## The Ekman spiral and point vortex motion around boundaries

**Abstract**

We present a very interesting phenomenon from geophysical fluid dynamics: the Ekman spiral. It can be thought of as a "spiral staircase down into the depths of the ocean". This is not only a fascinating oceanic phenomenon in itself, but also has far reaching effects on many coasts. The way the Ekman spiral transports water, known as upwelling, has a large influence on the life in the affected regions of the ocean. We also present vortex motion with boundaries (Kirchhoff-Routh theory) and extensions of this to multiply connected geometries.

### Ekman layers

In many boundary layers in non-rotating flow the dominant balance in the momentum equation is between the advective and viscous terms. In some contrast, in large-scale atmospheric and oceanic flow the effects of rotation are large, and this results in a boundary layer called the *Ekman layer*, in which the dominant balance is between Coriolis and frictional or stress terms. Non-geostrophic effects in the free-surface and rigid-bottom boundary layers are responsible for transferring momentum from the *wind* and *bottom stresses* to the interior geostrophic currents.

We now consider the corresponding Ekman layer at the *ocean surface.*

- Horizontal momentum is transferred down by vertical turbulent flux, which is commonly approximated by *vertical friction*:

$$\overline{w' \frac{\partial \boldsymbol{u'}}{\partial z}} = A_v \frac{\partial^2 \overline{\boldsymbol{u}}}{\partial z^2},$$

  where overbar indicates time mean and prime indicates fluctuating flow component.

- Consider the boundary layer correction, so that $\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_E$ in the thin layer with depth $h_E$

$$-f_0(v_g + v_E) = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial x} + A_v \frac{\partial^2 u_E}{\partial z^2}, \qquad f_0(u_g + u_E) = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y} + A_v \frac{\partial^2 v_E}{\partial z^2}.$$

  To make the friction term important in the balance, the *Ekman layer thickness* must be $h_E \sim [A_v/f_0]^{1/2}$, therefore, let's define $h_E \equiv [2A_v/f_0]^{1/2}$. Typical value of $h_E$ is $\sim 50\,\mathrm{m}$ in the ocean.

- The Ekman balance is

$$-f_0 v_E = A_v \frac{\partial^2 u_E}{\partial z^2}, \qquad f_0 u_E = A_v \frac{\partial^2 v_E}{\partial z^2}. \tag{1}$$

- The *boundary conditions* for the Ekman flow are zero at the bottom of the boundary layer and the stress condition at the upper surface

$$A_v \frac{\partial^2 u_E}{\partial z^2} = \frac{1}{\rho_0}\tau^x, \qquad A_v \frac{\partial^2 v_E}{\partial z^2} = \frac{1}{\rho_0}\tau^y. \tag{2}$$

  Now we look for a solution of (1) and (2) in the form

$$u_E = e^{z/h_E}\left[C_1 \cos\left(\frac{z}{h_E}\right) + C_2 \sin\left(\frac{z}{h_E}\right)\right], \quad v_E = e^{z/h_E}\left[C_3 \cos\left(\frac{z}{h_E}\right) + C_4 \sin\left(\frac{z}{h_E}\right)\right],$$

  and obtain the *Ekman spiral solution*:

$$u_E = \frac{\sqrt{2}}{\rho_0 f_0 h_E} e^{z/h_E}\left[\tau^x \cos\left(\frac{z}{h_E} - \frac{\pi}{4}\right) - \tau^y \sin\left(\frac{z}{h_E} - \frac{\pi}{4}\right)\right],$$
$$v_E = \frac{\sqrt{2}}{\rho_0 f_0 h_E} e^{z/h_E}\left[\tau^x \sin\left(\frac{z}{h_E} - \frac{\pi}{4}\right) + \tau^y \cos\left(\frac{z}{h_E} - \frac{\pi}{4}\right)\right].$$

In figure 1 we show an idealized Ekman spiral driven by the imposed wind-stress. Note that the net transport is at right angles to the wind, independent of the detailed form of the friction. The angles of the surface flow is 45° to the wind only for a Newtonian viscosity [Val17].

