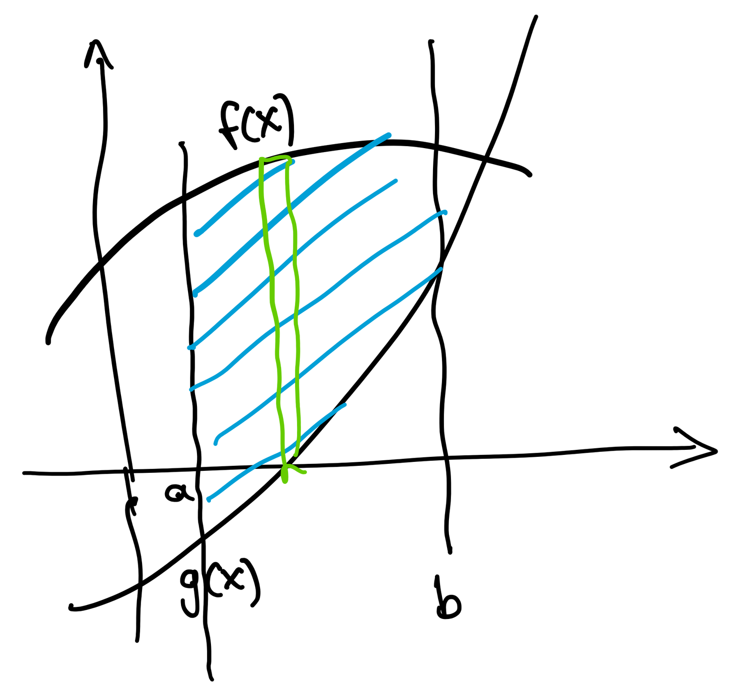
Areas and Volumes

Friday, July 17, 2020 9:29 AM

Areas between curves

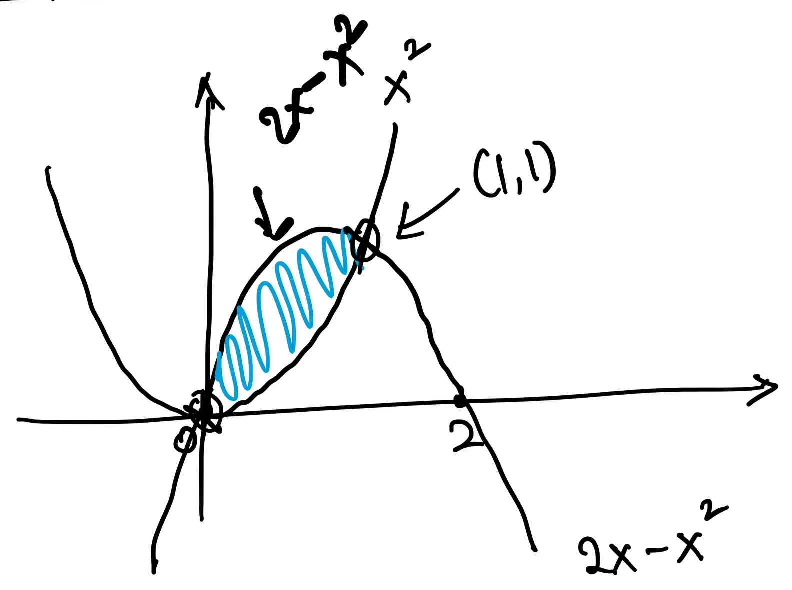


The area of a region bounded by the curves $y=f(x)$ and $y=g(x)$ and the lines $x=a$ and $x=b$, where f and g are continuous, with $f(x) \geq g(x)$ for all x in $[a,b]$ is

$$\text{area} = \int_a^b [f(x)-g(x)]\,dx$$

$$\left(= \int_a^b f(x)dx - \int_a^b g(x)dx \right)$$

Example Find the area enclosed by the parabolas $y=x^2$ and $y=2x-x^2$
$= x(2-x)$



