Polar coordinates

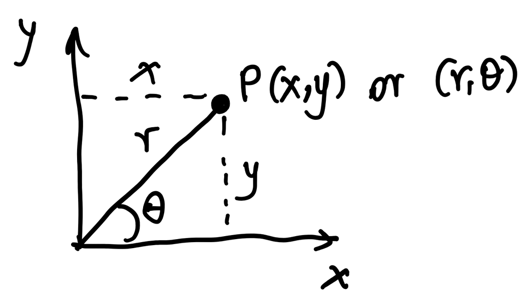
Tuesday, July 21, 2020 9:11 PM

Polar coordinates are an alternative way of describing a point P in a two-dimensional space.

You need two measurements to describe the position of this point.

a) the distance from the pole (usually the origin O), r

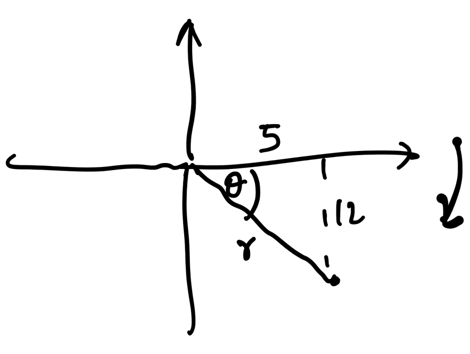
b) the angle measured anticlockwise from the initial line (usually the x-axis), θ



To convert between Cartesian coordinates and polar coordinates.

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}} \longleftrightarrow \boxed{\begin{aligned} r^2 &= x^2+y^2 \\ \theta &= \arctan\left(\tfrac{y}{x}\right) \end{aligned}}$$

e.g Find the polar coordinates of the point $(x,y) = (5,-12)$



$r = \sqrt{12^2+5^2} = 13$

$\theta = -\arctan\left(\frac{12}{5}\right) = -1.176$

$(r,\theta) = (13, -1.176)$.

e.g. Find the Cartesian coords of $(r,\theta) = \left(10, \frac{4\pi}{3}\right)$

$x = r\cos\theta = 10\cos\left(\frac{4\pi}{3}\right) = -5$

$y = r\sin\theta = 10\sin\left(\frac{4\pi}{3}\right) = -5\sqrt{3}$

$(x,y) = (-5, -5\sqrt{3})$

Polar equations of curves are usually given by $r = f(\theta)$. For example $r = 1+2\cos\theta$, $r=3$, $r = 2\sin\theta$, etc.

e.g Find the Cartesian equation of $r = 2+\cos 2\theta$ Use identity $\cos 2\theta = 2\cos^2\theta - 1$

$r = 2+\cos 2\theta$

$r = 2+2\cos^2\theta - 1$

$r = 1+2\cos^2\theta$

Use $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2+y^2$

Multiply both sides by r^2

$r^3 = r^2 + 2\underbrace{r^2\cos^2\theta}_{x^2}$ use $x = r\cos\theta$

$(x^2+y^2)^{3/2}$ $\quad x^2+y^2$

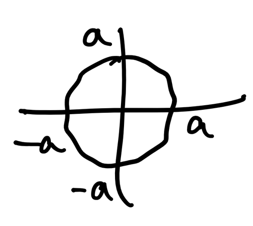
$(x^2+y^2)^{3/2} = x^2+y^2+2x^2$

$\boxed{(x^2+y^2)^{3/2} = 3x^2+y^2}$

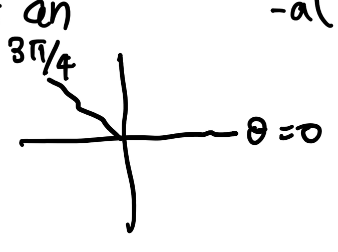
Sketching polar curves

Standard curves:

