Integration

Tuesday, July 14, 2020 10:42 AM

During last class we saw limits and differentiation. This time we will cover Integration methods. In particular we will use integration by parts, by substitution. Next time I'll quickly cover partial fractions at the beginning of class.

Integration by substitution

How does it work? It's based on reverse chain rule.

Assume you have an integral of the form $\int f(g(x))g'(x)\,dx$

If F is the antiderivative of f then $F' = f$ then by the chain rule you have

$$\frac{d}{dx}\big(F(g(x))\big) = f(g(x))g'(x)$$

Thus $\int f(g(x))g'(x)\,dx = F(g(x)) + C$

So for the substitution use $\boxed{u = g(x)} \Rightarrow \frac{du}{dx} = g'(x)$

$$\int f(u)\frac{du}{dx}\,dx = F(u) + C$$

Since $F' = f \Rightarrow \int f(u)\,du = F(u) + C$

SUBSTITUTION

Examples

① $\int t\cos(t^2)\,dt$ $\quad u = t^2 \Rightarrow du = 2t\,dt \Rightarrow t\,dt = \frac{1}{2}du$

$= \int \frac{1}{2}\cos(u)\,du = \frac{1}{2}\int\cos(u)\,du = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(t^2) + C.$

② $\int x(x^2+3)^2\,dx$ $\quad u = x^2+3 \Rightarrow du = 2x\,dx$

$\int \frac{1}{2}u^2\,du = \frac{1}{2}\frac{u^3}{3} + C = \frac{1}{6}u^3 + C = \frac{1}{6}(x^2+3)^3 + C$

Check $\frac{d}{dx}\left(\frac{1}{6}(x^2+3)^3 + C\right) = \frac{1}{\cancel{2}}(x^2+3)^2(\cancel{2}x) = x(x^2+3)^2$

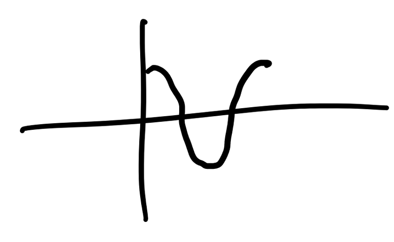
③ $\int_0^{\pi/2} e^{-\cos\theta}\sin\theta\,d\theta$ $\quad \boxed{u = -\cos\theta}$, $du = \sin(\theta)\,d\theta$

$\theta = 0 \Rightarrow u = -\cos 0 = -1$

$\theta = \frac{\pi}{2} \Rightarrow u = -\cos\frac{\pi}{2} = 0$

$= \int_{-1}^{0} e^u\,du$

$= \left[e^u\right]_{-1}^{0} = e^0 - e^{-1} = 1 - \frac{1}{e}$



Integration by parts:

This method is based on the product rule: $\frac{d}{dx}(uv) = uv' + u'v$

Rearrange this to write $uv' = \frac{d}{dx}(uv) - u'v$

Integrate both sides $\boxed{\int uv'\,dx = uv - \int u'v\,dx}$

How do you choose u and v':

- whatever you let v' be, you need to be able to get v
- It's good if u' is simpler than u
- It's good if v is simpler than v'

BY PARTS

Example $\int \ln(x)\,dx$ $\quad u = \ln(x)$, $\frac{dv}{dx} = 1$, $\frac{du}{dx} = \frac{1}{x}$, $v = x$

When you have $\ln(x)$ or something similar choose $u = \ln(x)$

$= x\ln(x) - \int \frac{1}{x}(x)\,dx$

$= x\ln(x) - \int 1\,dx$

$= x\ln(x) - x + C$

e.g. $\int x^2e^x\,dx$ $\quad u = x^2$, $\frac{dv}{dx} = e^x$, $\frac{du}{dx} = 2x$, $v = e^x$

$= x^2e^x - \int 2xe^x\,dx$

$u = 2x$, $\frac{dv}{dx} = e^x$, $\frac{du}{dx} = 2$, $v = e^x$

$= x^2e^x - \left[2xe^x - \int 2e^x\,dx\right]$

$= x^2e^x - 2xe^x + 2e^x + C$

Exercise. $\int \frac{1}{2+2\sqrt{x}}\,dx$ $\quad 2(1+\sqrt{x})$ $\quad \boxed{u = \sqrt{x}} \Rightarrow du = \frac{1}{2}x^{-1/2}\,dx$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$

$2\sqrt{x}\,du = dx$

$2u\,du = dx$

$\int \frac{\cancel{2}u\,du}{\cancel{2}(1+u)} = \int \frac{u}{1+u}\,du$

$= \int \frac{1+u-1}{1+u}\,du$

$= \int \frac{(1+u)-1}{1+u}\,du$

$= \int \left(\frac{1+u}{1+u} - \frac{1}{1+u}\right)du$

$= \int \left(1 - \frac{1}{1+u}\right)du$

$= u - \ln(1+u) + C$

$= \sqrt{x} - \ln(1+\sqrt{x}) + C$

HW1 (hints)

④ $S(x) = z(x)z(x+2)$ where $\boxed{S'(1) = 17\pi}$

x	-1	0	1	2	3	4
$z(x)$	2				π	3
$z'(x)$			5	π	12	

$S'(x) = z(x)z'(x+2) + z'(x)z(x+2)$

$\underbrace{S'(1)}_{17\pi} = \underbrace{z(1)}\,\underbrace{z'(3)}_{12} + \underbrace{z'(1)}_{5}\,\underbrace{z(3)}_{\pi}$

$17\pi = 12z(1) + 5\pi$

$12\pi = 12z(1)$

$\boxed{z(1) = \pi}$

$$u(x) = \begin{cases} z'(-1)\dfrac{x^2+1}{2^x} & \text{for } x<0 \\ 3 & \text{for } x=0 \\ e^{z(x)} & \text{for } x>0 \end{cases}$$

$\lim_{x\to 0^-} u(x) = \underbrace{z'(-1) = 3}$ by continuity

$\lim_{x\to 0^+} u(x) = \underbrace{e^{z(0)} = 3}$

$z(0) = \ln(3)$