Parametric equations

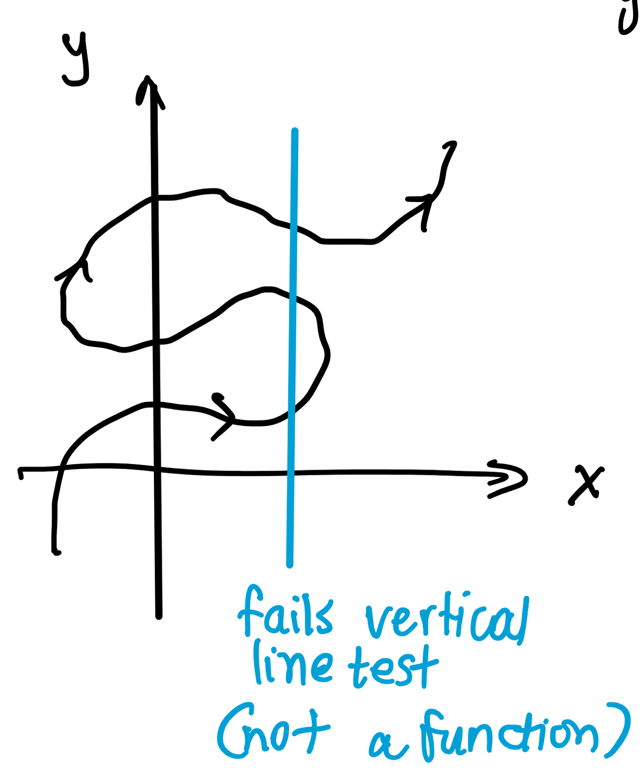
Saturday, July 18, 2020 11:00 PM

So far you have described curves by giving y as a function of x $(y=f(x))$ or by implicitly defining y as a function of x $(f(x,y)=0)$.

Some curves are best handled when x and y are both given in terms of a third variable t (we call this the parameter)

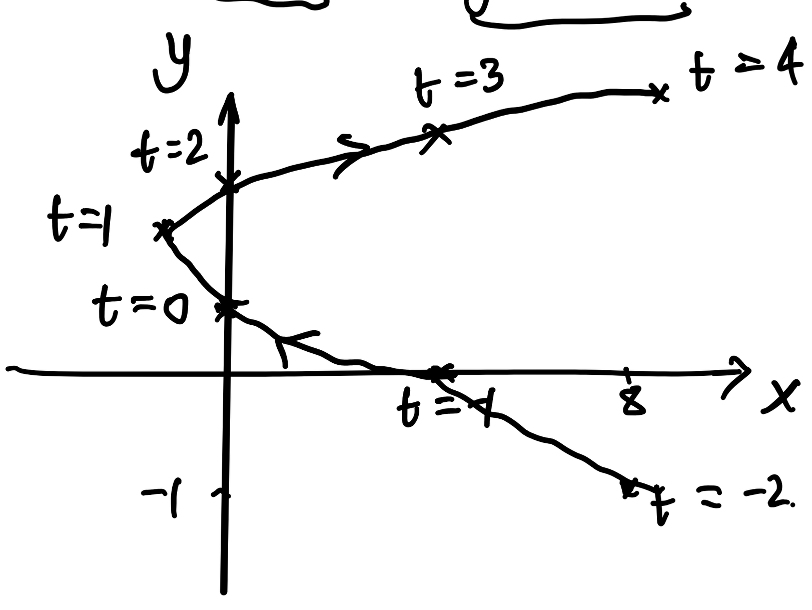
Parametric equations: $\boxed{x=f(t) \quad , \; y=g(t)}$

For each value of t you get an x and y and you can plot this



Example Sketch and identify the curve given by $\underbrace{x=t^2-2t}$, $\underbrace{y=t+1}$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



$\underline{x=f(y)}$

$x=t^2-2t \qquad y=t+1 \Rightarrow \boxed{t=y-1}$

$x=(y-1)^2-2(y-1)$

$=y^2-2y+1-2y+2$

$\boxed{x=y^2-4y+3}$

Speed and velocity

Recall $x=f(t)$, $y=g(t)$

The instantaneous speed of a moving particle is defined to be

$$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

where $\frac{dx}{dt}$ represents the instantaneous velocity in the x-direction

$\frac{dy}{dt}$ 〃 y-direction

velocity vector

$v = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$

unit vectors

Example

As a particle moves in the xy-plane with $x=2t^3-9t^2+12t$ and $y=3t^4-16t^3+18t^2$, where t is time, find:

(a) At what times is the particle stopped

(b) At what times is the particle moving parallel to the x- or y-axis

(c) The speed of the particle at time t.

Solution: (a) Both $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=0$ for particle to be stopped.

$x=2t^3-9t^2+12t \Rightarrow \frac{dx}{dt}=6t^2-18t+12=6(t-2)(t-1)$

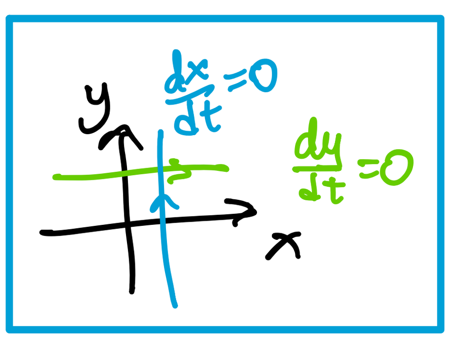
$y=3t^4-16t^3+18t^2 \Rightarrow \frac{dy}{dt}=12t^3-48t^2+36t=12t(t^2-4t+3)$

$=12t(t-3)(t-1)$

$\frac{dx}{dt}=0 \Rightarrow t=2, t=1$

$\frac{dy}{dt}=0 \Rightarrow t=0,3,1$ } stopped at $t=1$

(b)



$\frac{dx}{dt}=0$ when $t=2,1$

$\frac{dy}{dt}=0$ when $t=0,\cancel{1},3$

Particle is parallel to the x-axis if $\frac{dy}{dt}=0$ and $\frac{dx}{dt}\neq 0$ so when $\boxed{t=0,3}$

Particle is parallel to the y-axis if $\frac{dx}{dt}=0$ and $\frac{dy}{dt}\neq 0$ so when $\boxed{t=2}$

(c) speed $=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{(6t^2-18t+12)^2+(12t^3-48t^2+36t)^2}$

Slope & concavity of parametric curves

Slope from chain rule is given by $\boxed{\frac{dy}{dx}=\frac{dy/dt}{dx/dt}}$

$y=h(x)$ and $x=f(t)$, $y=g(t)$

$\underline{y(t)=y(x(t))} \Rightarrow \frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$

Rearranging $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Concavity (you need the 2nd derivative, $\frac{d^2y}{dx^2}$).

$w=\frac{dy}{dx}$

If you are given $\underline{w=\frac{dy}{dx}}$, then $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dw}{dx}=\frac{dw/dt}{dx/dt}$

$w=\frac{dy}{dx}$

Concavity: $\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$