Partial derivatives

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Suppose f is a function of two variables x and y. If only x varies and y is constant, say at $y=b$ then we're considering a function of one variable x, i.e. $g(x) = f(x,b)$

If g has a derivative at a, then we call it the partial derivative of f wrt x at (a,b) and denote it by $f_x(a,b) = \frac{\partial f}{\partial x}(a,b)$

Thus $f_x(a,b) = g'(a)$ where $g(x) = f(x,b)$

By definition a derivative is $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h} = g'(a)$

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Similarly, the partial derivative of f wrt y at (a,b) (keep constant $x=a$)

$$\left(\frac{\partial f}{\partial y}(a,b)\right) f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Rule for finding the partial derivatives

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ wrt x
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ wrt y

Example If $f(x,y) = x^3 + x^2y^3 - 2y^2$ find $f_x(2,1)$ and $f_y(2,1)$

$f_x(x,y) = 3x^2 + 2xy^3$

$f(x,y) = x^3 + x^2 y^3 - 2y^2$ "constants"

$f_x(x,y)$ keep y as a constant and treat x as a variable.

$f_x(x,y) = 3x^2 + 2xy^3 \cancel{+0}$

$f_y(x,y) = 3x^2y^2 - 4y$

$f_x(2,1) = 3(2)^2 + 2(2)(1)^3 = 12 + 4 = 16$

x y

$f_y(2,1) = 3(2)^2(1)^2 - 4(1) = 8$

$f(T,H)$

Table 1 Heat index I as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°F) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Find the partial derivative when $H = 70\%$

$\boxed{f_H(96,70)}$ keep 1st variable constant & vary the second.

Use limit definition of a derivative

$f_H(96,70) = \lim_{h\to 0} \frac{f(96,70+h) - f(96,70)}{h}$

$f(T,H)$

take $H = \pm 5$

$H=5$

$f_H(96,70) \approx \frac{f(96,75) - f(96,70)}{5}$

$= \frac{130 - 125}{5} = 1$

used $H=-5$

$f_H(96,70) \approx \frac{f(96,65) - f(96,70)}{-5}$

$= \frac{121-125}{-5} = 0.8$

average the two

$f_H(96,70) = \frac{0.8+1}{2} = 0.9$

Interpretation When the temperature is 96°F and the relative humidity is 70%, the heat index increases by about 0.9°F for every percent that the relative humidity rises.

NB $D_x f = f_x = \frac{\partial f}{\partial x}$

$\frac{df}{dx}$ if f is a function of a single variable

$\frac{d^2f}{dx^2}, \frac{d^3f}{dx^3}, \ldots$

Higher derivatives

$(f_x)_x = \frac{\partial^2 f}{\partial x^2}$ $(f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$

$(f_y)_y = \frac{\partial^2 f}{\partial y^2}$ $(f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$ ← first differentiate wrt y and then wrt x

e.g $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3$

$f_y = 3x^2y^2 - 4y$

$f_{xx} = 6x + 2y^3$

$f_{yy} = 6x^2y - 4$

$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 + 2xy^3\right) = 0 + 6xy^2$

$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x^2y^2 - 4y\right) = 6xy^2$

Theorem Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D then $f_{xy}(a,b) = f_{yx}(a,b)$

$\left(\frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial^2 f}{\partial x \partial y}\right)$

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notations for Partial Derivatives If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$