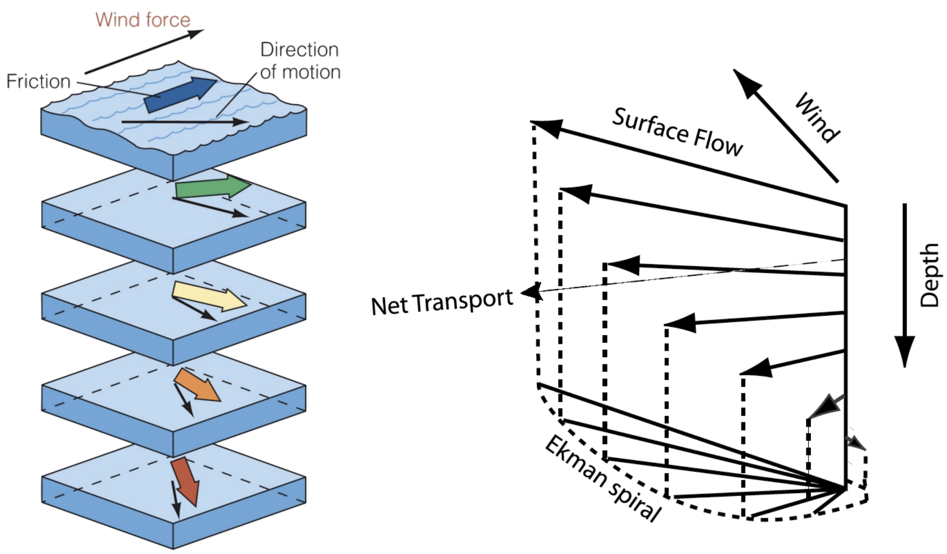



Figure 1: A body of water can be thought of as a set of layers. The top layer is driven forward by the wind, and each layer below is moved by friction. Each succeeding layer moves with a slower speed and at an angle to the layer immediately above it–to the right on the Northern Hemisphere, to the left in the Southern Hemisphere–until water motion becomes negligible.

Now, we change the topic and study point vortex dynamics. Here we make use of complex variables.

## Point vortex dynamics

We denote the position of a point vortex of circulation $\Gamma$, at time $t$ by $z_1(t)$. Then assuming the flow (including any background flow) is irrotational except for point vortex singularities, it has a complex potential $w(z,t)$. Local to $z_1(t)$,

$$w(z,t) = -\frac{i\Gamma}{2\pi}\log(z-z_1) + \hat{w}_1(z,t),$$

where $\hat{w}_1(z,t)$ is analytic at $z = z_1(t)$.

It can be shown that

$$\frac{\mathrm{d}\overline{z_1}(t)}{\mathrm{d}t} = \left.\frac{\partial \hat{w}_1(z,t)}{\partial z}\right|_{z=z_1(t)},$$

i.e. the point vortex moves with its **non-self-induced velocity**.

The simplest example of point vortex motion is that of systems of point vortices (i.e. no background flows or solid boundaries present).

Suppose we have $N$ point vortices at positions $z_j(t)$, $h = 1, \dots, N$, where that at $z_j(t)$ has circulation $\Gamma_j$. Then by superposition the complex potential of the flow is

$$w(z,t) = \sum_{j=1}^{N} -\frac{i\Gamma_j}{2\pi}\log(z - z_j(t)).$$

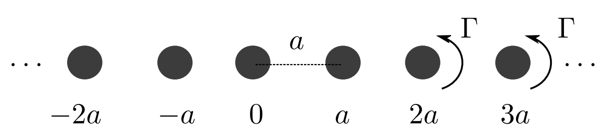
Thus, we have

$$\frac{\mathrm{d}\overline{z_j}(t)}{\mathrm{d}t} = \sum_{\substack{j=1\\ j\neq k}}^{N} -\frac{i\Gamma_j}{2\pi}\frac{1}{z_k(t)-z_j(t)}, \quad k = 1, \dots, N.$$

This gives a system of $N$ coupled ODEs.

