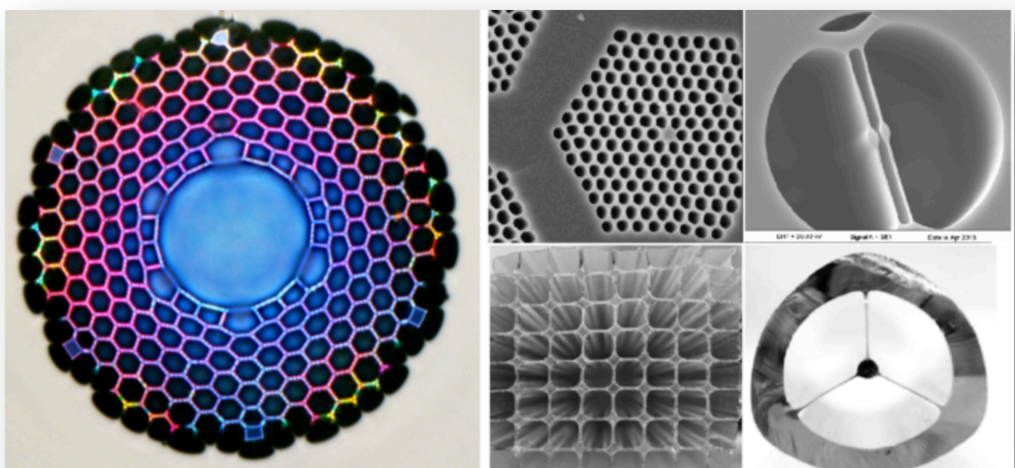
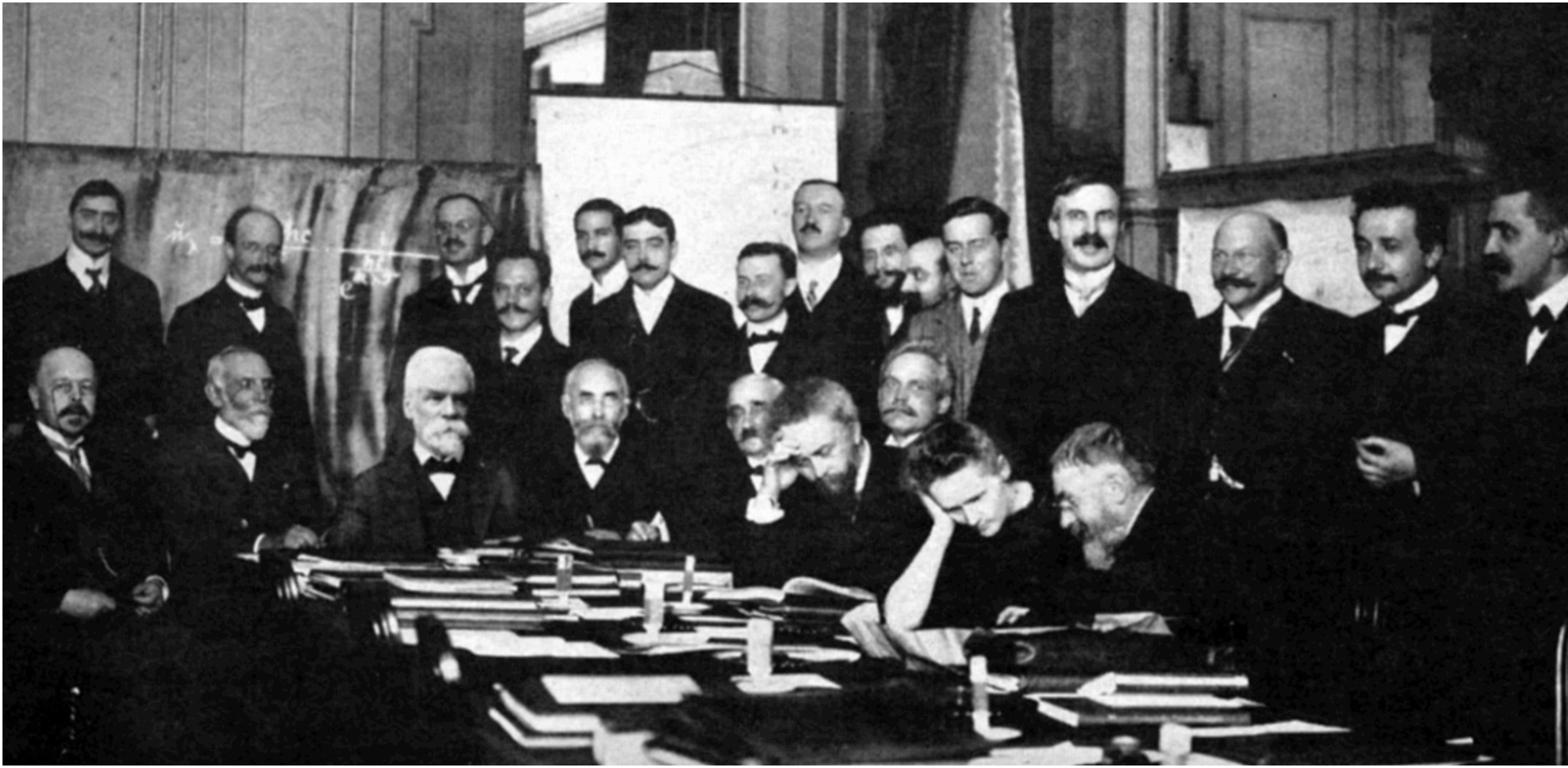
Mathematical modeling of microstructured optical fibers

Christiana Mavroyiakoumou

Student AIM Seminar, December 1, 2017

Supervisors: Dr. Ian Griffiths, Prof. Peter Howell
(University of Oxford)

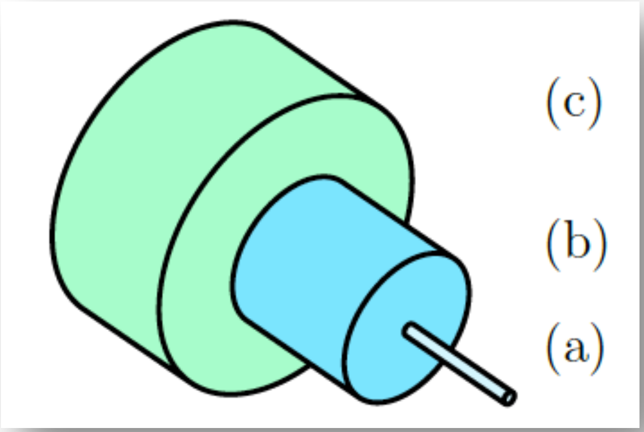




Source: Dancing Vortices, Talk by Etienne Ghys at the University of Oxford

Traditional optical fibers

- Consist of layers of **different** materials.
- These have different refractive indices.



