Tangent planes

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Suppose a surface S has equation $z = f(x,y)$ and let $P(x_0, y_0, z_0)$ be a point on S (see figure). Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at point P. Then the tangent plane to the surface S at P is the plane that contains both tangent lines T_1 and T_2.

Eqn of a plane through the point $P(x_0, y_0, z_0)$ is of the form

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

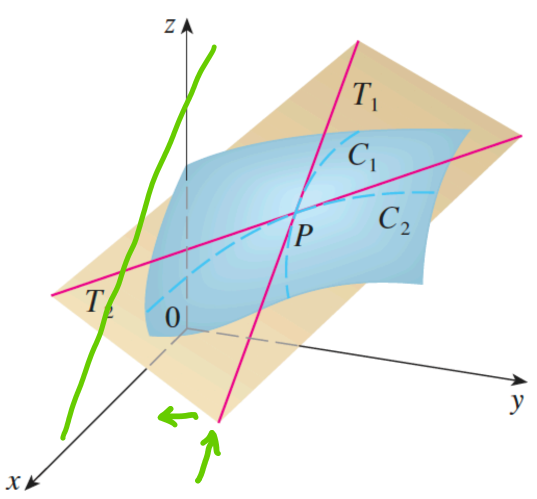


FIGURE 1
The tangent plane contains the tangent lines T_1 and T_2.

Divide through by C: $\frac{A}{C}(x-x_0) + \frac{B}{C}(y-y_0) + z - z_0 = 0$

$$z - z_0 = -\frac{A}{C}(x-x_0) - \frac{B}{C}(y-y_0)$$

let $a = -\frac{A}{C}$ and $b = -\frac{B}{C}$

$$\Rightarrow \quad z - z_0 = a(x-x_0) + b(y-y_0) \qquad (*)$$

If $(*)$ represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be tangent to T_1.

If we substitute $y = y_0$ into $(*)$ we obtain

$$z - z_0 = a(x - x_0) \quad \leftarrow \text{point slope formula of a line with slope } a.$$

The slope of this tangent line is $f_x(x_0, y_0)$ and thus $a = f_x(x_0, y_0)$

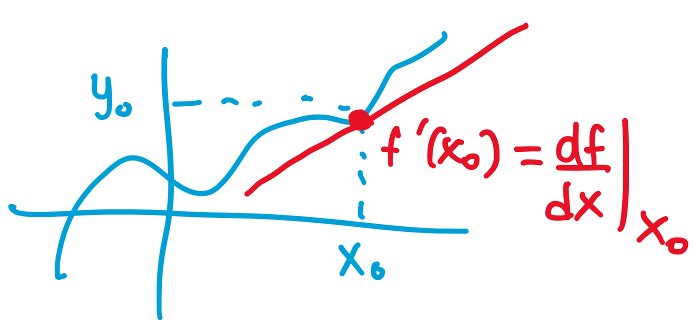
Similarly $b = f_y(x_0, y_0)$.

Thus, $(*)$ becomes

$$\boxed{z - z_0 = \underbrace{f_x(x_0, y_0)}_{\text{evaluate } \frac{\partial f}{\partial x} \text{ at } (x_0, y_0)} \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0)}$$

equation of a tangent plane through (x_0, y_0, z_0)

$$y - y_0 = \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0)$$



<u>Example</u> Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $\underbrace{(1,1,3)}_{(x_0, y_0, z_0)}$

Let $\underline{z = f(x,y)} = 2x^2 + y^2$

$\frac{\partial f}{\partial x} = f_x = 4x \qquad \frac{\partial f}{\partial x}(x_0, y_0) = 4(1) = 4$

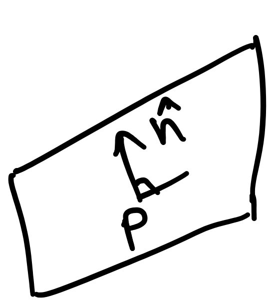
$\frac{\partial f}{\partial y} = f_y = 2y \qquad \frac{\partial f}{\partial y}(x_0, y_0) = 2(1) = 2$

$$\underbrace{z}_{} - \underbrace{3}_{z_0} = \underbrace{4}_{f_x(x_0,y_0)}(x - \overset{x_0}{1}) + \underbrace{2}_{f_y(x_0,y_0)}(y - \overset{y_0}{1})$$

where $(x_0, y_0, z_0) = (1,1,3)$

$$z - 3 = 4x - 4 + 2y - 2$$

$$\boxed{z = 4x + 2y - 3} \quad \text{tangent plane}$$



$$x \cdot \hat{n} = p \cdot \hat{n}$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 1(2) + 2(-1) + 3(0)$$

$$= 2 - 2 + 0$$

$$= 0$$