## A single street

Consider infinitely many point vortices of equal circulation $\Gamma$, equally spaced along the real axis.



The complex potential at $t = 0$ is

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z) + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} -\frac{i\Gamma}{2\pi}\log\left(1 - \frac{z}{na}\right)$$

One can simplify this as follows (see next example):

$$\begin{aligned} w(z) &= -\frac{i\Gamma}{2\pi}\log\left[z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2a^2}\right)\right] \\ &= -\frac{i\Gamma}{2\pi}\log\left(\sin\left(\frac{\pi z}{a}\right)\right) + \text{constant}, \end{aligned}$$

where we use the identity $\sin(\pi z) = \pi z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$.

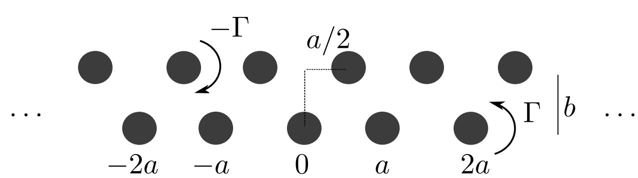
Now, without loss of generality, consider the point vortex initially at $z = 0$. Label the position of this as $z_0(t)$. Then at $t = 0$,

$$\frac{\mathrm{d}\overline{z_0}}{\mathrm{d}t} = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} -\frac{i\Gamma}{2\pi}\left.\frac{-\frac{1}{na}}{1 - \frac{z}{na}}\right|_{z=0} = 0.$$

So this point vortex does not move. By symmetry none of the others do either, i.e. the street is stationary.

## A double street (von Kármán vortex street)

Asymmetric (i.e. staggered streets). The initial configuration consists of two staggered single streets.




Consider the point vortex initially at $z = \frac{a}{2} + ib$. Label the position of this at time $t$ as $z_0(t)$. Initially, this point vortex will move with the velocity induced by the lower street (there is no net contribution to its velocity from the other point vortices in the upper street).

Complex potential of lower street is

$$w(z) = -\frac{i\Gamma}{2\pi}\log\left(\sin\left(\frac{\pi z}{a}\right)\right).$$

Therefore, at $t = 0$,

$$\begin{aligned}
\frac{\mathrm{d}\overline{z_0}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{i\Gamma}{2\pi}\log\left(\sin\left(\frac{\pi z}{a}\right)\right)\right)\bigg|_{z=z_0(0)} \\
&= -\frac{i\Gamma}{2\pi}\left(\cot\left(\frac{\pi z}{a}\right)\right)\frac{\pi}{a}\bigg|_{z=\frac{a}{2}+ib} \\
&= -\frac{i\Gamma}{2\pi}\cot\left(\frac{\pi}{2}+\frac{\pi i b}{a}\right) \\
&= \frac{i\Gamma}{2\pi}\tan\left(\frac{i\pi b}{a}\right) \\
&= -\frac{\Gamma}{2\pi}\tanh\left(\frac{\pi b}{a}\right).
\end{aligned}$$

This is real and negative if $\Gamma > 0$. By symmetry, al the point vortices in the upper street move with the same velocity. One can also check that the lower street moves with the same velocity. In other words, the whole array moves together.

This pattern is observed in many fluid flows. For example, the photo shown in figure 2 is that of the von Kármán vortex street visualized by clouds off the Chilean coast near the Juan Fernandez Islands (also known as the Robinson Crusoe Islands) photographed by the Landsat 7 satellite on September 15, 1999 https://en.wikipedia.org/wiki/K%C3%A1rm%C3%A1n_vortex_street.