intersection

$x^2 = 2x - x^2$

$2x^2 = 2x$

$x^2 - x = 0$

$x(x-1) = 0$

$x=0 \quad x=1$

$x=0 \Rightarrow y=0 \qquad (0,0)$

$x=1 \Rightarrow y=1 \qquad (1,1)$

"top - bottom"

$$\text{area} = \int_0^1 (2x-x^2) - x^2\,dx$$

$$= \int_0^1 (2x-2x^2)\,dx$$

$$= \left[x^2 - \frac{2}{3}x^3\right]_0^1$$

$$= 1 - \frac{2}{3}$$

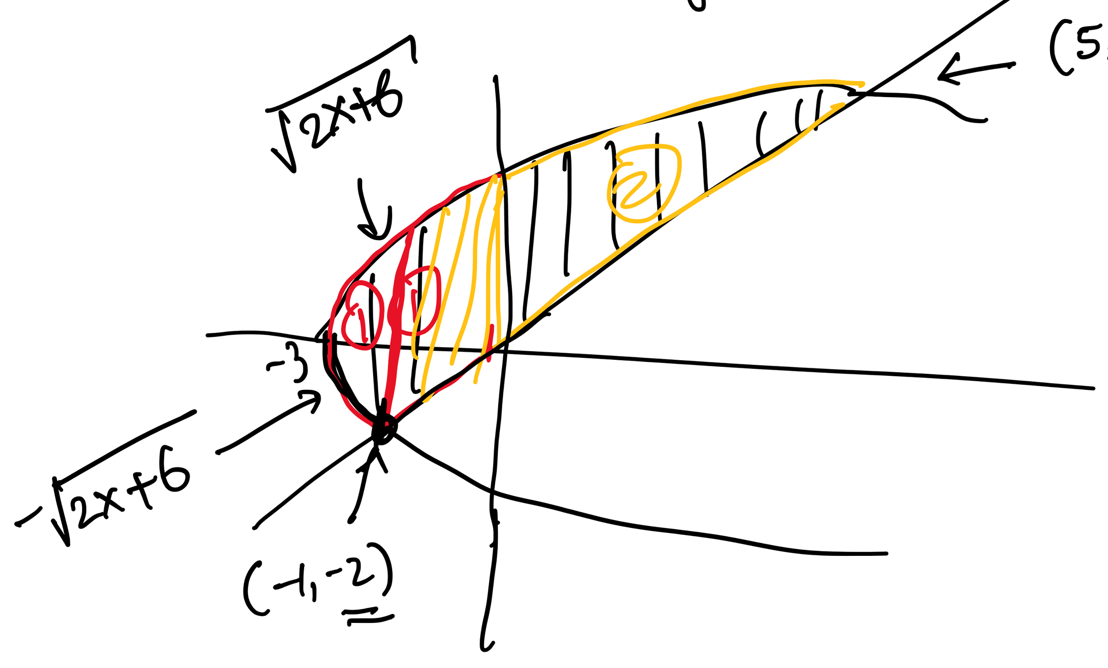
$$= \frac{1}{3} \quad //$$

Examples

$y = x-1$

$y^2 = 2x+6$

$x-1=y \Rightarrow x=y+1$



$y^2 = 2x+6$

$\Rightarrow x = \frac{y^2-6}{2}$

" right - left "