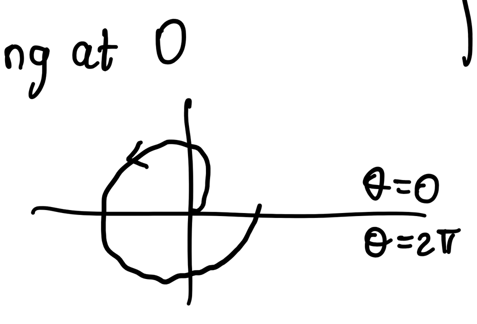
- $r=a$ is a circle of radius a centered at the origin



- $\theta = \alpha$ is a half-line through O and that makes an angle α w/ the x-axis. e.g. $\theta = \frac{3\pi}{4}$



- $r = a\theta$. This is a spiral starting at O

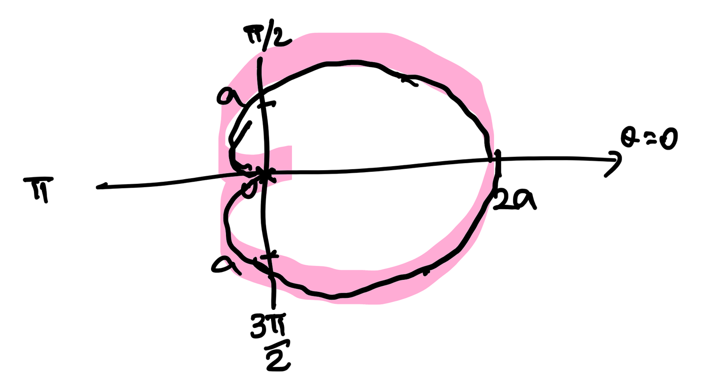


Note

You can sketch curves by drawing up a table of values of r for particular values of θ. It's common to use values of θ that result in positive r.

e.g Sketch $r = (1+\cos\theta)a$

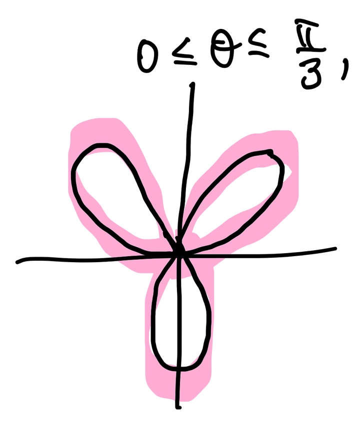
θ	0	$\pi/2$	π	$3\pi/2$	2π
r	$2a$	a	0	a	$2a$



Cardioid.

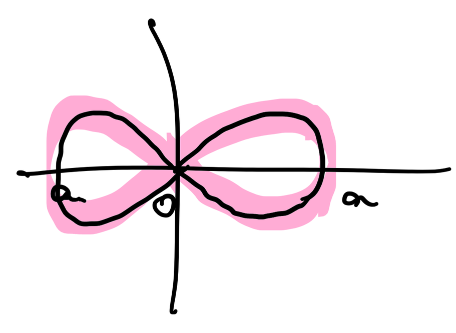
Jamboards

b) $r = a\sin 3\theta$, $\quad 0 \le \theta \le \frac{\pi}{3}$, $\frac{2\pi}{3} \le \theta \le \pi$, $\frac{4\pi}{3} \le \theta \le \frac{5\pi}{3}$ give positive r



symmetric

c) $r^2 = a^2\cos 2\theta$ $\quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, $\frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}$.

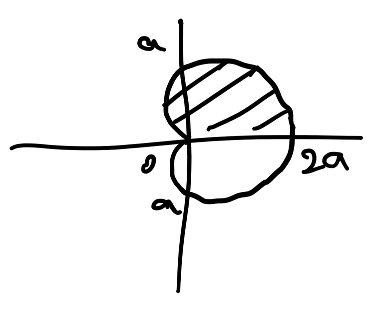


Area enclosed by polar curve

The area of a sector bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$, where θ is in radians, is given by

$$\left[\text{Area} = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta = \frac{1}{2}\int_\alpha^\beta (f(\theta))^2\,d\theta \right]$$ (recall curve is given by $r = f(\theta)$).

e.g Find the area enclosed by $r = a(1+\cos\theta)$



$\text{Area} = 2\left(\frac{1}{2}\int_0^\pi (a(1+\cos\theta))^2\,d\theta\right)$

Tangents of polar curves.