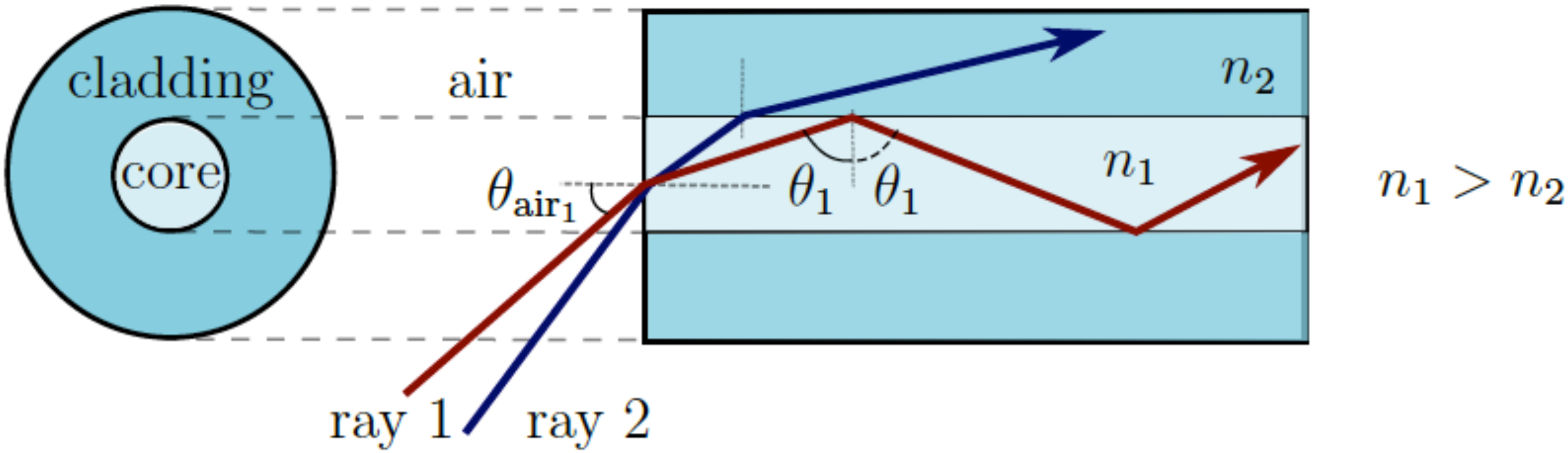
Protective buffer layer

Outer cladding

Inner core

Figure: Schematic diagram of optical fiber

Total internal reflection (TIR)



Horizontal cross-section of fiber

Vertical cross-section of fiber

- Light travels by TIR: core and cladding have different refractive indices (n_1, n_2)

Microstructured optical fibers (MOFs)

- Consist of a thread of a **single** material, usually glass or silica.
- Has a solid or air core surrounded by an array of air channels.

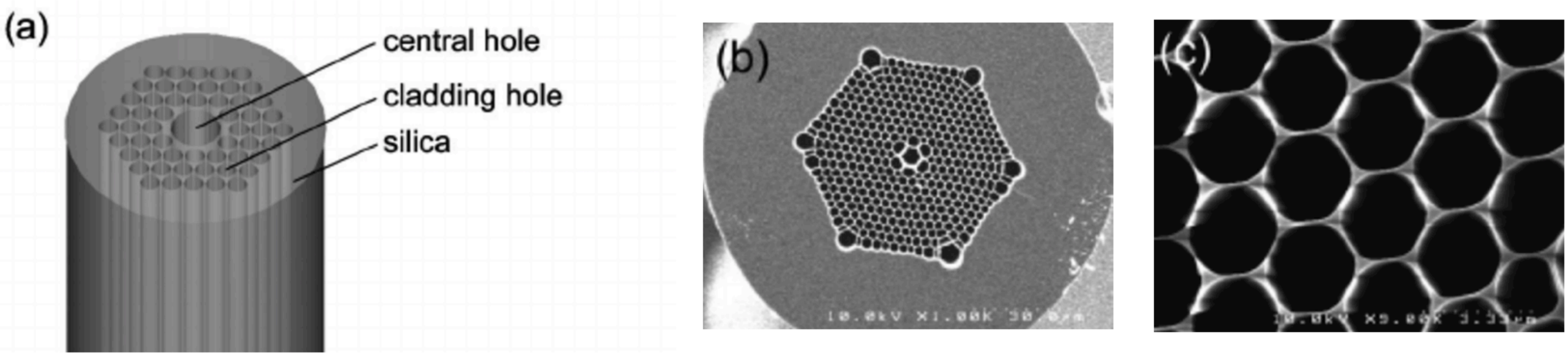
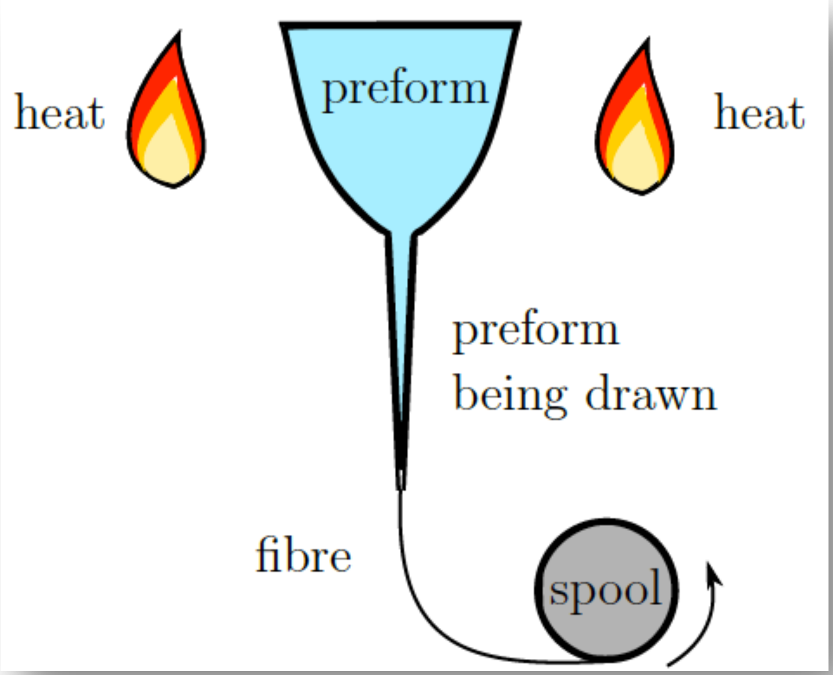


Figure: a) A MOF with a hexagonal arrangement. b) Diameters of central holes and cladding holes are 5.8 and 2.8 μm. c) Silica layers between adjacent air holes are of order 50—100 nm.

