Lagrange multipliers

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(Not in Exam)

This is an example of where you can use partial derivatives.

Maximum and minimum values

Theorem If $f(x,y)$ has a local minimum or maximum at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

Notation: This can be written as $\boxed{\nabla f(a,b)=\vec{0}}$ where $\nabla f = \text{grad } f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$ (grad)

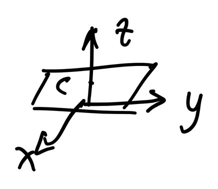
(nabla : ∇)

Note From the previous class we saw that the tangent plane at the point (a,b,c)

$$z-c = \cancel{f_x(a,b)}\cdot(x-a) + \cancel{f_y(a,b)}\cdot(y-b)$$
$$0 \qquad\qquad 0$$

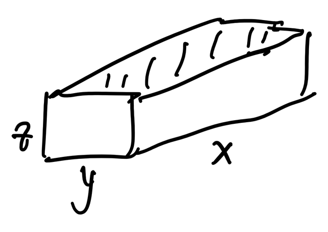
$$\Rightarrow z=c$$

The geometric interpretation is that if the graph of f has a tangent plane at a local minimum or maximum then the tangent plane is horizontal.



A point (a,b) is called a critical point of f if $\boxed{f_x(a,b)=0}$ and $\boxed{f_y(a,b)=0}$ or if one of f_x or f_y does not exist.

Example A rectangular box without a lid is made of $12\,m^2$ of cardboard. Find the maximum volume of this box.



Sol^n $V = xyz$

Constraint: $2xz + 2yz + xy = 12$

$\Rightarrow z(2x+2y)+xy=12$

$\Rightarrow \boxed{z = \frac{12-xy}{2(x+y)}}$

Objective function: $V = xyz$

$$= xy\left(\frac{12-xy}{2(x+y)}\right)$$

$$= \frac{12xy - x^2y^2}{2(x+y)} \leftarrow$$

Critical points: $\frac{\partial V}{\partial x}=0$ and $\frac{\partial V}{\partial y}=0$

$$\frac{\partial V}{\partial x} = \frac{\cancel{2}(x+y)\left[12y-2xy^2\right] - \cancel{2}\left[12xy-x^2y^2\right]}{2^{\cancel{2}}(x+y)^2} = \frac{\cancel{12xy}+12y^2-2x^2y^2-2xy^3-\cancel{12xy}+x^2y^2}{2(x+y)^2}$$

$$= \frac{12y^2-x^2y^2-2xy^3}{2(x+y)^2}$$

$$= \frac{y^2(12-x^2-2xy)}{2(x+y)^2}$$

$$\frac{\partial V}{\partial y} = \frac{12x^2-x^2y^2-2yx^3}{2(x+y)^2} = \frac{x^2(12-y^2-2yx)}{2(x+y)^2}$$

$x=0=y \Rightarrow V=0$

$12-y^2-2yx=0$ and $12-x^2-2xy=0 \Rightarrow x=y$

using $x=y \Rightarrow$ $12-x^2-2x^2=0$

$12-3x^2=0$

$x^2=4$

$\boxed{x=2} \Rightarrow y=2 \Rightarrow z = \frac{12-(2)(2)}{2(2+2)} = 1$

$x=2,\ y=2,\ z=1 \Rightarrow V = 2(2)(1) = 4\ m^3$

To find the absolute maximum and minimum values of a continuous function on a closed, bounded set D:

① Find the values of f at the critical points of f in D

② Find the values of f on the boundary of D.

③ The largest of the values from ① and ② is the maximum and the smallest is the minimum

Lagrange multipliers

This is used for maximizing or minimizing an objective function $f(x,y,z)$ subject to a constraint of the form $g(x,y,z)=k$.

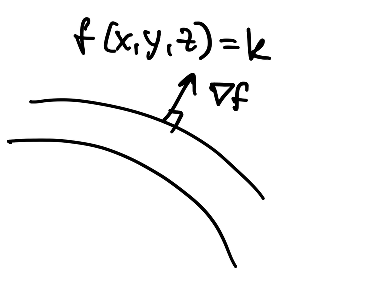
Step 1 Find all values of x,y,z and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
and $g(x,y,z)=k$

λ: Lagrange multiplier

$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$

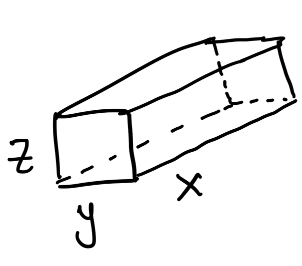
Aside.



Step 2 Evaluate f at all points (x,y,z) that result from step ①

The largest of these values is the maximum of f.

Example



$V = xyz$

Constraint: $2xz + 2yz + xy = 12$ (recall the lid is missing)

$f = xyz$

$g = 2xz + 2yz + xy$

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} \qquad \nabla g = \begin{pmatrix} 2z+y \\ 2z+x \\ 2x+2y \end{pmatrix}$$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} 2z+y \\ 2z+x \\ 2x+2y \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix}$$

multiply

$(\times x)$ $yz = \lambda(2z+y)$ $\quad xyz = \lambda(2xz+xy)$

$(\times y)$ $xz = \lambda(2z+x)$ $\rightarrow xyz = \lambda(2yz+xy)$

$(\times z)$ $xy = \lambda(2x+2y)$ $\rightarrow \boxed{xyz = \lambda(2xz+2yz)}$

$\boxed{\lambda(2xz+xy) = \lambda(2yz+xy)}$

$\lambda=0 \Rightarrow xy = xz = yz = 0$

this cannot be true because the constraint is not satisfied.