If you are given a curve $r = f(\theta)$ then you can use

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

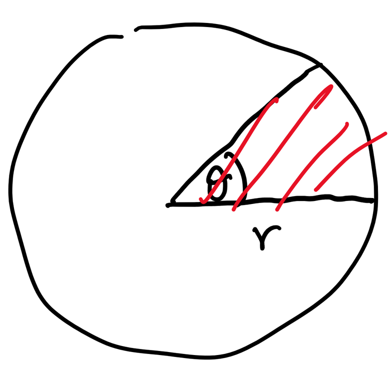
Parametric eqns: $\boxed{\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}}$

When $\frac{dy}{d\theta} = 0$, the tangent to the curve is horizontal

When $\frac{dx}{d\theta} = 0$, the tangent to the curve is vertical.

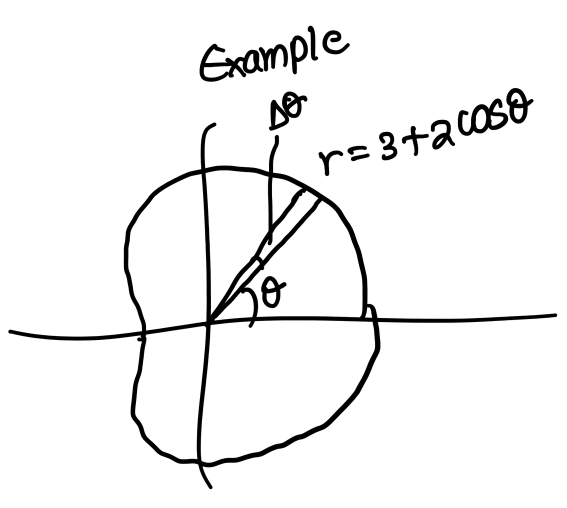
Deriving the area in polar coordinates

$\text{Area} = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$



area of sector $= \frac{\theta}{2\cancel{\pi}}\cancel{\pi} r^2 = \frac{1}{2}r^2\theta$

circle.



Area of sector $\approx \frac{1}{2}r^2\Delta\theta$

$= \frac{1}{2}(3+2\cos\theta)^2\Delta\theta$

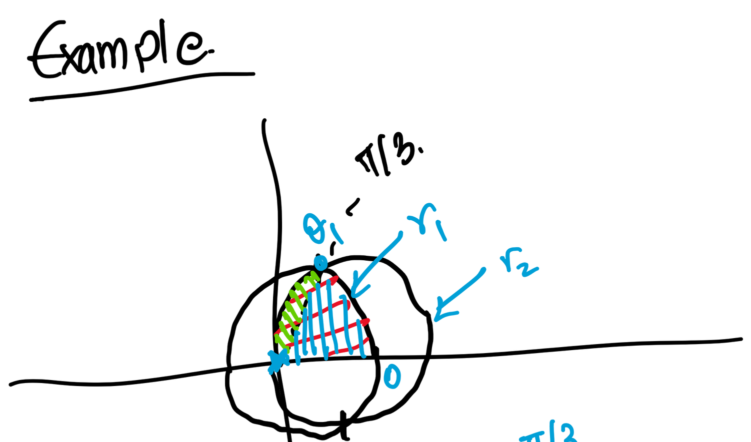
Area of the whole region is

$\sum \frac{1}{2}(3+2\cos\theta)^2\,\Delta\theta$

As $n \to \infty$ and $\Delta\theta \to 0$

$\text{Area} = \int_0^{2\pi} \frac{1}{2}(3+2\cos\theta)^2\,d\theta$

Example



$\frac{1}{2}\int_0^{\pi/3} r_1^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} r^2\,d\theta$