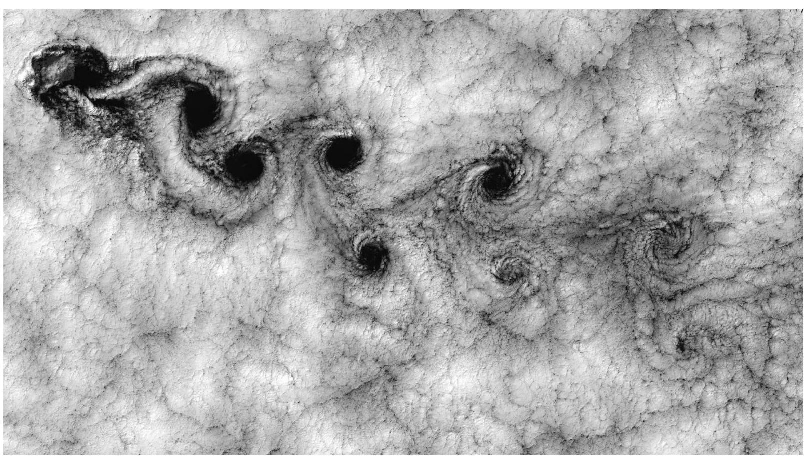


Figure 2: Von Kármán vortex street as seen in clouds.

## Point vortex motion around boundaries

**Example.** Consider a point vortex of circulation $\Gamma$ initially at $z = z_0$ in the UHP, with an infinite solid wall along the real axis. We want to find the complex potential, $w(z)$, for the flow in the UHP generated by the point vortex.

The complex potential $w(z)$ has three requirements:

1. $w(z)$ must be analytic everywhere in the UHP except that local to $z_0$
2. singularity: $w(z) \sim -\frac{i\Gamma}{2\pi}\log(z-z_0)$+non-singular terms
3. the real axis must be a streamline, i.e.

$$\text{Im}\{w(z)\} \text{ must be constant for } z \text{ real.}$$

Consider the following possibilities:

1. $w(z) = w_0(z)$. This satisfies, (1) and (2), but not (3). Note, assume $z_0 \neq \mathbb{R}$.
2. $w(z) = w_0(z) + \overline{w_0(z)}$. This satisfies (2) and (3), but not (1), as $\overline{w_0(z)}$ is a function of $z$ and thus not analytic anywhere.
3. $w(z) = w_0(z) + \overline{w}_0(z)$, where we define $\overline{w}_0(z) = \overline{w_0(\overline{z})}$, i.e.

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z-z_0) + \frac{i\pi}{2\pi}\log(z-\overline{z}_0).$$

   Note that $\overline{w}_0(z)$ is analytic in the UHP as $\overline{z_0}$ is in the LHP. Thus, this form for $w(z)$ satisfies (1) and (2). Also $z \in \mathbb{R}$.

$$\begin{aligned}\overline{w(z)} &= \overline{w_0(z)} + \overline{\overline{w}_0(z)}\\ &= 2\text{Re}\{w_0(z)\}.\end{aligned}$$

   This implies that $\text{Im}\{w(z)\} = \text{const}$ and so (3) also holds.

Therefore, the complex potential is given by

$$w(z) = -\frac{i\Gamma}{2\pi}\log(z-z_0) + \frac{i\Gamma}{2\pi}\log(z-\overline{z_0}).$$

This is consistent with what is known as the "**method of images**". This can be described as follows:

For a point vortex in a region $D$ with boundary $\partial D$, find a distribution of additional "image" point vortices such that the flow in $D$ generated by the original point vortex and all its images but *no* boundary, is the same as that generated in $D$ by just the original point vortex and $\partial D$. Note that these image vortices lie in the *exterior* of $D$.