$2xz + xy = 2yz + xy \Rightarrow x=y$

$x=y \Rightarrow$ $2yz + xy = 2xz + 2yz \Rightarrow \cancel{2yz} + y^2 = \cancel{2yz} + 2yz \Rightarrow y^2 - 2yz = 0$

$y(y-2z)=0$

$z = \frac{y}{2}$

Using the constraint $2xz + 2yz + xy = 12$

$\cancel{2}y\left(\frac{y}{\cancel{2}}\right) + \cancel{2}y\left(\frac{y}{\cancel{2}}\right) + y^2 = 12$

$y^2+y^2+y^2 = 12 \Rightarrow 3y^2 = 12$

$y=2 \Rightarrow x=2 \Rightarrow z=1$ $\quad V = 4\ m^3$

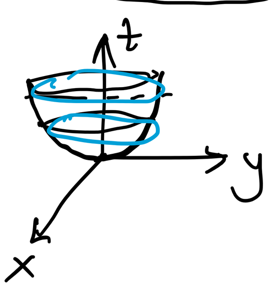
Example Find the extreme values of $f(x,y) = x^2+2y^2$ on the circle $x^2+y^2=1$

sketching

$x^2+2y^2 = z$ where k is a constant

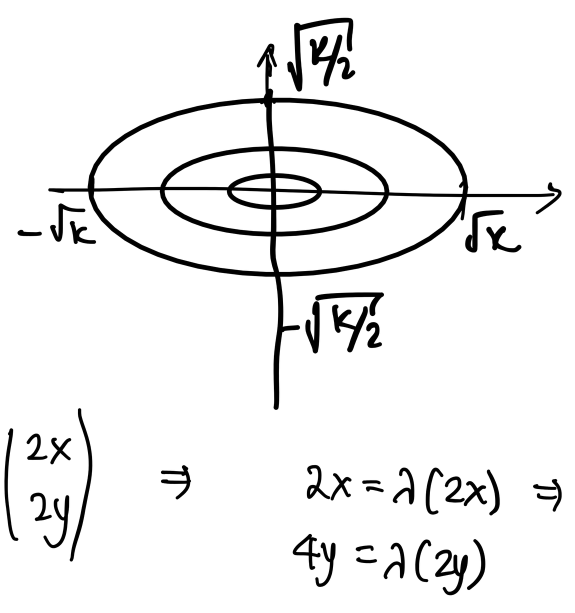
Setting $z = \text{constant} = k$

$x^2+2y^2=k$ $\quad$ (Ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$)



$\frac{x^2}{k} + \frac{2y^2}{k} = 1$

$\left(\frac{x}{\sqrt{k}}\right)^2 + \left(\frac{y}{\sqrt{k/2}}\right)^2 = 1$



the cross-sections are ellipses!

$$\left.\begin{matrix} \nabla f = \lambda \nabla g \\ \text{and } x^2+y^2=1 \end{matrix}\right] \Rightarrow \begin{pmatrix} 2x \\ 4y \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix} \Rightarrow \begin{matrix} 2x = \lambda(2x) \\ 4y = \lambda(2y) \end{matrix} \Rightarrow \lambda = 1 \text{ or } x=0$$

$\cancel{x^2}+y^2=1$, 0

$y=\pm 1$

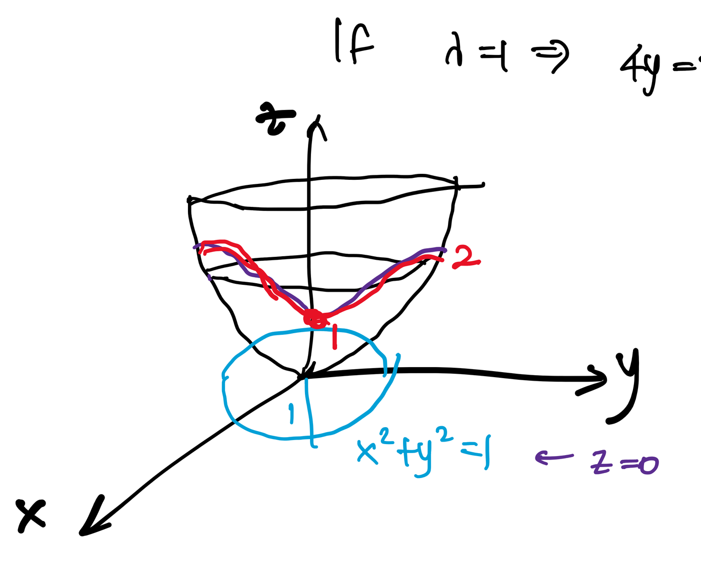
recall $\boxed{f = x^2+2y^2}$

max $f(0,\pm1) = 0+2(\pm1)^2 = 2$

min $f(\pm1,0) = (\pm1)^2 + 2(\cancel{0})^2 = 1$

If $\lambda=1 \Rightarrow 4y=2y \Rightarrow y=0$

then $x=\pm1$ from constraint



If constraint $x^2+y^2 \le 1$

Extra step find critical points of f

$f_x=0$ $\quad f_x = 2x = 0$

$f_y=0$ $\quad f_y = 4y = 0$

$\Rightarrow x=0$ and $y=0$

$f(0,0)=0$ min

$f(0,\pm1)=2$ max