -N \quad \text{on } y = 0, x \in C. \quad (4.15)$$

Therefore, from Equations (4.1) and (A.1), (A.2) and (4.5) we obtain

$$\begin{aligned} v_y &= -i \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \bar{z}^2} \right) [\bar{z}\theta(z) + \theta(z) + z\overline{\phi(z)} + \overline{\phi(z)}] \\ &= -i[\bar{z}\theta''(z) + \phi''(z) - z\overline{\theta''(z)} - \overline{\phi''(z)}]. \end{aligned} \quad (4.16)$$

Using Equations (4.12) and (4.15) we have on $x \in C$ and $y = 0$, $z = x = \bar{z}$ that

$$\begin{aligned} p - 2v_y &= 4i(\theta'(x) - \overline{\theta'(x)}) + 2i[x\theta''(x) + \phi''(x) - x\overline{\theta''(x)} - \overline{\phi''(x)}]^+ = -N \\ 2i[x\theta''(x) + 2\theta'(x) - 2\overline{\theta'(x)} + \phi''(x) - x\overline{\theta''(x)} - \overline{\phi''(x)}]^+ &= -N. \end{aligned} \quad (4.17)$$

In a similar way, the compressive normal stress in $x \in C'$ is found by

$$2i[x\theta''(x) + 2\theta'(x) - 2\overline{\theta'(x)} + \phi''(x) - x\overline{\theta''(x)} - \overline{\phi''(x)}]^+ = N_{\text{loc}} - N. \quad (4.18)$$

The last condition is that the shear stress vanishes everywhere at the bed, so $u_y + v_x = 0$ at $y = 0$. Using Equations (4.1), (A.2) and (4.5) we get

$$\begin{aligned} u_y &= - \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)^2 [\bar{z}\theta(z) + \theta(z) + z\overline{\phi(z)} + \overline{\phi(z)}] \Big|_{y=0} \\ &= -\bar{z}\theta''(z) - \phi''(z) + 2\theta'(z) + 2\overline{\theta'(z)} - z\overline{\theta''(z)} - \overline{\phi''(z)} \Big|_{y=0} \\ &= [-x\theta''(x) - \phi''(x) + 2\theta'(x) + 2\overline{\theta'(x)} - x\overline{\theta''(x)} - \overline{\phi''(x)}]^+ \end{aligned} \quad (4.19)$$

and so, on $y = 0$ and $x \in C \cup C'$, the boundary condition $u_y + v_x = 0$ becomes

$$[x\theta''(x) + x\overline{\theta''(x)} + \overline{\phi''(x)} + \phi''(x)]^+ = 0. \quad (4.20)$$

4.2 Hilbert problem

We will now formulate the problem into a Hilbert problem as shown in [11]. This enables the sliding law to be reduced to the solution of a boundary value problem of a single holomorphic function.

Define the following holomorphic functions in the upper half plane, S^+

$$\Omega(z) = z\theta''(z) + 2\theta'(z) + \phi''(z) \quad \text{and} \quad \omega(z) = z\theta''(z) + \phi''(z). \quad (4.21)$$

Note that if we have a point z in S^+ then its reflection in S^- will be \bar{z} . The functions $\Omega(z)$ and $\omega(z)$ are defined in S^+ and the corresponding functions in S^- are $\overline{\Omega(\bar{z})} =: \Omega_\star(z)$ and $\overline{\omega(\bar{z})} =: \omega_\star(z)$. For a more detailed description of some basic complex variable techniques see Appendix A.

At the boundary Ω satisfies

$$\overline{\Omega(x)}^+ = \lim_{y \rightarrow 0^+} \overline{\Omega(x + iy)} = \lim_{y \rightarrow 0^-} \overline{\Omega(x - iy)} = \Omega_\star^-(x). \quad (4.22)$$

and ω satisfies

$$\overline{\omega(x)}^+ = \lim_{y \rightarrow 0^+} \overline{\omega(x + iy)} = \lim_{y \rightarrow 0^-} \overline{\omega(x - iy)} = \omega_\star^-(x). \quad (4.23)$$

Using these results, the boundary condition (4.14) in $x \in C'$ becomes

$$-[\Omega^+(x) + \Omega_\star^-(x)] = u_b h''(x) \quad x \in C' \quad (4.24)$$

where we have used $[x\theta''(x) + 2\theta'(x) + \phi''(x)]^+ = \Omega^+(x)$ and $[\overline{x\theta''(x) + 2\theta'(x) + \phi''(x)}]^+ = \overline{\Omega^+(x)} = \Omega_\star^-(x)$.