**Example.** Point vortex inside a cylinder.

We want to find $w(z)$. Let $w_0(z) = -\frac{i\Gamma}{2\pi}\log(z-z_0)$. Consider now $\tilde{w}(z) = w_0(z)+\overline{w}\left(\frac{1}{z}\right)$, where $\overline{w}_0(z) \equiv \overline{w_0(\overline{z})}$. Note that if the boundary of the cylinder is given by $|z| = 1$, this implies that $z\overline{z} = 1$ and so $\overline{z} = 1/z$. Therefore,

$$\overline{\overline{w}_0(z)} = \overline{w}_0(\overline{z}) = \overline{w}_0\left(\frac{1}{z}\right).$$

So, $\tilde{w}(z)$ is real for $z \in \partial D$, but

$$\begin{aligned}\tilde{w}(z) &= -\frac{i\Gamma}{2\pi}\log(z-z_0) + \frac{i\Gamma}{2\pi}\log\left(\frac{1}{z} - \overline{z_0}\right)\\ &= -\frac{i\Gamma}{2\pi}\log(z-z_0) + \frac{i\Gamma}{2\pi}\log\left(z - \frac{1}{\overline{z_0}}\right) - \frac{i\Gamma}{2\pi}\log z + \text{constant}.\end{aligned}$$

Therefore, $\overline{z_0}$ has an unwanted singularity at $z = 0$ due to the term $\frac{i\Gamma}{2\pi}\log z$. However, this term has constant (in fact 0) imaginary part on $|z| = 1$, so we can disregard it, to deduce that

$$\tilde{w}(z) = -\frac{i\Gamma}{2\pi}\log(z-z_0) + \frac{i\Gamma}{2\pi}\log\left(z - \frac{1}{\overline{z_0}}\right).$$

Note that $1/\overline{z_0} \notin D$.

For simple boundaries (such as the real axis, or a circle) it is straightforward to find where to place "image" vortices. More generally, however, we use *conformal mapping* to construct complex potentials. We do this as follows:

Consider a point vortex of circulation $\Gamma$ at $z = z_\alpha$ in some arbitrary (simply-connected) domain $D_z$, with boundary $\partial D_z$.

Suppose we know the complex potential, $w(\zeta)$, say for a point vortex of circulation $\Gamma$ at $\zeta = \alpha$ in some domain $D_\zeta$, with boundary $\partial D_\zeta$. If we can find a function $\zeta(z)$, say which maps $D_z$ into $D_\zeta$ with $\zeta(z_\alpha) = \alpha$ and furthermore is analytic in $D_\zeta^\star$ and $\zeta'(z_\alpha) \neq 0$. Then

$$W(z) = w(\zeta(z)).$$

Check:

1. $w(\zeta(z))$ is the composition of two analytic functions and is thus analytic for $z \in D_z$, except:

2. For $z$ local to $z_\alpha$, Taylor expanding $\zeta(z)$, gives

$$\zeta(z) = \zeta(z_\alpha) + \zeta'(z_\alpha)(z-z_\alpha) + \mathcal{O}(z-z_\alpha)^2.$$

   This implies that $(\zeta - \alpha) \sim \zeta'(z_\alpha)(z-z_\alpha) + \mathcal{O}(z-z_\alpha)^2$.

Thus, since $\zeta$ is local to $\alpha$,

$$\begin{aligned} w(\zeta(z)) &\sim -\frac{i\Gamma}{2\pi}\log(\zeta-\alpha) \\ &\sim -\frac{i\Gamma}{2\pi}\log[\zeta'(z_\alpha)(z-z_\alpha)(1+\mathcal{O}(z-z_\alpha))] \\ &\sim -\frac{i\Gamma}{2\pi}\log(z-z_\alpha) + \text{non-singular terms.} \end{aligned}$$

Now, we need to check the third requirement of $W(z)$: For $z \in \partial D_z$, we have $\zeta \in \partial D_\zeta$ and thus $\mathrm{Im}\{w(\zeta(z))\} = \text{const}$, as required. (In other words, if the boundary $\partial D_\zeta$ is a streamline in the $\zeta$-plane, then the corresponding boundary $\partial D_z$ is a streamline in the $z$-plane. Note also that a conformal mapping, maps a boundary to a boundary.)

This confirms that $W(z) = w(\zeta(z))$.

Since $z_\alpha$ is an arbitrary point in $D_z$, we want $\zeta'(z) \neq 0$ for all $z \in D_z$. For arbitrary $D_z$ and $D_\zeta$ does such a $\zeta(z)$ exist? Yes! This is known as the *Riemann mapping theorem*, and $\zeta(z)$ is a conformal map.