Y. Huang and Y. Xu and A. Yariv. Fabrication of functional microstructured optical fibers through a selective filling technique. Applied Physics Letters. 2004

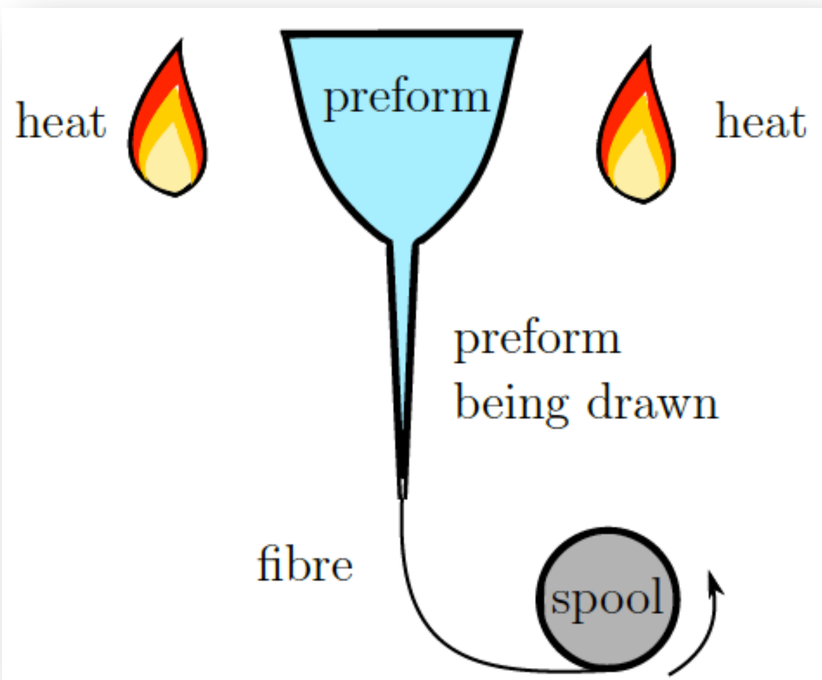
Fabrication process

1. Preform is fed into the furnace.
2. It gets heated up.
3. Preform is drawn into fiber.
4. Wrapped around spool.



Fabrication process

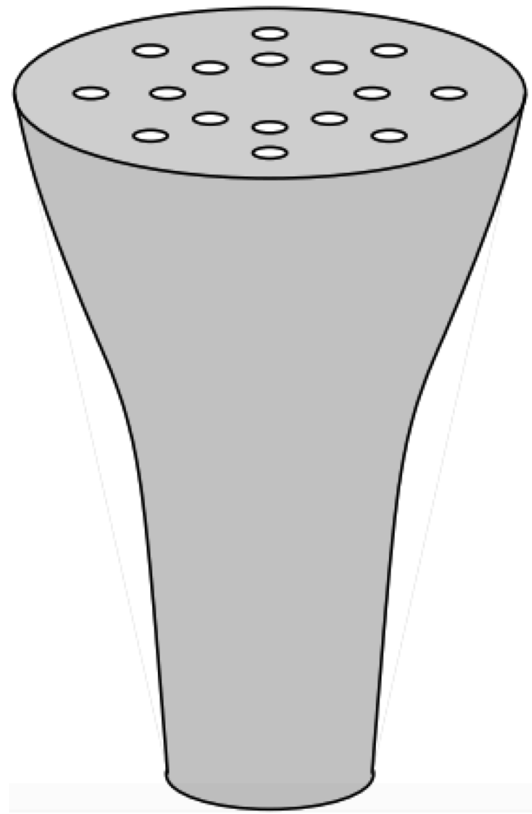
1. Preform is fed into the furnace.
2. It gets heated up.
3. Preform is drawn into fiber.
4. Wrapped around spool.

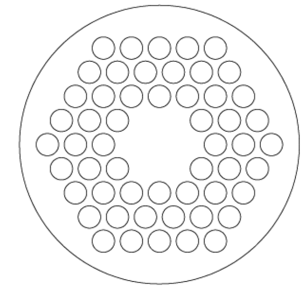


3D problem

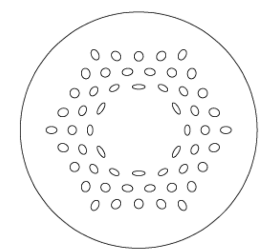
2D cross-plane problem

Preform is currently constructed through trial and error.





initial preform



drawn fibre

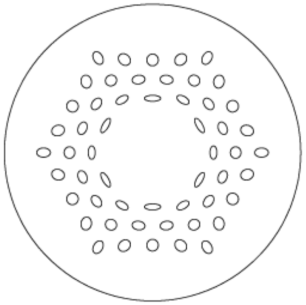
The cost of trial and error:

Preform cost: ~$8,000
Fiber draw: ~$5,500

AIM: develop mathematical models to tell experimentalists how to design the preform.

The problem: final fiber product differs from preform geometry

- A preform that has circular channels yields a fiber that has channels that are not circular.
- Surface tension **deforms** the geometrical cross section.
- These shape deformations are undesired.