Similarly, the boundary condition (4.17) in $x \in C$ can be written as

$$2i[\Omega^+(x) - \Omega_\star^-(x)] = -N \quad x \in C. \quad (4.25)$$

Following the same procedure, the rest of the boundary conditions become

$$2i[\Omega^+(x) - \Omega_\star^-(x)] = N_{\text{loc}}(x) - N \quad x \in C' \quad (4.26)$$

and finally we can write Equation (4.20) as

$$\omega^+(x) + \omega_\star^-(x) = 0 \quad x \in C \cup C'. \quad (4.27)$$

Recall that as $\Im(z) \rightarrow \infty$ we have $\theta, \phi \rightarrow 0$ and from the definitions (4.21), this boundary condition becomes

$$\Omega(z), \omega(z) \rightarrow 0 \quad \text{as} \quad \Im(z) \rightarrow \infty \quad (4.28)$$

Assuming that ω has no singularities at the endpoints of C and C' , we conclude that

$$\begin{aligned} \omega^+(x) + \omega_\star^-(x) &= 0 \quad x \in C \cup C' \\ \omega(z) &\rightarrow 0 \quad \text{as} \quad \Im(z) \rightarrow \pm\infty \end{aligned} \quad (4.29)$$

and this implies that $\omega(z) \equiv 0$.

To determine the function Ω we need to define the following function

$$F(z) = \begin{cases} \frac{-\Omega(z) + \frac{1}{4}iN}{u_b}, & z \in S^+ \\ \frac{-\Omega_\star^-(z) - \frac{1}{4}iN}{u_b}, & z \in S^-. \end{cases} \quad (4.30)$$

So $F(z)$ is holomorphic in $S^+ \cup S^-$ and satisfies $F(z) = F_\star(z) := \overline{F(\bar{z})}$ (recall that $\overline{\Omega(\bar{z})} =: \Omega_\star(z)$).

Using the definition (4.30) and after rearrangement we obtain

$$\begin{aligned} \Omega^+(x) &= -F^+(x)u_b + \frac{1}{4}iN, \\ \Omega_\star^-(x) &= -F^-(x)u_b - \frac{1}{4}iN. \end{aligned} \quad (4.31)$$

Thus $-[\Omega^+(x) + \Omega_\star^-(x)] = u_b h''(x)$ becomes

$$F^+(x) + F^-(x) = h''(x) \quad x \in C'. \quad (4.32)$$

In addition, we have $2i[\Omega^+ - \Omega_\star^-] = -N$ in the cavities, $x \in C$ such that on substituting Equations (4.31) we obtain

$$F^+(x) - F^-(x) = 0 \quad x \in C. \quad (4.33)$$

And lastly, we have $\Omega(z) \rightarrow 0$ as $\Im(z) \rightarrow \pm\infty$ which together with (4.30) imply that

$$F(z) \rightarrow \pm \frac{iN}{4u_b} \quad \Im(z) \rightarrow \pm\infty. \quad (4.34)$$

In the same way as in Equation (4.24) we have for the cavity roof shape

$$-[\Omega^+(x) + \Omega_\star^-(x)] = u_b h''_C(x) \quad x \in C \quad (4.35)$$

and on dividing by u_b and using Equations (4.31) we get

$$F^+(x) + F^-(x) = h''_C(x) \quad x \in C. \quad (4.36)$$

Since in the cavities we have shown that $F^+(x) - F^-(x) = 0$, it follows that F is continuous across C . Therefore we have $F^+(x) = F^-(x) = F(x)$ and we can rewrite Equation (4.36) as

$$F(x) = \frac{h_C''(x)}{2} \quad x \in C. \quad (4.37)$$

Moreover, we can derive the sliding law using the normal stress boundary condition on the cavity roof (4.15). Since the shear stress on the base is zero we have $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$ on $y = 0$. Using the method of asymptotic expansions, as it appears in [11], the effective basal shear stress τ_b at leading order becomes

$$\tau_b = \frac{1}{a} \int_0^a (p - 2v_y)|_{y=0} h'(x) dx. \quad (4.38)$$

Note that the effective pressure, N , is a constant and $h(x)$ has a period a , thus we can write

$$\tau_b = \frac{1}{a} \int_0^a (p - 2v_y + N)|_{y=0} h'(x) dx = \frac{1}{a} \int_0^a N_{\text{loc}}(x) h'(x) dx \quad (4.39)$$

Using Equation (4.26) for $x \in C'$ and (4.31) we have

$$-2iu_b[F^+(x) - F^-(x)] = N_{\text{loc}}(x). \quad (4.40)$$

But we have shown that $F^+(x) - F^-(x) = 0$ for $x \in C$ and so the local effective pressure in $x \in C$ is $N_{\text{loc}}(x) = 0$.

