Partial fractions

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The integrand of some rational functions can be obtained by splitting the integrand into partial fractions.

e.g. Find $\int \frac{1}{(x-3)(x-7)}\,dx$

Write $\frac{1}{(x-3)(x-7)} = \frac{A}{x-3} + \frac{B}{x-7}$ where A and B are constants to be found.

$= \frac{A(x-7)+B(x-3)}{(x-3)(x-7)}$

Get identity $1 = A(x-7) + B(x-3)$ ☆

Two ways:

① Eliminate B and solve for A:
let $x=3 \Rightarrow 1 = A(3-7) \Rightarrow 1 = -4A \Rightarrow A = -\frac{1}{4}$

② Eliminate A and solve for B:
let $x=7 \Rightarrow 1 = B(7-3) \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$

OR Equate coefficients (☆) $1 = (A+B)x - 7A - 3B$

const: $1 = -7A - 3B$
x: $0 = A+B \Rightarrow A = -B$

$1 = -7(-B) - 3B = 4B \Rightarrow B = 1/4$

$A = -1/4$

$\int \frac{1}{(x-3)(x-7)}\,dx = \int \left(\frac{-1/4}{x-3} + \frac{1/4}{x-7}\right)dx$

$= -\frac{1}{4}\ln|x-3| + \frac{1}{4}\ln|x-7| + C$

$= \frac{1}{4}\ln\left|\frac{x-7}{x-3}\right| + C$

$\ln|A| - \ln|B| = \ln\left|\frac{A}{B}\right|$

$\ln|A-B| \neq \frac{\ln|A|}{\ln|B|}$

e.g. $\int \frac{x}{(x-1)^2(x-2)}\,dx = \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}\right)dx$

Multiply through by $(x-1)^2(x-2)$:

$x = A(x-1)(x-2) + B(x-2) + C(x-1)^2$

let $x=2 \Rightarrow 2 = C(2-1)^2 \Rightarrow 2 = C$

let $x=1 \Rightarrow 1 = B(1-2) \Rightarrow B = -1$

let $x=0 \Rightarrow 0 = A(-1)(-2) + B(-2) + C(1)$ $\quad (0-1)^2 = (-1)^2 = 1$

$\Rightarrow 0 = 2A + 2 + 2$

$2A = -4$

$A = -2$

$\int \left(\frac{-2}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{x-2}\right)dx = -2\ln|x-1| + (x-1)^{-1} + 2\ln|x-2| + C$

$\int -(x-1)^{-2}\,dx$

$= 2\ln\left|\frac{x-2}{x-1}\right| + \frac{1}{x-1} + C$

Examples from Jamboard

(40) $\int \frac{1}{(x+7)(x-2)}\,dx = -\frac{1}{9}\ln|x+7| + \frac{1}{9}\ln|x-2| + C$

(44) $\int \frac{1}{3P - 3P^2}\,dP = \int \frac{1}{3P(1-P)}\,dP = \int \frac{A}{3P} + \frac{B}{1-P}\,dP$ where A & B are constants to be found

$1 = A(1-P) + B(3P)$

let $P=1 \Rightarrow 1 = 3B \Rightarrow B = 1/3$

let $P=0 \Rightarrow 1 = A$

$\int \left(\frac{1}{3P} + \frac{1}{3}\frac{1}{1-P}\right)dP = \frac{1}{3}\ln|P| - \frac{1}{3}\ln|1-P| + C$

check answer

$\frac{d}{dP}\left(\frac{1}{3}\ln|P| - \frac{1}{3}\ln|1-P| + C\right)$

$= \frac{1}{3}\frac{1}{P} - \frac{1}{3}\frac{1}{1-P}(-1) = \frac{1}{3P} + \frac{1}{3}\frac{1}{1-P}$ ✓

chain rule

(45) $\int \frac{3x+1}{x^2-3x+2}\,dx = \int \frac{3x+1}{(x-2)(x-1)}\,dx$

$A = 7$

$B = -4$

$\frac{A}{x-2} + \frac{B}{x-1} = \frac{3x+1}{(x-2)(x-1)}$

Final answer

$7\ln|x-2| - 4\ln|x-1| + C$