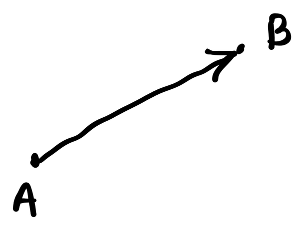
Vectors and vector functions

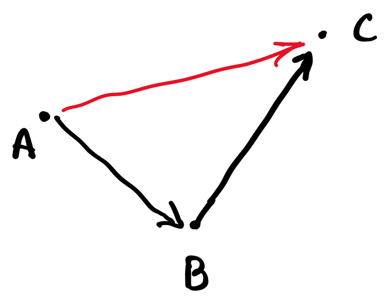
Friday, July 31, 2020 9:00 AM

The term vector is used to describe a quantity that has both magnitude and direction



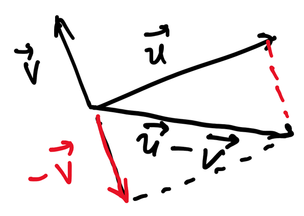
the directed line segment is the vector

$\vec{v} = \vec{AB}$

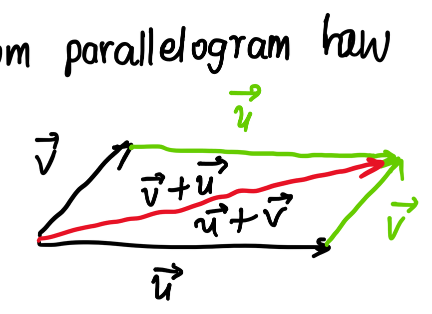


$\vec{AC} = \vec{AB} + \vec{BC}$

You can subtract vectors.



from parallelogram law

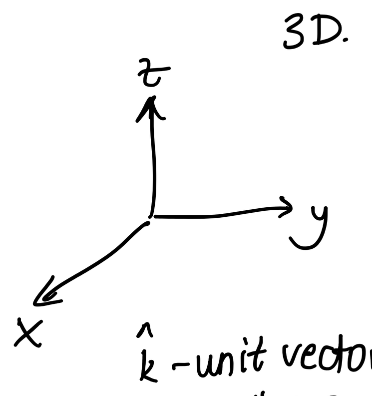


Vector-valued function is a set of vectors. It's a function whose domain is a set of real numbers and whose range

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k} = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$

where $f(t), g(t), h(t)$ are the components of the vector $\vec{r}(t)$ and t is the independent variable.

3D.



$\hat{k}$ - unit vector in the z-direction

$\hat{j}$ - unit vector in the y-direction

$\hat{i}$ - unit vector in the x-direction

Example $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$. $f(t) = t^3$, $g(t) = \ln(3-t)$, $h(t) = \sqrt{t}$

$3 - t > 0 \Rightarrow t < 3$

$t \geq 0 \Rightarrow$ $t \in [0,3)$

t is an element of the interval $[0,3)$

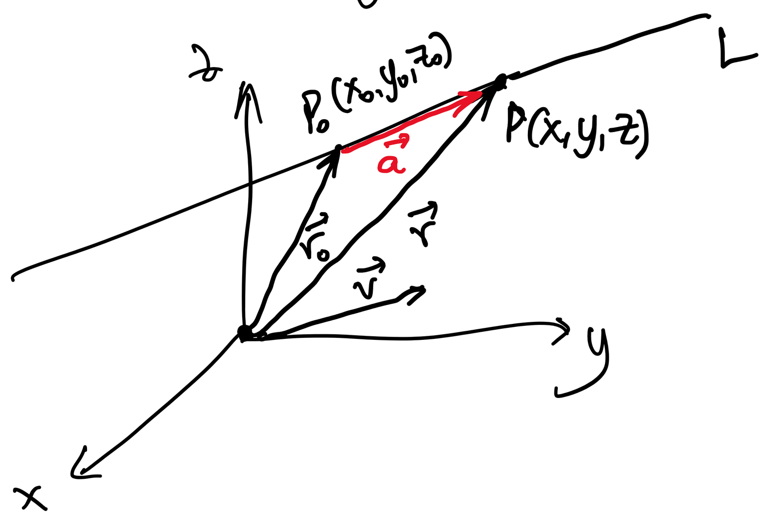
Equation of a line

As in 2D space, a line in 3D is determined when we know

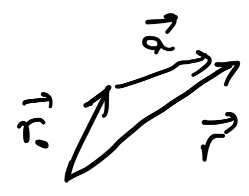
- a point $P_0(x_0, y_0, z_0)$ on L
- the direction of L (its slope).

In 3D the direction of a line is described by a vector, so we let $\vec{v}$ be a vector parallel to L.

Let $P(x,y,z)$ be a point on L and let $\vec{r}_0$ and $\vec{r}$ be the position vectors of P_0 and P.



If $\vec{a}$ is the vector for $\vec{P_0P}$ then $\vec{r} = \vec{r}_0 + \vec{a}$



But $\vec{a}$ and $\vec{v}$ are parallel and so $\vec{a} = t\vec{v}$

($\vec{a}$ is a scalar multiple of $\vec{v}$)

Use $\vec{r} = \vec{r}_0 + \vec{a}$ and $\vec{a} = t\vec{v}$ to write

$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$ vector equation for a line

If $\vec{r} = \langle x, y, z \rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\vec{v} = \langle a, b, c \rangle$ then

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the direction vector (a, b, c).

EXAMPLE 1

(a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

(b) Find two other points on the line.

a) $\vec{r} = \vec{r}_0 + t\vec{v}$ where $\vec{r}_0 = \langle 5, 1, 3 \rangle$ and $\vec{v} = \langle 1, 4, -2 \rangle$

$\vec{r} = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle$

$= \langle 5+t, 1+4t, 3-2t \rangle$

b) Choose a parameter $t = 1$ $x = 6, y = 5, z = 1$ so $(6, 5, 1)$ is a point on L

and similarly $t = -1$ $x = 4, y = -3, z = 5$ so $(4, -3, 5)$ is another point on L.

Showing if two lines intersect

Example $\vec{r}_1 = (3\hat{i} + 8\hat{j} - 2\hat{k}) + t(2\hat{i} - \hat{j} + 3\hat{k}) = \begin{pmatrix} 3 \\ 8 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$\vec{r}_2 = (7\hat{i} + 4\hat{j} + 3\hat{k}) + s(2\hat{i} + \hat{j} + 4\hat{k}) = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

At the intersection point $\begin{pmatrix} 3+2t \\ 8-t \\ -2+3t \end{pmatrix} = \begin{pmatrix} 7+2s \\ 4+s \\ 3+4s \end{pmatrix}$ $\leftarrow x$, $\leftarrow y$, $\leftarrow z$

Equate the x and y-components :

$3 + 2t = 7 + 2s \rightarrow 3 + \cancel{2t} = 7 + 2s$

$8 - t = 4 + s \rightarrow 16 - \cancel{2t} = 8 + 2s$ +

$19 = 15 + 4s$

$4 = 4s$

$s = 1$

$8 - t = 4 + 1$

$t = 3$

Check that the z-components are also equal

$-2 + 3t = -2 + 9 = 7$ ✓

$3 + 4s = 3 + 4 = 7$ ✓

If the z-coordinate does not agree then the lines do not intersect!

Intersection point has position vector $\begin{pmatrix} 3+2t \\ 8-t \\ -2+3t \end{pmatrix}$ with $t = 3 \Rightarrow \vec{r} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$