Given the form of the local effective pressure in the contact areas and cavities in terms of the function F , it is apparent that we can substitute it in the sliding law (4.39) to compute the sliding law once the function F is known.

4.3 Representing cavities in the ζ -plane

The purpose of transforming the problem into the complex plane and of using the method of solution is to allow for solutions to be found for any finite number of cavities per period. Since the problem we are considering is periodic in x with period a , when we transform into the complex unit circle the complex variables are

$$\zeta = \exp\left(\frac{2\pi iz}{a}\right) \quad \text{and} \quad \xi = \exp\left(\frac{2\pi ix}{a}\right) \quad (4.41)$$

where $z = x + iy$ with $x, y \in \mathbb{R}$ and $\zeta = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}$.

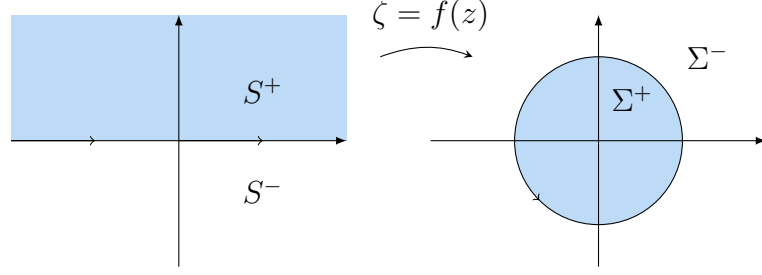


Figure 5: Conformal map from the upper half plane to the interior of the unit circle.

Using this transformation, if C corresponds to a section in the real x -axis then this is mapped to an arc, Γ , in the complex ζ -plane and it represents the location of cavities. The section C' which denotes the contact areas is mapped to Γ' by (4.41). Furthermore, let the interior of the unit circle be Σ^+ and the exterior be Σ^- .

We define

$$G(\zeta) = F(z) \quad (4.42)$$

and since we showed that F is continuous at $x \in C$ we have that G is a single valued function and holomorphic across the Γ .

All in all, we have the following boundary conditions under the various transformations from the (x, y) -plane to the complex plane and the ζ -plane. Note that we define the $h_2(\xi) = h''(x)$ and also now $^+$ is used to indicate that we take the limit as ζ approaches the boundary from the interior of the circle, Σ^+ and similarly for $^-$.

Table 1: Summary of boundary conditions.

Real plane	Complex plane	ζ -plane
$-\left[\Omega^+(x) + \Omega_\star^-\right] = u_b h_C''(x)$	$F^+(x) - F^-(x) = 0$	$G^+(\xi) + G^-(\xi) = h_2(\xi)$
$2i[\Omega^+(x) - \Omega_\star^-] = -N$	$F^+(x) + F^-(x) = h''(x)$	$G(0) = \frac{iN}{4u_b}$
$2i[\Omega^+(x) - \Omega_\star^-] = N_{\text{loc}}(x) - N$	$F(z) \rightarrow \pm \frac{iN}{4u_b}$ as $\Im(z) \rightarrow \pm\infty$	$G(\infty) = -\frac{iN}{4u_b}$
$\omega^+(x) + \omega_\star^-(x) = 0$	$F(z) = F_\star(z) := \overline{F(\bar{z})}$	$G(\zeta) = G_\star(\zeta) = G\left(\frac{1}{\bar{\zeta}}\right)$
$\Omega_\star = \overline{\Omega(\bar{z})}$		