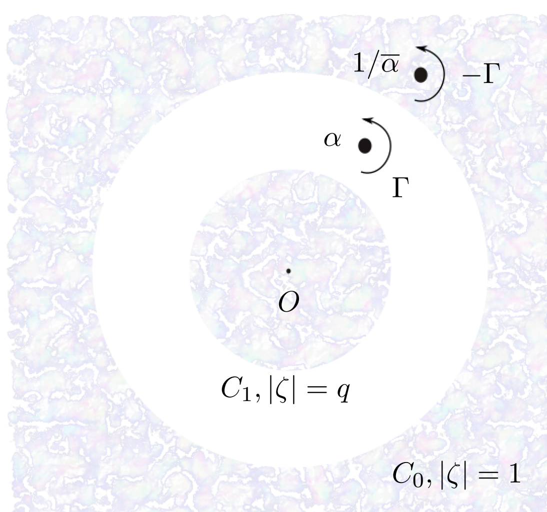
**Example.** Single point vortex in a concentric circular annulus.

We want to find the complex potential $w(\zeta)$.

Recall that the complex potential for a point vortex inside $|\zeta| = 1$ (without inner boundary $C_1$), is

$$w_0(\zeta) = -\frac{i\Gamma}{2\pi}\log\left(1-\frac{\zeta}{\alpha}\right) + \frac{i\Gamma}{2\pi}\log\left(1-\overline{\alpha}\zeta\right).$$

To get a constant imaginary part on $C_1$, add image vortices that are reflections in $C_1$ of those we already have but with opposite circulations.



Reflection in $C_1$ is $\zeta \mapsto \dfrac{q^2}{\overline{\zeta}}$. Our next guess is

$$w_1(\zeta) = -\frac{i\Gamma}{2\pi}\log\left[\left(\frac{1-\zeta/\alpha}{1-\overline{\alpha}\zeta}\right)\left(\frac{1-\zeta/(q^2\alpha)}{1-\overline{\alpha}\zeta/q^2}\right)\right].$$

The reflection in $C_0$ is $\zeta \mapsto \dfrac{1}{\overline{\zeta}}$.

There are infinitely many image vortices.

To write down $w(\zeta)$ in this case, we need to introduce the following special function

$$P(\zeta, q) = (1-\zeta)\prod_{k=1}^{\infty}(1-q^{2k}\zeta)\left(1-\frac{q^{2k}}{\zeta}\right). \tag{3}$$

One can show that for $0 < q < 1$, this infinite product converges and represents a function analytic for all $\zeta$, except at $\zeta = 0, \infty$. Also, one can see it has simple zeros at $\zeta = 1$, and in fact at $\zeta = q^{2k}$ for all $k \in \mathbb{Z}$.

We claim that $w(\zeta) = -\dfrac{i\Gamma}{2\pi}\log\left[\dfrac{P(\zeta/\alpha, q)}{P(\overline{\alpha}\zeta, q)}\right]$. Note that the reflection in $C_0$ followed by the reflection in $C_1$ is given by $\zeta \mapsto 1/\overline{\zeta} \mapsto q^2\zeta$. Also, reflection in $C_1$ followed by reflection in $C_0$ is given by $\zeta \mapsto q^2/\overline{\zeta} \mapsto \zeta/q^2$.

So we expect to need point vortices of circulation $\Gamma$ at $\alpha$ and all its images under compositions of $\theta_1(\zeta)$ and its inverse $\theta^{-1}(\zeta)$.

In addition, we need point vortices of circulation $-\Gamma$ at $1/\overline{\alpha}$ and all its images under compositions of $\theta_1(\zeta)$ and $\theta^{-1}(\zeta)$.