$$\text{area} = \int_{-2}^{4} (y+1) - \left(\frac{y^2-6}{2}\right)\,dy$$

$$= \int_{-2}^{4} y+1 - \frac{y^2}{2} + 3\,dy$$

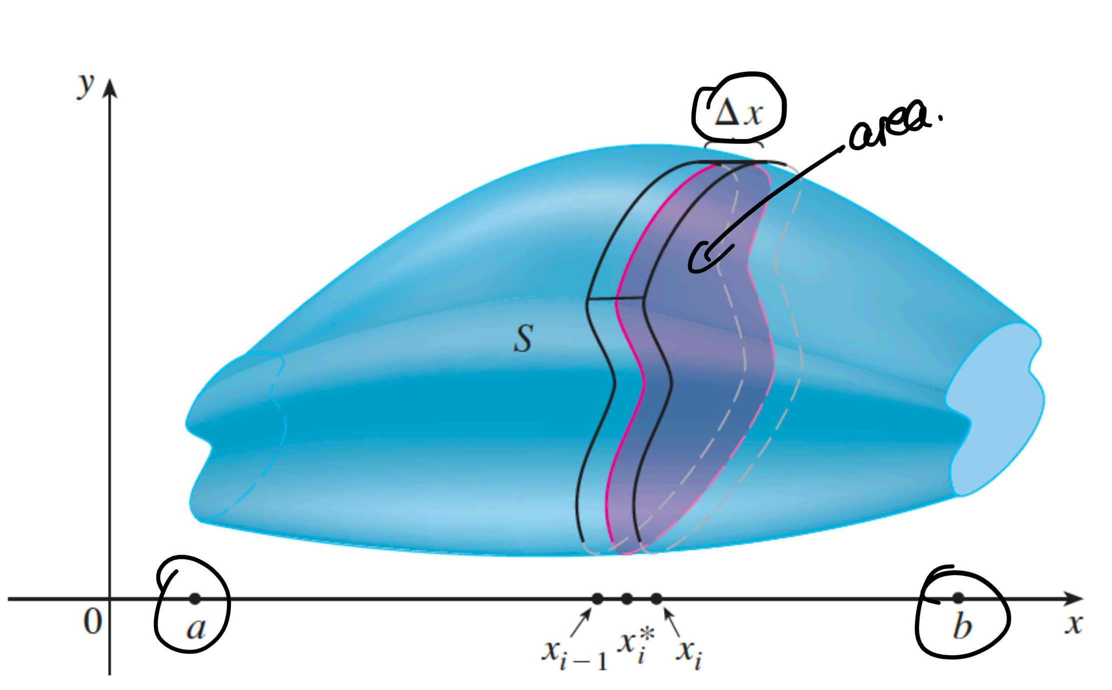
$$= \int_{-2}^{4} y + 4 - \frac{y^2}{2}\,dy$$

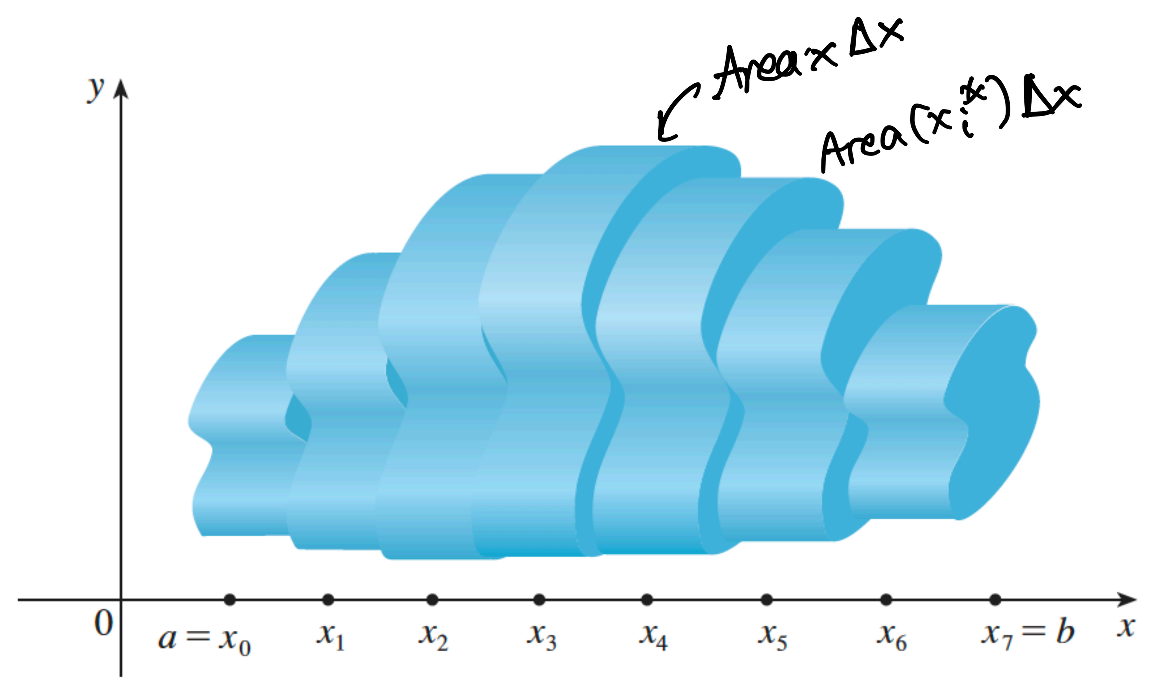
$$= \left[\frac{y^2}{2} + 4y - \frac{y^3}{6}\right]_{-2}^{4}$$