Using the Schwarz reflection principle $G(\zeta)$ can be continued analytically across the real axis by defining $G(\zeta) = \overline{G\left(\frac{1}{\bar{\zeta}}\right)}$. Note that the complex conjugate in unit circle comes from $|\zeta| = 1$ which implies that $\zeta\bar{\zeta} = 1$ and thus $\zeta = \frac{1}{\bar{\zeta}}$. Moreover, the period of the unit circle is $a = 2\pi$. Using

$$N_{\text{loc}}(\xi) = -2iu_b(G^+(\xi) - G^-(\xi)) \quad (4.43)$$

and (4.39) and defining, as in [11], $h_1(\xi) = h'(x)$, we can write the sliding law in terms of the function G as

$$\begin{aligned}\tau_b &= \frac{1}{2\pi} \int_{\Gamma'} -2iu_b(G^+(\xi) - G^-(\xi))h_1(\xi)d\xi = -\frac{iu_b}{\pi} \int_{\Gamma'} \frac{G^+(\xi) - G^-(\xi)}{i\xi} h_1(\xi)d\xi \\ &= -\frac{u_b}{\pi} \int_{\Gamma'} \frac{G^+(\xi) - G^-(\xi)}{\xi} h_1(\xi)d\xi.\end{aligned}\quad (4.44)$$

and also determine the shape of the cavity roof using

$$h_{2C}(\xi) = G^+(\xi) + G^-(\xi) = 2G(\xi) \quad \xi \in \Gamma. \quad (4.45)$$

4.4 Determining cavity endpoint positions

The task is to determine the endpoints of the cavities which are not known in advance.

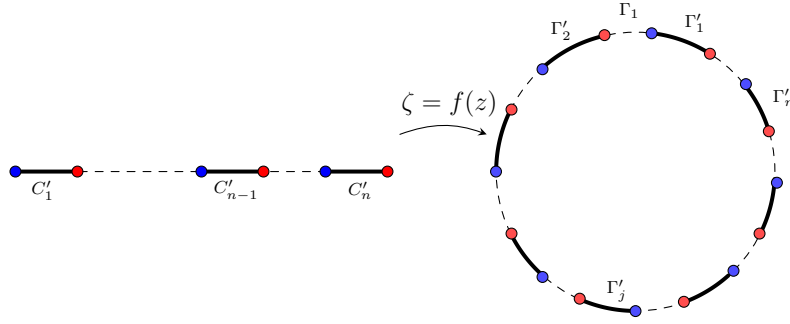


Figure 6: The blue dots correspond to points ξ_{b_i} and the red dots correspond to points ξ_{c_i} . The dashed lines are the cavities and the solid lines are the contact areas. And the union of the Γ_i is the union of disjoint cavities.

The method shown in [11] finds the positions of the endpoints of the cavities in one period. Consider the j -th cavity, then the upstream endpoint of the cavity is labelled by ξ_{b_j} and the downstream endpoint by ξ_{c_j} . Since the period of the bed is a , it is possible to write $\xi_{b_{j+n}} = \xi_{b_j+a}$ and similarly $\xi_{c_{j+n}} = \xi_{c_j+a}$. Assuming that the cavities are disjoint, the endpoints of the cavities are ordered as follows

$$\xi_{c_0} < \xi_{b_1} < \xi_{c_1} < \dots < \xi_{b_n} < \xi_{c_n}, \quad (4.46)$$

where the difference between the first and last endpoints is given by a and so ξ_{c_0} coincides with ξ_{c_n} . Notice that the number of cavity endpoint positions is $2n$ since we have n upstream and n downstream.

Muskhelishvili in [8] presents the solution to the problem

$$G^+(\xi) + G^-(\xi) = h_2(\xi) \quad \xi \in \Gamma', \quad G(0) = \frac{iN}{4u_b}, \quad G(\infty) = -\frac{iN}{4u_b} \quad (4.47)$$

as

$$G(\zeta) = \frac{\chi(\zeta)}{2\pi i} \int_{\Gamma'} \frac{h_2(\xi)}{\chi^+(\xi)(\xi - \zeta)} d\xi + G(\infty)\chi(\zeta), \quad (4.48)$$

where we have $\chi(\zeta) = \prod_{j=1}^n \chi_j(\zeta)$ such that $\chi_j(\zeta) = \left[\frac{\zeta - \xi_{b_j}}{\zeta - \xi_{c_{j-1}}} \right]^{\frac{1}{2}}$. It is also required that $\chi_j(\zeta) \sim 1$ as $\zeta \rightarrow \infty$ so that $G(\zeta) \rightarrow G(\infty)$ as $\zeta \rightarrow \infty$.