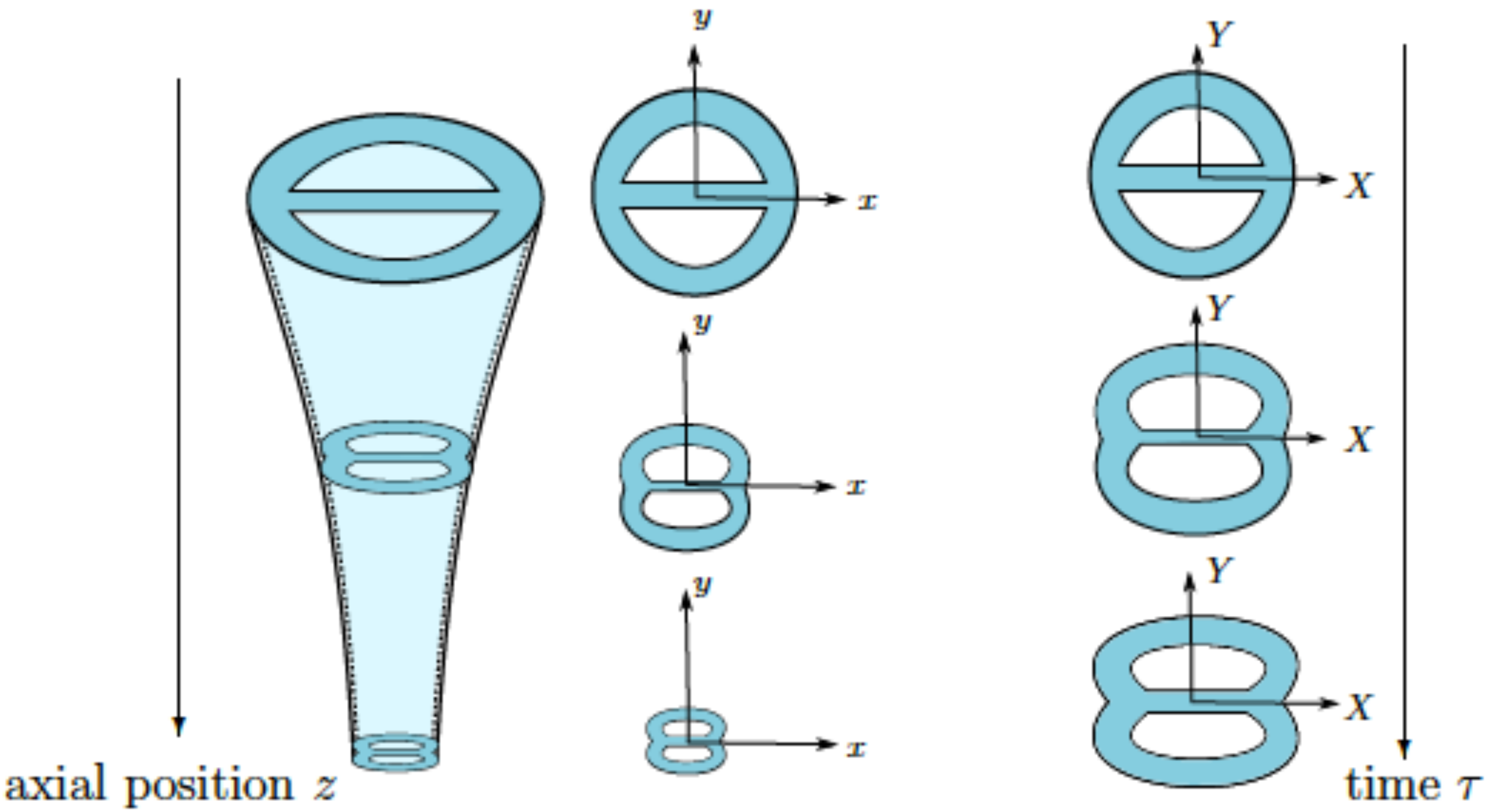
drawn fibre

Goal: Predict these deformations

What has been done?

- Griffiths and Howell studied the evolution of a single closed viscous loop.
- Key features:
  - Very viscous
  - Asymptotic approximation: long and thin geometry
  - Surface tension on free boundaries

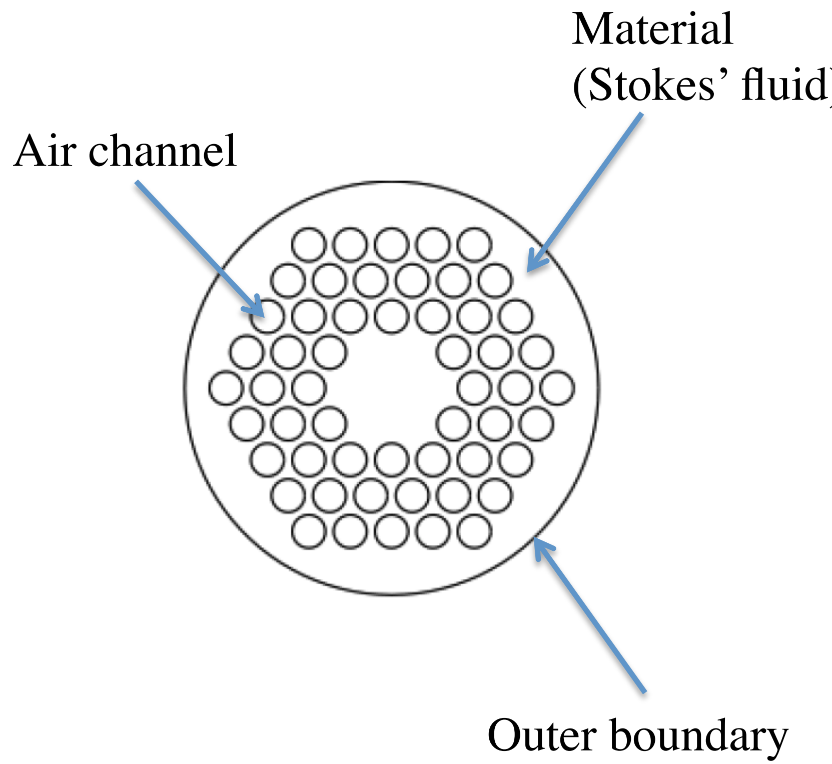
I.M. Griffiths and P. D. Howell. The surface-tension-driven evolution of a two-dimensional annular viscous tube, Journal of Fluid Mechanics. 2007



2D model describing the shape evolution of the cross-section of a MOF

- Crowdy uses complex variables and numerical methods to solve free BVPs with multiple holes.

Preform drawn into fiber



D.G. Crowdy. et al. Elliptical pore regularisation of the inverse problem for microstructured optical fiber fabrication, Journal of Fluid Mechanics. 2015

Model answers the following

Forward problem:

- For a given preform geometry, what will the final MOF look like?

Model answers the following

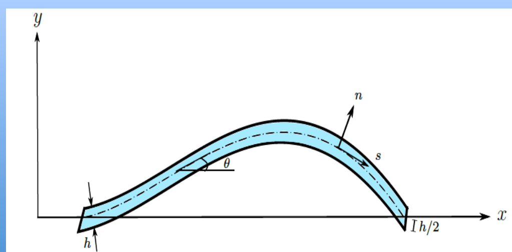
Forward problem:

- For a given preform geometry, what will the final MOF look like?

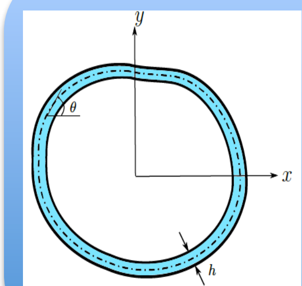
Inverse problem:

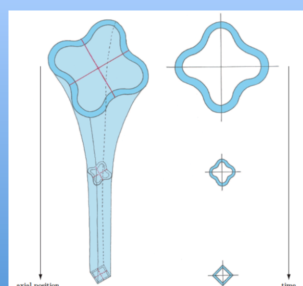
- For a desired MOF geometry, what preform will produce it?

