More on polar coordinates

Thursday, July 23, 2020 3:18 PM

Arc length in polar coordinates

We can calculate the arclength of the curve $r=f(\theta)$ by expressing x and y in terms of θ as a parameter

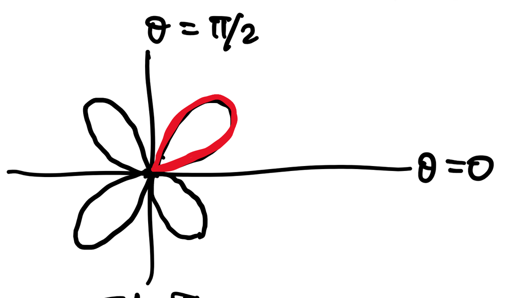
$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

and using the formula for arclength in parametric equations

$$\boxed{\text{arclength} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta}$$

Example Find the arclength of one petal of the "rose" curve $\boxed{r = 3\sin 2\theta} \leftarrow f(\theta)$ $\quad f'(\theta) = 6\cos 2\theta$

for $0 \le \theta \le \frac{\pi}{2}$



$$x = r\cos\theta = 3\sin 2\theta\cos\theta$$
$$y = r\sin\theta = 3\sin 2\theta\sin\theta$$

$$\text{arclength} = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/2} \sqrt{(6\cos 2\theta\cos\theta - 3\sin 2\theta\sin\theta)^2 + (6\cos 2\theta\sin\theta + 3\sin 2\theta\cos\theta)^2}\, d\theta$$

$$\frac{dx}{d\theta} = 6\cos 2\theta\cos\theta - 3\sin 2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 6\cos 2\theta\sin\theta + 3\sin 2\theta\cos\theta$$

OR $\displaystyle\int_0^{\pi/2} \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta$

$$= \int_0^{\pi/2} \sqrt{(6\cos 2\theta)^2 + (3\sin 2\theta)^2}\, d\theta$$

The calculations can be simplified if we use instead

$$\text{arclength} = \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta$$

Proof

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \quad \text{using product rule}$$
$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\text{arclength} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2$$

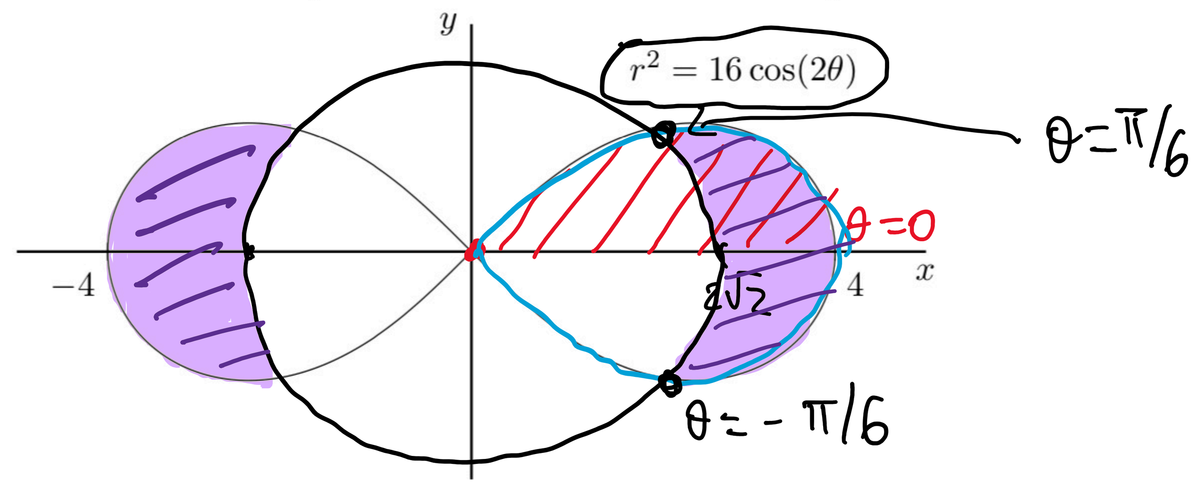
$$= (f'(\theta))^2\cos^2\theta - \cancel{2f'(\theta)f(\theta)\cos\theta\sin\theta} + (f(\theta))^2\sin^2\theta$$
$$+ (f'(\theta))^2\sin^2\theta + \cancel{2f'(\theta)f(\theta)\sin\theta\cos\theta} + (f(\theta))^2\cos^2\theta$$

using $\cos^2\theta + \sin^2\theta = 1$

$$= (f'(\theta))^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + (f(\theta))^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$$

$$= (f'(\theta))^2 + (f(\theta))^2 \quad ✓$$

2. [12 points] Chancelor was doodling in his coloring book one Sunday afternoon when he drew an infinity symbol, or lemniscate. The picture he drew is the polar curve $r^2 = 16\cos(2\theta)$, which is shown on the axes below. (The axes are measured in inches.)



Example
$$y^2 = 4x$$
$$2y\frac{dy}{dx} = 4$$

a. [4 points] Chancelor decides to color the inside of the lemniscate red. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total area, in square inches, that he has to fill in with red.

$$\text{Area} = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

$r^2 = 16\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}$

$\theta = \pi/4$

$$\text{area} = 4\left(\frac{1}{2}\int_0^{\pi/4} 16\cos 2\theta\, d\theta\right) \text{ square inches}$$

b. [4 points] He decides he wants to outline the right half (the portion to the right of the y-axis) of the lemniscate in blue. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total length, in inches, of the outline he must draw in blue.

$$\text{arclength} = \int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

$$f(\theta) = \sqrt{16\cos 2\theta}$$

$$2r\frac{dr}{d\theta} = 16(-2\sin 2\theta)$$

$$\frac{dr}{d\theta} = -\frac{\cancel{32}^{16}\sin 2\theta}{\cancel{2}r} = -\frac{16\sin 2\theta}{\sqrt{16\cos 2\theta}} = -\frac{4\sin 2\theta}{\sqrt{\cos 2\theta}}$$

either use symmetry or

$$\int_{-\pi/4}^{\pi/4} \sqrt{16\cos 2\theta + \left(-\frac{4\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2}\, d\theta$$

inches

c. [4 points] Chancelor draws another picture of the same lemniscate, but this time also draws a picture of the circle $r = 2\sqrt{2}$. He would like to color the area that is inside the lemniscate but outside the circle purple. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total area, in square inches, that he must fill in with purple.

$16\cos 2\theta = (2\sqrt{2})^2$ intersection

$$16\cos 2\theta = 8$$
$$\cos 2\theta = \frac{1}{2}$$
$$2\theta = -\frac{\pi}{3}, \frac{\pi}{3} \Rightarrow \theta = -\frac{\pi}{6}, \frac{\pi}{6}$$

$$2\left(\frac{1}{2}\int_{-\pi/6}^{\pi/6} (16\cos 2\theta - 8)\, d\theta\right) \text{ square inches.}$$

by symmetry