Equation (4.48) has n integrable singularities and since we have n cavities, we expect one singularity per cavity located at each cavity endpoint, ξ_{b_j} or ξ_{c_j} .

Consider the integral

$$I = \frac{a}{2\pi i} \int_{\Gamma \cup \Gamma'} \frac{G^+(\xi) + G^-(\xi)}{\xi} d\xi. \quad (4.49)$$

Also recall that the conformal transformation $\xi = \exp\left(\frac{2\pi i x}{a}\right)$ is used and so $\frac{d\xi}{\xi} = \frac{2\pi i}{a} dx$. In addition, for $\xi \in \Gamma$ we have $h_2(\xi) = h''(x) = G^+(\xi) + G^-(\xi)$ and the integral becomes

$$\begin{aligned} I &= \sum_{j=1}^n \int_{b_j}^{c_j} h''_C(x) dx + \sum_{j=1}^n \int_{c_j}^{b_{j+1}} h''(x) dx \\ &= \sum_{j=1}^n [h'_C(c_j) - h'_C(b_j)] + \sum_{j=1}^n [h'(b_{j+1}) - h'(c_j)] \\ &= \sum_{j=1}^n [h'_C(c_j) - h'(c_j)] + \sum_{j=1}^n [h'(b_j) - h'_C(b_j)] \\ &= 0 \end{aligned} \quad (4.50)$$

where in the third equality we use the property $h'(b_{n+1}) = h'(b_1)$ due to the periodicity of the bed. To show the fourth equality we argue using Cauchy's theorem and the boundary conditions

$$G(0) = \frac{iN}{4u_b} \quad \text{and} \quad G(\infty) = -\frac{iN}{4u_b}. \quad (4.51)$$

If L is a closed contour around the origin in Figure 6, then depending on whether L is entirely inside or outside the unit circle and after deforming L to be along the unit circle, we obtain

$$\frac{1}{2\pi i} \oint_L \frac{G(\zeta)}{\zeta} d\zeta = \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma'} \frac{G^+(\xi)}{\xi} d\xi = G(0) \quad \text{and} \quad \frac{1}{2\pi i} \int_{\Gamma \cup \Gamma'} \frac{G^-(\xi) - G(\infty)}{\xi} d\xi = 0 \quad (4.52)$$

respectively. Using the boundary conditions (4.51), Equation (4.49) becomes

$$I = a[G(0) + G(\infty)] = 0. \quad (4.53)$$

Now let us determine what happens to the terms in Equation (4.50). At the cavity endpoints, the cavity roof must coincide with the height of the bed so $h_C(x) = h(x)$ for all $x \in \{b_j, c_j\}$ and for all j . Also, in the cavities, the height of the cavity roof must be greater than $h(x)$. So at the endpoints b_j we have $h'_C(b_j) \geq h'(b_j)$ and at c_j we have $h'_C(c_j) \leq h'(c_j)$. This shows that we require the terms in the parentheses in (4.50) to vanish, leading to $h'_C(x) = h'(x)$ for all $x \in \{b_j, c_j\}$ for all j . Therefore, the cavity roof and the bed of the glacier meet tangentially at the cavity endpoints.

Integrating $h''_C(x) = h_{2C}(\xi) = G^+(\xi) + G^-(\xi) = 2G(\xi)$ with respect to x and using $\xi = \exp\left(\frac{2\pi ix}{a}\right)$, yields

$$\int_{b_j}^{c_j} h''_C(x) dx = \int_{b_j}^{c_j} 2G\left(\exp\left(\frac{2\pi ix}{a}\right)\right) dx \quad (4.54)$$

which is

$$h'_C(c_j) - h'_C(b_j) = h'(c_j) - h'(b_j) = \int_{b_j}^{c_j} 2G\left(\exp\left(\frac{2\pi ix}{a}\right)\right) dx \quad (4.55)$$

for $j = 1, \dots, n-1$. If we integrate this a second time with respect to x we get

$$h(c_j) - h'(b_j)(c_j - b_j) - h(b_j) = \int_{b_j}^{c_j} 2(c_j - x)G\left(\exp\left(\frac{2\pi ix}{a}\right)\right) dx \quad (4.56)$$

for $j = 1, \dots, n-1$. For further details on how to derive this, see Appendix C. Also, note that G is a real function on $\xi \in \Gamma$ and so Equations (4.55) and (4.56) represent $2n-1$ real constraints on the positions of the $2n$ cavity endpoints. We need an extra real constraint to determine the position of the cavity endpoints. This comes from the requirement that $G(\zeta) = \overline{G\left(\frac{1}{\zeta}\right)}$. For details on this, see [10].