Remainder of the talk

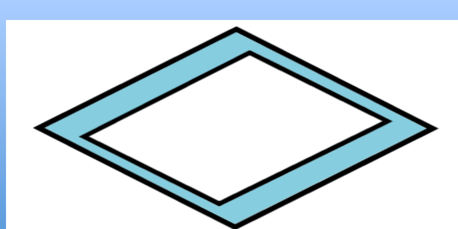


- Model derivation of uniformly thick viscous sheet





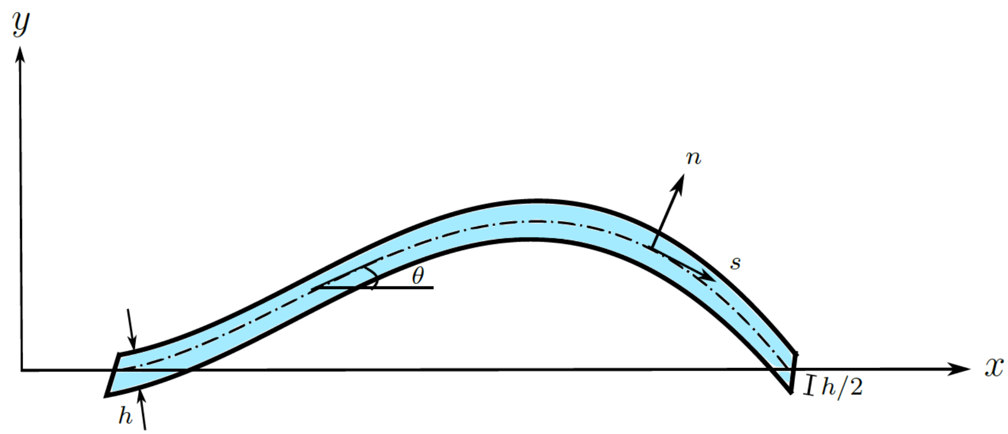
- Free viscous sheet
- Simply-supported viscous sheet
- Clamped viscous sheet
- Closed-loop viscous sheet



- Generalization to non-uniform thickness

Model derivation

<u>Governing equations</u>



- Geometrical identities: $\dfrac{\partial x}{\partial s} = \cos\theta, \quad \dfrac{\partial y}{\partial s} = \sin\theta$

- Moment balance: $\dfrac{\partial \mathcal{M}}{\partial s} = -\mathcal{A}\sin\theta + \mathcal{B}\cos\theta$

θ(s, t) is the centerline
A(t), *B(t)* are the total tensions in the *x*-, *y*-direction, respectively

- Evolution equation for thickness: $4\mu \frac{Dh}{Dt} = 2\gamma$
- Constitutive relation for bending moment:

$$\mathcal{M} = -\frac{\mu}{3} h^3 \frac{D}{Dt}\left(\frac{\partial \theta}{\partial s}\right)$$

- **Non-dimensionalization** of governing equations:

$$\boxed{\frac{\partial}{\partial s}\left[\frac{1}{3} h^3 \frac{D}{Dt}\left(\frac{\partial \theta}{\partial s}\right)\right] = \mathcal{A} \sin\theta - \mathcal{B} \cos\theta}$$

$$\boxed{\frac{Dh}{Dt} = \frac{1}{2}}$$

Evolution equation used in numerical simulations:

$$\frac{\partial}{\partial \tau}\left[\left(1+\frac{\tau}{2}\right)\frac{\partial^2\theta}{\partial \xi^2}\right] = A(\tau)\cos\theta + B(\tau)\sin\theta,$$

$$\text{where} \quad \xi = (1+\tau/2)s \quad \text{and} \quad \tau = t.$$

4th order in space ➔ four boundary conditions (BCs)
1st order in time ➔ one initial condition

Assume an initial uniform thickness, h, of magnitude one.

Evolution equation used in numerical simulations:

$$\frac{\partial}{\partial \tau}\left[\left(1+\frac{\tau}{2}\right)\frac{\partial^2\theta}{\partial \xi^2}\right] = A(\tau)\cos\theta + B(\tau)\sin\theta,$$

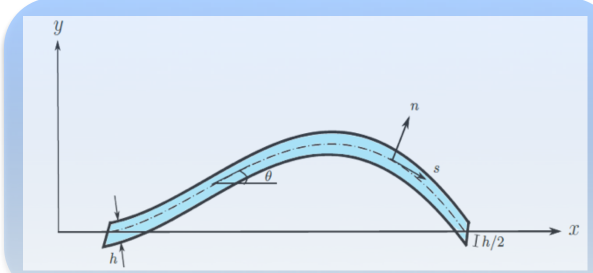
$$\text{where}\quad \xi = (1+\tau/2)s \quad \text{and} \quad \tau = t.$$

4^{th} order in space ➔ four boundary conditions (BCs)
1^{st} order in time ➔ one initial condition

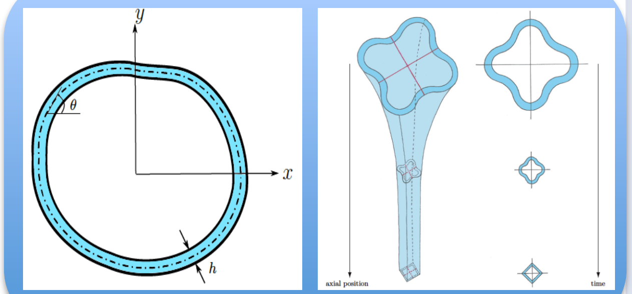
Assume an initial uniform thickness, h, of magnitude one.

Eulerian to Lagrangian coordinates: $\xi = s/L(t)$,
with length of viscous sheet, $L(t) = 1/(1+t/2)$, found by mass-conservation.

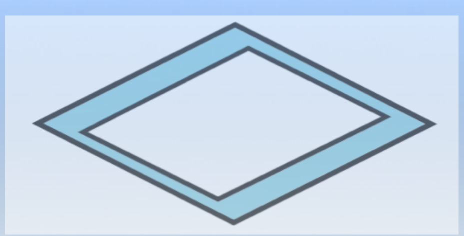
Remainder of the talk



- Model derivation of uniformly thick viscous sheet



- Free viscous sheet
- Simply-supported viscous sheet
- Clamped viscous sheet
- Closed-loop viscous sheet



- Generalization to non-uniform thickness

Free viscous sheet

- Total tensions are zero: $A(\tau) = 0 = B(\tau)$.
- BCs: Moments should vanish at both ends

$$\frac{\partial}{\partial \tau}\left[(1+\tau/2)\,\frac{\partial \theta}{\partial \xi}\right] = 0 \qquad \text{at } \xi = 0, 1.$$

- Evolution equation becomes

$$\frac{\partial}{\partial \tau}\left[(1+\tau/2)\,\frac{\partial^2 \theta}{\partial \xi^2}\right] = 0$$

Shape predictions

Choose the following initial corner condition:

$$\theta_0(\xi) = \begin{cases} \pi/4, & 0 \leq \xi \leq 1/2, \\ -\pi/4, & 1/2 \leq \xi \leq 1. \end{cases}$$