*Claim.*

$$w(\zeta) = -\frac{i\Gamma}{2\pi}\log\left[\frac{P\left(\frac{\zeta}{\alpha}, q\right)}{P(\overline{\alpha}\zeta, q)}\right], \tag{4}$$

$$\text{where } P(\zeta, q) = (1-\zeta)\prod_{k=1}^{\infty}(1-q^{2k}\zeta)\left(1-\frac{q^{2k}}{\zeta}\right). \tag{5}$$

$P(\zeta, q)$ is analytic for all $\zeta$, except $\zeta = 0, \infty$. Furthermore, it has simple zeros at $\zeta = q^{2k}$ $\forall\, k \in \mathbb{Z}$. So $w(\zeta)$ has precisely the singularities that we expect of $w(\zeta)$. To check rigorously the above claim, we proceed as follows:

First note that it defines a function that is analytic everywhere in the annulus, except at $\zeta = \alpha$, local to which it gives $w(\zeta) \sim \frac{i\Gamma}{2\pi}\log(\zeta - \alpha)$, as required.

Local to $\zeta = \alpha$, $P(\zeta/\alpha, q) \sim (1 - \zeta/\alpha)$. Furthermore, for all $k \in \mathbb{Z}_{>0}$, since $q < |\alpha| < 1$, then $0 < |q^{2k}\alpha| < q^{2k} < q$.

This implies that $q^{2k}\alpha \notin D_\zeta$. Similarly, for $k \in \mathbb{Z}_{<0}$,

$$1 < q^{2k+1} < |q^{2k}\alpha| \Rightarrow q^{2k}\alpha \notin D_\zeta.$$

Similarly, we can check that $\frac{q^{2k}}{\overline{\alpha}} \in D_\zeta$, for all $k \in \mathbb{Z}$.

It remains to check that $w(\zeta)$ as given by (4) has constant imaginary part on $C_0$ and $C_1$. To do this, we use the following properties of $P(\zeta, q)$.

1. $P\left(\frac{1}{\zeta}, q\right) = -\frac{1}{\zeta} P(\zeta, q)$
2. $P(q^2\zeta, q) = -\frac{1}{\zeta} P(\zeta, q)$.

Now let $R(\zeta) = \dfrac{P(\zeta/\alpha)}{P(\overline{\alpha}\zeta)}$.

For $\zeta \in C_0$,

$$\overline{R(\zeta)} = \frac{P(\overline{\zeta}/\overline{\alpha})}{P(\alpha\overline{\zeta})} = \frac{P(1/(\overline{\alpha}\zeta))}{P(\alpha/\zeta)} = \frac{-\frac{1}{\overline{\alpha}\zeta} P(\overline{\alpha}\zeta)}{-\frac{\alpha}{\zeta} P\left(\frac{\zeta}{\alpha}\right)} = \frac{1}{|\alpha|^2} \frac{1}{R(\zeta)}.$$

So (4) gives for $\zeta \in C_0$

$$\overline{w(\zeta)} = \frac{i\Gamma}{2\pi} \log(\overline{R(\zeta)} = -\frac{i\Gamma}{2\pi} \log(|\alpha|^2 R(\zeta)) = w(\zeta) - \frac{i\Gamma}{\pi} \log|\alpha|. \tag{6}$$

Therefore, we can write

$$\begin{aligned} w(\zeta) - \overline{w(\zeta)} &= \frac{i\Gamma}{\pi} \log|\alpha| \\ 2i\mathrm{Im}\{w(\zeta)\} &= \frac{i\Gamma}{\pi} \log|\alpha| \\ \mathrm{Im}\{w(\zeta\} &= \mathrm{const} \qquad \text{for} \quad \zeta \in C_0. \end{aligned}$$

Similarly, one can check that (4) gives$\mathrm{Im}\{w(\zeta)\} = \mathrm{const}$ for $\zeta \in C_1$. Note that one should use the fact that $\overline{\zeta} = q^2/\zeta$ for $\zeta \in C_1$, and so we can use property 2. of $P(\zeta)$.

# References

[Val17] Geoffrey K Vallis. *Atmospheric and oceanic fluid dynamics*. Cambridge University Press, 2017.