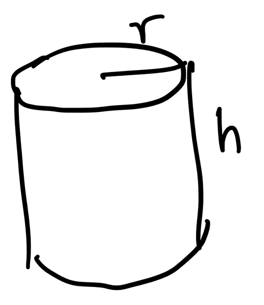
$$= \frac{4^2}{2} + 4(4) - \frac{4^3}{6} - \left(\frac{(-2)^2}{2} + 4(-2) - \frac{(-2)^3}{6}\right)$$

$$= 18 \quad //$$

Volumes







$\pi r^2 h = \text{Area} \times h$

|| area

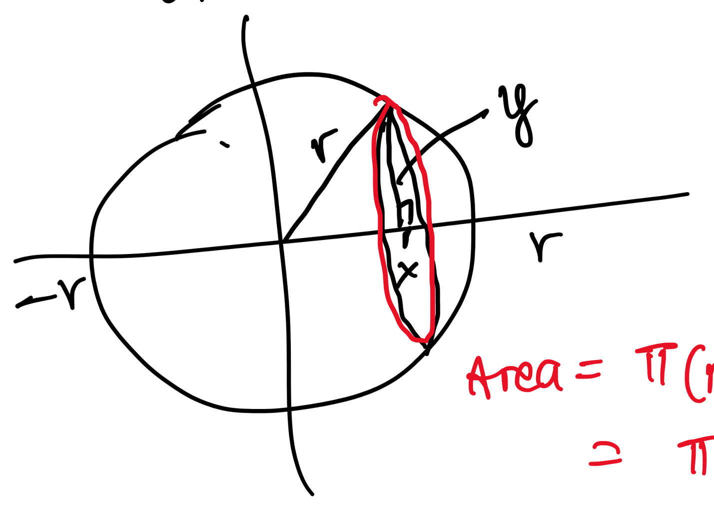
Def^n of volume

let S be the solid that lies between $x=a$ and $x=b$. If the cross-sectional area of S perpendicular to the x-axis is $A(x)$ and (A is a continuous function) volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

Example

Show that the volume of a sphere of radius r is given by $V = \frac{4\pi r^3}{3}$.



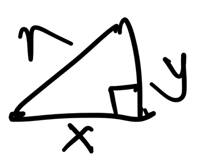
Area $= \pi (\text{radius})^2$

$= \pi y^2$

$= \pi \left(\sqrt{r^2-x^2}\right)^2$

$= \pi (r^2 - x^2)$

Pythagorean theorem: $y = \sqrt{r^2-x^2}$



$$\text{Volume} = \int_{-r}^{r} \pi(r^2-x^2)\,dx$$

$$= \left[\pi r^2 x - \frac{\pi x^3}{3}\right]_{-r}^{r}$$

$$= \pi r^3 - \frac{\pi r^3}{3} - \left(\pi r^2(-r) - \frac{\pi(-r)^3}{3}\right)$$

$$= \frac{2\pi r^3}{3} - \left(-\frac{2\pi r^3}{3}\right)$$

$$= \frac{4\pi r^3}{3}$$