Exact analytical solution:

$$\theta(\xi, \tau) = \frac{\theta_0(\xi)}{1 + \tau/2} + C(\tau)$$

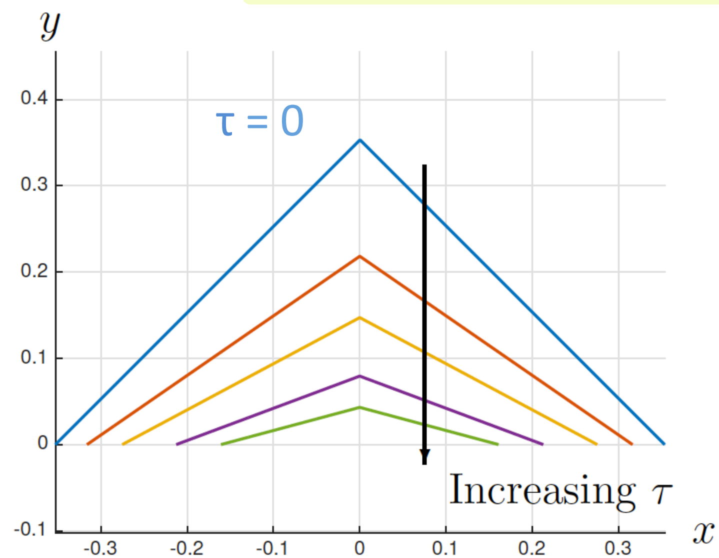
Shape predictions

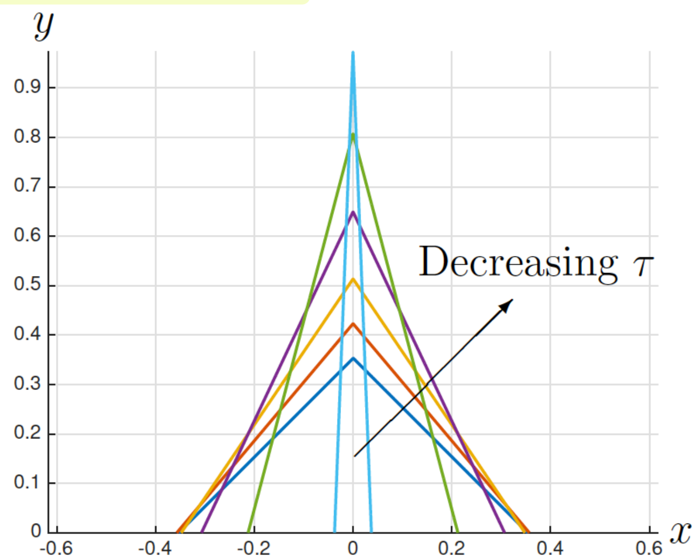
Choose the following initial corner condition:

$$\theta_0(\xi) = \begin{cases} \pi/4, & 0 \leq \xi \leq 1/2, \\ -\pi/4, & 1/2 \leq \xi \leq 1. \end{cases}$$

Exact analytical solution:

$$\theta(\xi, \tau) = \frac{\theta_0(\xi)}{1 + \tau/2} + C(\tau)$$





Simply-supported viscous sheet

<u>Boundary conditions</u>

- Specify the position of each end

$$X(0,\tau)=0, \quad X(1,\tau)=\int_0^1 \cos\theta\,\mathrm{d}\xi=\Delta x(1+\tau/2),$$

$$Y(0,\tau)=0, \quad Y(1,\tau)=\int_0^1 \cos\theta\,\mathrm{d}\xi=0$$

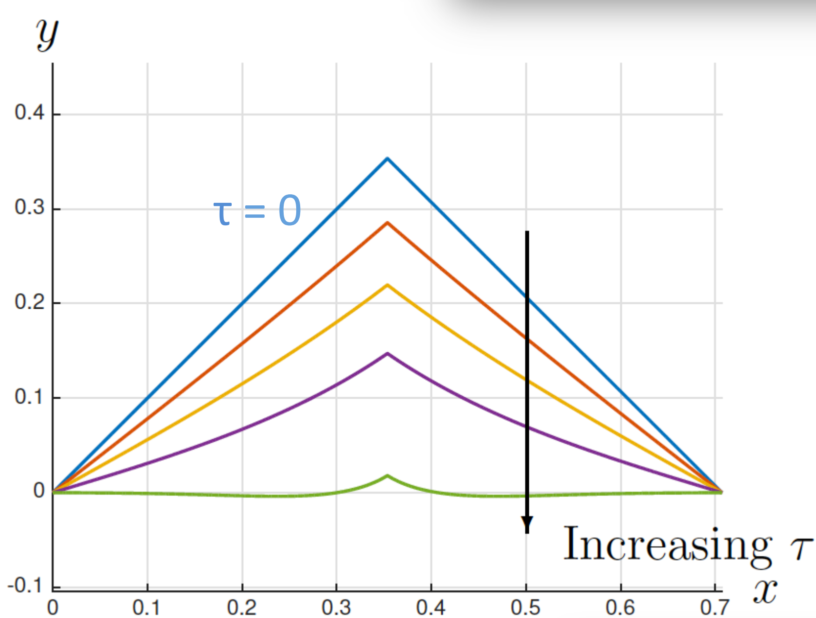
- Zero bending moments at both ends

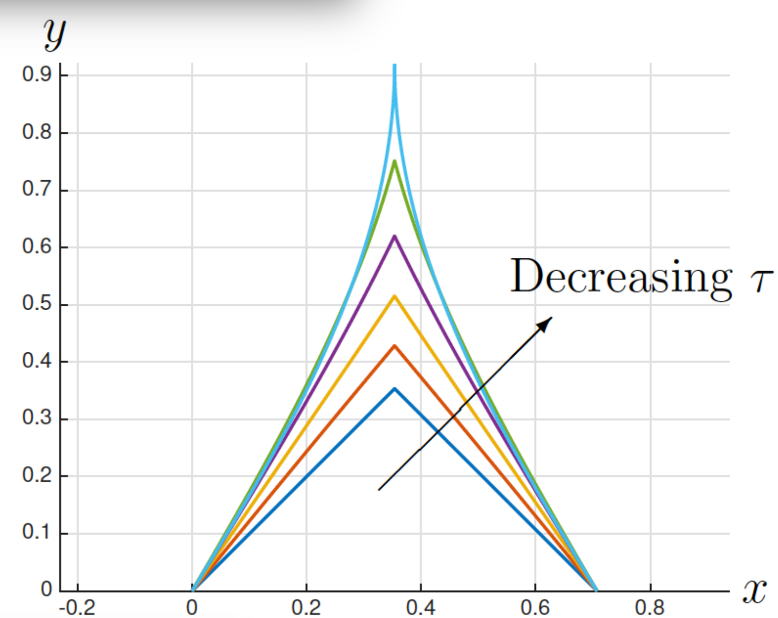
$$\frac{\partial}{\partial\tau}\left[(1+\tau/2)\frac{\partial\theta}{\partial\xi}\right]=0 \ \text{ at } \xi=0,1.$$

Shape predictions

Choose the following initial corner condition:

$$\theta_0(\xi) = \begin{cases} \pi/4, & 0 \leq \xi \leq 1/2, \\ -\pi/4, & 1/2 \leq \xi \leq 1. \end{cases}$$