5 Conclusion

Glacier sliding has been studied extensively over the years and various mathematicians and glaciologists have contributed to our understanding of the topic. At the beginning, the models devised considered only simple bed geometries and neglected many processes like cavitation and regelation. This article is meant as a summary of the evolution of sliding laws and deals mainly with the problem of glacier sliding with cavitation over beds, allowing for a finite number of cavities per bedrock period as shown in [10].

Iken first used a simple staircase model to show that the basal shear stress is bounded and her bound was determined by the bed geometry and the effective pressure. This model is not only rather unrealistic, but also breaks down in the case

of a bed obstacle having a face perpendicular to the ice flow, since then the basal shear stress becomes unbounded ($\frac{\tau_b}{N} \leq \tan \beta = \tan\left(\frac{\pi}{2}\right) = \infty$). Schoof's derivation for more general bed geometries requires slopes which are bounded and therefore cannot resolve the issue in Iken's method.

Fowler first reformulated the sliding process in terms of complex variables and reduced this problem to a Hilbert problem. The results found, showed that the basal shear stress decreases to zero as the velocity tends to infinity. Schoof has also verified his results using numerical methods. His results are shown in Figure 7. A parameter α is used to describe the irregularity of the bed geometry.

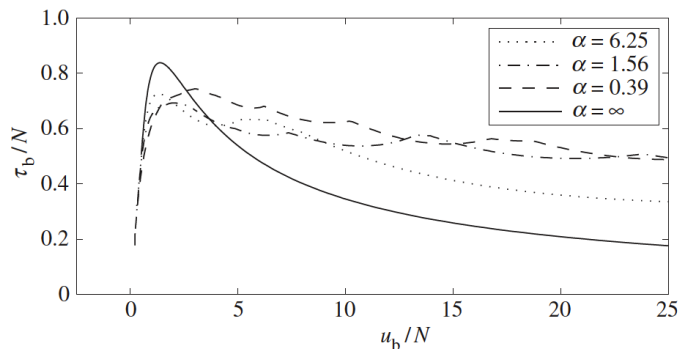


Figure 7: Sliding laws calculated for beds of different shapes and after the onset of cavitation. More irregular shapes have smaller α and $\alpha = \infty$ corresponds to a sinusoidal bed. Results agree with Iken's bound since $\frac{\tau_b}{N}$ is bounded. For details of the numerical procedures used see [10] and [11].

To conclude, Lliboutry was the first to introduce the effect of cavitation in glacier sliding, and then Fowler and Schoof showed that the sliding law takes the form

$$\frac{\tau_b}{N} = f\left(\frac{u_b}{N}\right). \quad (5.1)$$

They have shown that the basal shear stress and extent of cavitation increase with sliding velocity up to a global maximum and then begin to drop.

Acknowledgements. I would like to thank Professor Ian Hewitt for suggesting the topic and introducing me to the field of Mathematical Geoscience.

Appendix A

In this Appendix, we include some preliminaries for complex variables. Let us represent a complex number by $z = x + iy$ where x and y are real numbers. Then its complex conjugate is $\bar{z} = x - iy$.

Using chain rule, we can write

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x} \frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x} \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \quad (\text{A.1})$$

and similarly

$$\frac{\partial}{\partial y} = i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right). \quad (\text{A.2})$$

Assuming that the flow is incompressible, we have $\nabla \cdot \underline{u} = 0$. This is equivalent to $u_x + v_y = 0$. This implies that there exists a stream function $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

This stream function satisfies the biharmonic equation

$$\nabla^4 \psi(x, y) = 0 \quad \text{for } y \in (0, \infty).$$

Lemma A.1. Let f be a holomorphic function on a connected open set $U \subset \mathbb{C}$. If f is real valued, then it is constant on U .

