Invertible and properties of inverses (sec. 10.2)

Tuesday, December 1, 2020 12:47 AM

Given a function $Q = f(x)$ with the property that each value of Q determines exactly one value of t then the function f has an inverse function f^{-1} and

$$f^{-1}(Q) = t \quad \text{if and only if} \quad Q = f(t)$$

Such a function is said to be invertible

eg. $\log(x)$ is the inverse of 10^x ($x = \log y$ if and only if $y = 10^x$)

e.g. $f^{-1}(100) = 2$ if and only if $f(2) = 100$

Note $(f(x))^{-1} = \frac{1}{f(x)}$ reciprocal $\neq$ (not equal to) $f^{-1}(x)$ inverse.

Finding the inverse from a formula

e.g. $g(x) = \frac{4x}{3x+2}$. Find the inverse $g^{-1}(x)$.

STEP1 : Write $g(x) = y$: $y = \frac{4x}{3x+2}$

STEP2 Rearrange for x to be the subject of the formula:

$(3x+2)\cdot(y) = 4x$

remember to put parenthesis

$3xy + 2y = 4x$

$2y = 4x - 3xy$ factorize x

$2y = x(4-3y)$

$x = \frac{2y}{4-3y}$

STEP3 Switch x with y and vice versa:

$y = \frac{2x}{4-3x}$

STEP 4 Write y as $g^{-1}(x)$:

$\boxed{g^{-1}(x) = \frac{2x}{4-3x}}$

Example Consider $h(t) = e^t - 3$. Find $h^{-1}(t)$.

STEP1 Replace $h(t)$ by y: $y = e^t - 3$

STEP2 Rearrange to make t subject of the formula:

$y + 3 = e^t$

$\ln(y+3) = t$

STEP3 Switch t with y & vice versa

$\ln(t+3) = y$

STEP4 Write y as $h^{-1}(t)$

$\boxed{h^{-1}(t) = \ln(t+3)}$

Note If $y = f(x)$ is an invertible function and $f^{-1}(x)$ is its inverse then

- $f^{-1}(f(x)) = x$ composition for all values of x for which $f(x)$ is defined. $\cancel{f^{-1}}(\cancel{f}(x))$
- $f(f^{-1}(x)) = x$ for all values of x for which $f^{-1}(x)$ is defined.

e.g. Recall $h(t) = e^t - 3$ and $h^{-1}(t) = \ln(t+3)$

Check $h^{-1}(h(t)) = t$

$h^{-1}(e^t - 3) = \ln(e^t \cancel{-3} \cancel{+3}) = \ln(e^t) = t$

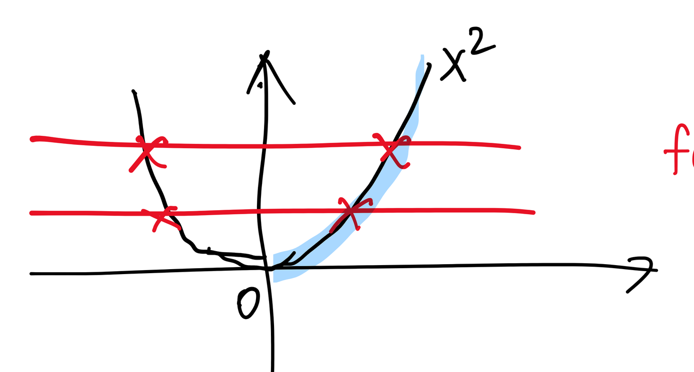
Check $h(h^{-1}(t)) = t$

$h(\ln(t+3)) = e^{\ln(t+3)} - 3 = t + \cancel{3} - \cancel{3} = t$ ✓

Non-invertible functions can be determined from the HORIZONTAL LINE TEST

The horizontal line test states that if a horizontal line intersects a function's graph in more than one point, then the function does not have an inverse.

Q Is $f(x) = x^2$ invertible?



fails the horizontal line test

$\Rightarrow$ NOT invertible.

If we restrict the domain such that $x \geq 0$, then x^2 becomes invertible.

$f(x) = x^2$

$y = x^2$

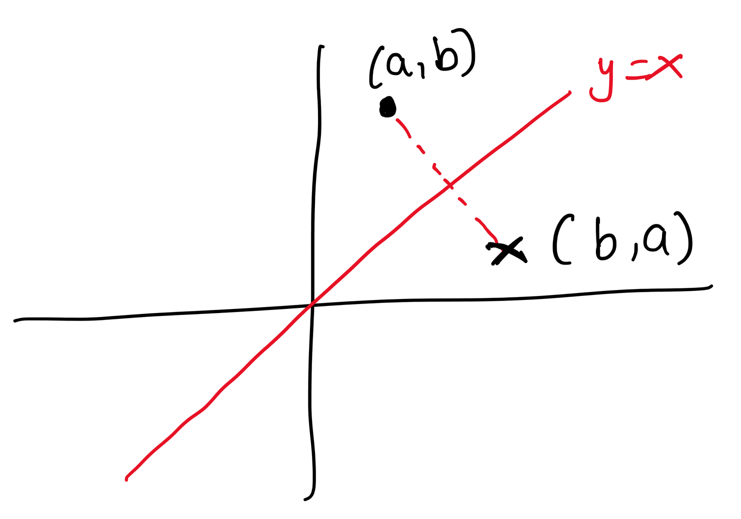
$\cancel{\pm}\sqrt{y} = x$

$\sqrt{x} = y$

$\boxed{f^{-1}(x) = \sqrt{x}}$

The graph, the domain and the range of an inverse

Graph of f^{-1} is the reflection of f across the line $y = x$



(switch x with y and vice versa)

Domain of f^{-1} is the range of f

Range of f^{-1} is the domain of f.

e.g.