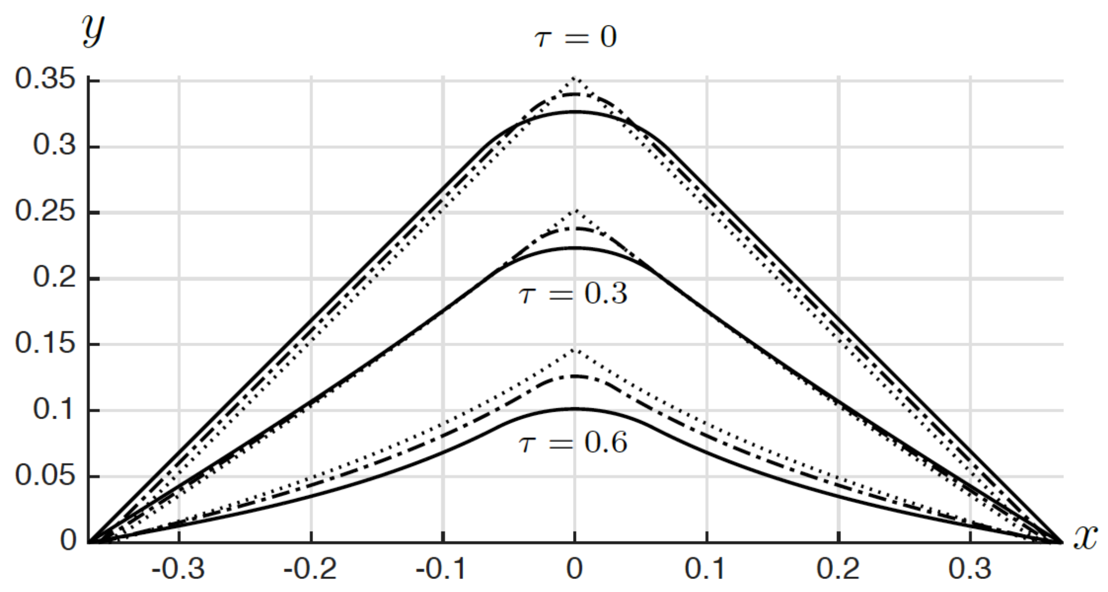


Reflectional symmetry:
$B(\tau) = 0$

Effect of corner smoothing

Focus on the following initial configurations

$$\theta_0(\xi) = \begin{cases} \pi/4, & 0 \le \xi \le 1/2 - \delta, \\ \pi(1-2\xi)/(8\delta), & 1/2 - \delta \le \xi \le 1/2 + \delta, \\ -\pi/4, & 1/2 + \delta \le \xi \le 1 \end{cases}$$



Clamped viscous sheet

Boundary conditions

- Specify the position at each end

$$X(0,\tau) = 0, \quad X(1,\tau) = \int_0^1 \cos\theta \, \mathrm{d}\xi = \Delta x(1+\tau/2),$$

$$Y(0,\tau) = 0, \quad Y(1,\tau) = \int_0^1 \cos\theta \, \mathrm{d}\xi = 0$$

- Fix the angle at each end

$\theta(0,\tau) = \theta(0,0) = \alpha$ and $\theta(1,\tau) = \theta(1,0) = \beta$.

Shape predictions

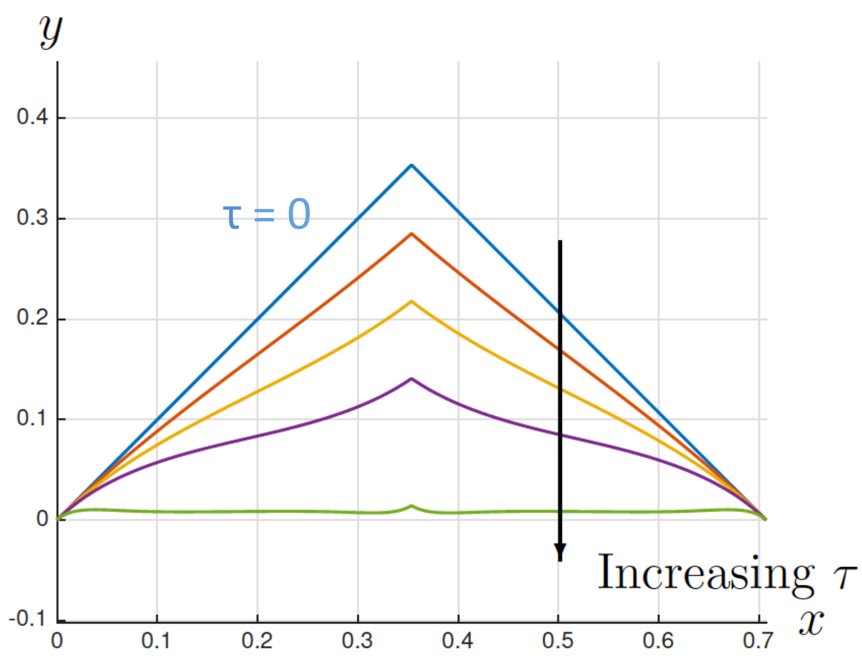
Choose as initial condition:

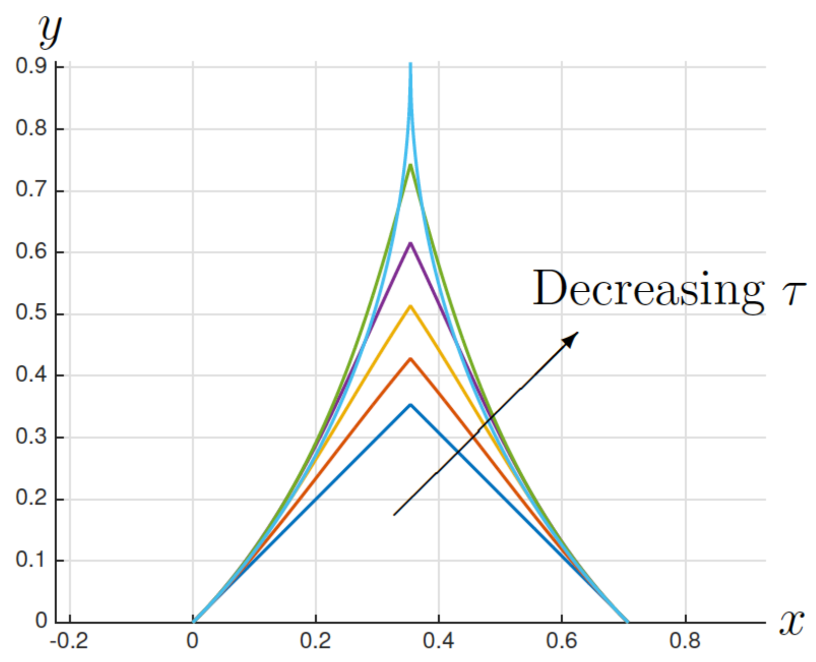
$$\theta_0(\xi) = \begin{cases} \pi/4, & 0 \leq \xi \leq 1/2, \\ -\pi/4, & 1/2 \leq \xi \leq 1. \end{cases}$$

Clamped conditions:

$$\theta(0,\tau) = \alpha,\ \theta(1,\tau) = -\alpha$$

$$\alpha = \pi/4$$





Closed-loop viscous sheet

Boundary conditions

- The ends of the viscous sheet meet

$$X(1,\tau) \equiv X(0,\tau), \quad Y(1,\tau) \equiv Y(0,\tau)$$

- The angle at the ends is given by

$$\theta(1,\tau) \equiv \theta(0,\tau) + 2\pi$$

where we choose $\theta(0,\tau) = 0$.