Proof. If $f = u + iv$ is real valued then $v \equiv 0$ on U . By the Cauchy-Riemann equations, we have that on U

$$u_x = v_y = 0 \quad \text{and} \quad u_y = -v_x = 0. \quad (\text{A.3})$$

Therefore, $f'(z) = u_x + iv_y = 0$ on U and so $f = \text{constant}$ because U is a connected open set. \square

A.1 Functions given on a half-plane

In this article, we denote the upper half-plane by S^+ and the lower half-plane by S^- . Let $F(z)$ be a function of the point z in S^+ . If we want to relate this to a function defined in S^- then we can define a new function as

$$F_*(z) := \overline{F(\bar{z})}. \quad (\text{A.4})$$

Thus, by definition we have that $F(z)$ and $F_*(z)$ take conjugate complex values at points $z = x + iy$ and $\bar{z} = x - iy$ which are the reflections of each other in the x -axis.

We have

$$F(z) = u(x, y) + iv(x, y) \quad (\text{A.5})$$

and

$$\overline{F(\bar{z})} = \overline{u(x, -y) + iv(x, -y)} = u(x, -y) - iv(x, -y) \quad (\text{A.6})$$

where u, v are real functions.

$F(z)$ is assumed to take a definite limiting value $F^+(t)$ as $z \rightarrow t$ from S^+ , where t is a point on the x -axis, i.e. a real number.

Then also $F_\star^-(t)$ exists and

$$F_\star^-(t) = \overline{F^-(t)} = \overline{F^+(t)} \quad (\text{A.7})$$

because for $z \rightarrow t$ from S^- , $\bar{z} \rightarrow t$ from S^+ , and hence $F_\star(z) = \overline{F(\bar{z})}$ tends to $\overline{F^+(t)}$.

Appendix B

The functions considered in this article are all considered to satisfy the Hölder condition and here we give a definition of this as presented by Muskhelishvili in Chapter 1 of [8].

Definition B.1. Let there be given on the arc L a function of position $\phi(t)$ (which is in general complex). Also, let t denote both the point $t(x, y)$ and the corresponding complex number $t = x + iy$.

The function $\phi(t)$ is said to satisfy the *Hölder condition* on L , if for any two points t_1, t_2 of L

$$|\phi(t_2) - \phi(t_1)| \leq A|t_2 - t_1|^\mu, \quad (\text{B.1})$$

where A is the Hölder constants and μ is the Hölder index and are both positive constants.

Appendix C

For the integral (4.56) we use

$$\int_b^c \int_b^x f(x') dx' dx = \int_b^c (c-x)f(x) dx. \quad (\text{C.1})$$

To show this, first consider Figure 8.

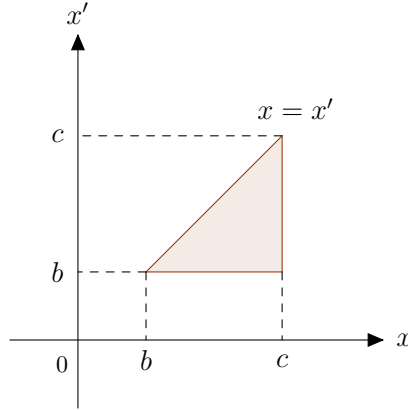


Figure 8: The region is given by $b \leq x \leq c$ and $b \leq x' \leq x$ which is equivalent to $b \leq x' \leq c$ and $x' \leq x \leq c$.

Therefore we have

$$\int_b^c \int_b^x f(x') dx' dx = \int_b^c \int_{x'}^c f(x') dx dx' = \int_b^c (c - x') f(x') dx' = \int_b^c (c - x) f(x) dx \quad (\text{C.2})$$

where in the second equality we use the fact that f is independent of x .

C.1 Cauchy stress tensor

The stress tensor consists of nine components t_{ij} that define the state of stress at a point inside a material,

$$\mathcal{T} = \begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix}. \quad (\text{C.3})$$

The diagonal entries are called the normal stresses. All the non-diagonal components of the stress tensor are the tangential and shear stresses.

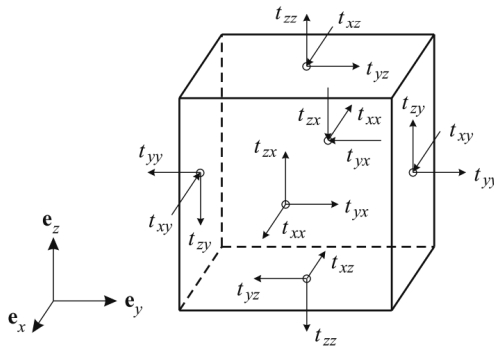


Figure 9: Components of the Cauchy stress tensor [3].

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