Logarithms and their properties (sec 5.1)

Monday, November 2, 2020 5:55 PM

Reminder: Exponential models ① $y = ab^t$: a initial amount

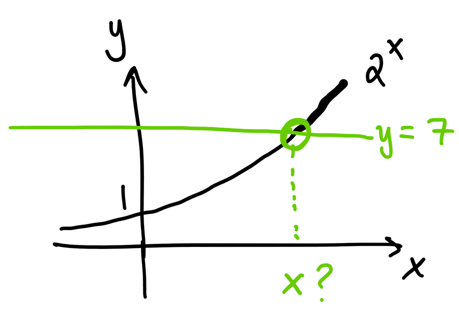
b growth factor

$b = 1+r$ where r is growth rate

② $y = ae^{kt}$: a initial amount

k continuous growth rate.

Example: Solve $\underbrace{2^x = 7}$



What is a logarithm?

If $x>0$ $\log_{10} x$ is the exponent of 10 that gives x.

i.e. $y = \log x \underset{\text{means}}{\Leftrightarrow} 10^y = x$

exponent, result: $\left(y = \log_{10} x \right)$, base

$10^y = x$

Eg. (a) $\log 0.01 = -2$

$\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

(b) $\log 100 = ?$

$10^{?} = 100$

$? = 2$

Note: Logarithmic and exponential functions are inverses.

For any N, $\log(10^N) = N$ and $10^{\log N} = N$

Recall $f(f^{-1}(x)) = x$

$f^{-1}(f(x)) = x$

e.g. Find • $10^{\log(3)} = 3$

• $\log(10^{2x}) = 2x$

Properties of logarithms

By definition $y = \log x$ means $10^y = x$

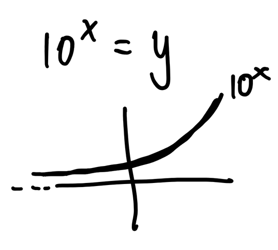
We also have that $\boxed{\log 1 = 0}$ and $\boxed{\log 10 = 1}$

Since logs and exponentials are inverses $\log(10^x) = x$

and $10^{\log x} = x$ for $x>0$

$\log(y) = x$

$10^x = y$



$10^x > 0$

$\Rightarrow y > 0$

so input of log is always positive

★ For a and b both positive and any t

① $\log(ab) = \log(a) + \log(b)$

② $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$

③ $\log(b^t) = t\log(b)$

Example Solve $3 \cdot 5^t = 7$

$5^t = \frac{7}{3}$

Take logs on both sides: $\log(5^t) = \log\left(\frac{7}{3}\right)$

$t\log(5) = \log\left(\frac{7}{3}\right)$

$\boxed{t = \frac{\log\left(\frac{7}{3}\right)}{\log(5)}}$

The natural logarithm

For $x>0$ $\ln(x)$ is the power of e that give x, or

$y = \ln(x)$ means $e^y = x$

and $y = \ln(x)$ is called the natural logarithm.

NB

exponent, result: $y = \ln x$, base e

$e^y = x$

Properties of natural logarithms:

a) $\ln(1) = 0$

b) $\ln(e) = 1$

c) $\ln(e^x) = x$

$e^{\ln x} = x$ } e^x and $\ln(x)$ are inverses, so they "undo" each other

① $\ln(ab) = \ln(a) + \ln(b)$

② $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

③ $\ln(b^t) = t\ln(b)$

Examples a) Solve $\ln(x+5) = 8$

$e^8 = x+5$

$x = e^8 - 5$

b) Solve $2^x = 1000$

Take $\ln$ on both sides: $\ln(2^x) = \ln(1000)$

$x\ln(2) = \ln(1000)$

$x = \frac{\ln(1000)}{\ln(2)}$

c) Solve $5e^{2x} = 50$

$e^{2x} = 10$

Take $\ln$ on both sides: $\ln(e^{2x}) = \ln(10)$

$2x = \ln(10)$

$x = \frac{\ln(10)}{2}$

Note: $\log(a+b) \neq \log(a) + \log(b)$

$\log(a-b) \neq \log(a) - \log(b)$

$\log\left(\frac{a}{b}\right) \neq \frac{\log(a)}{\log(b)}$

Examples Solve the equations using logs:

a) $84(0.74)^t = 38$

$(0.74)^t = \frac{38}{84}$

Take logs on both sides $\log(0.74^t) = \log\left(\frac{38}{84}\right)$

$t\log(0.74) = \log\left(\frac{38}{84}\right)$

$t = \frac{\log\left(\frac{38}{84}\right)}{\log(0.74)}$

b) $0.4\left(\frac{1}{3}\right)^{3x} = 7 \cdot 2^{-x}$

$\left(\frac{1}{3}\right)^{3x} = \frac{7}{0.4} \cdot 2^{-x}$

take logs on both sides $\log\left(\left(\frac{1}{3}\right)^{3x}\right) = \log\left(\overset{a}{\frac{7}{0.4}} \cdot \overset{b}{2^{-x}}\right)$

use $\log(a \cdot b) = \log(a) + \log(b)$

$3x\log\left(\frac{1}{3}\right) = \log\left(\frac{7}{0.4}\right) + \underbrace{\log(2^{-x})}_{= -x\log(2)}$

$3x\log\left(\frac{1}{3}\right) = \log\left(\frac{7}{0.4}\right) - x\log(2)$

Since we're solving for x, bring all the x's together:

$3x\log\left(\frac{1}{3}\right) + x\log(2) = \log\left(\frac{7}{0.4}\right)$

$x\left[3\log\left(\frac{1}{3}\right) + \log(2)\right] = \log\left(\frac{7}{0.4}\right)$

$\boxed{x = \frac{\log\left(\frac{7}{0.4}\right)}{3\log\left(\frac{1}{3}\right) + \log(2)}}$