Steady solutions

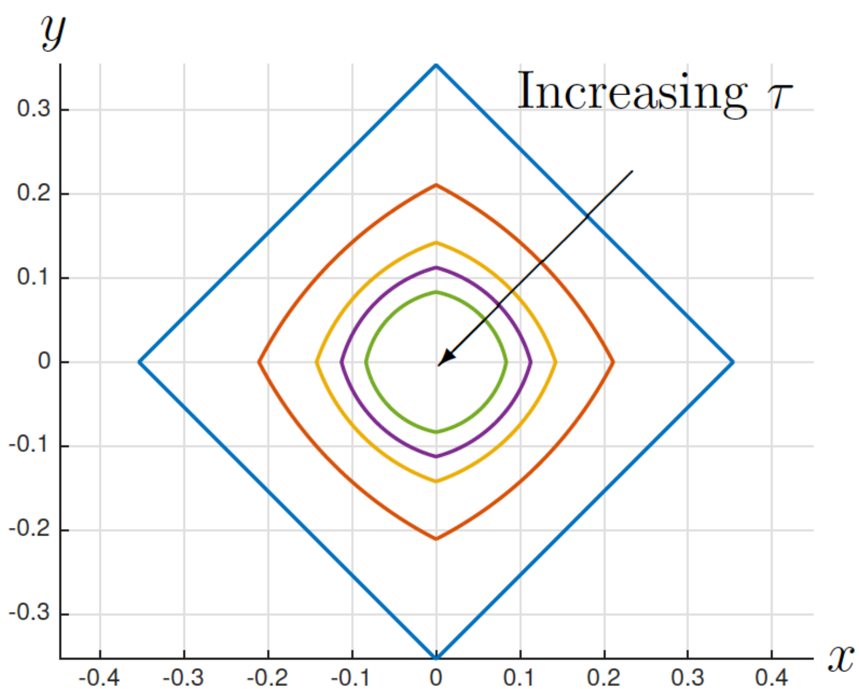
- Focus on profiles with zero total tensions in either direction.
- Profiles that possess rotational symmetry:

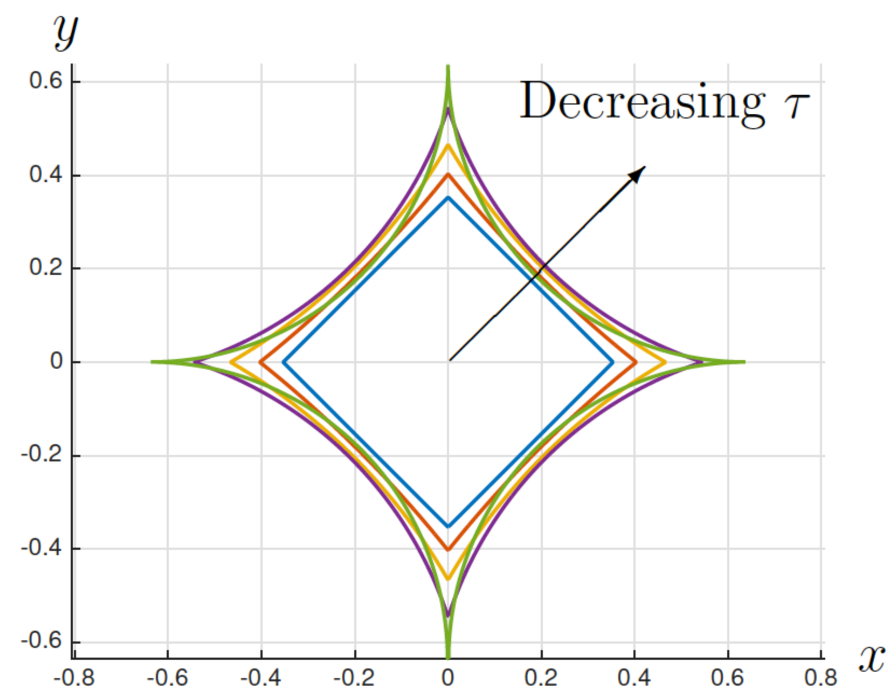
$$\theta\left(\frac{1}{n}+\xi\right)=\theta(\xi)+\frac{2\pi}{n}$$

- Equation solved analytically as

$$\theta(\xi,\tau)=\frac{\theta(\xi,0)-2\pi\xi}{1+\tau/2}+2\pi\xi$$

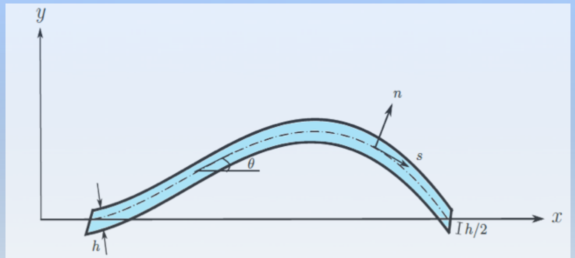
As τ increases, the centerline tends to a circle, regardless of the initial shape.



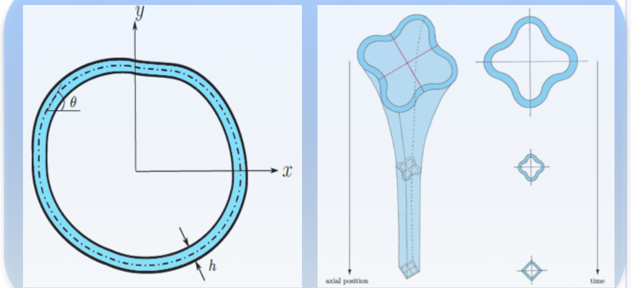


- Corners persist but are not preserved.

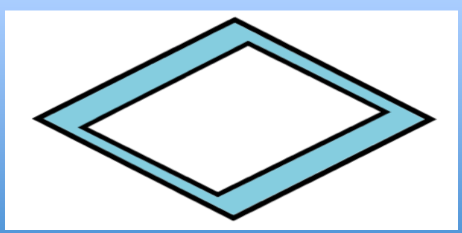
Remainder of the talk



- Model derivation of uniformly thick viscous sheet



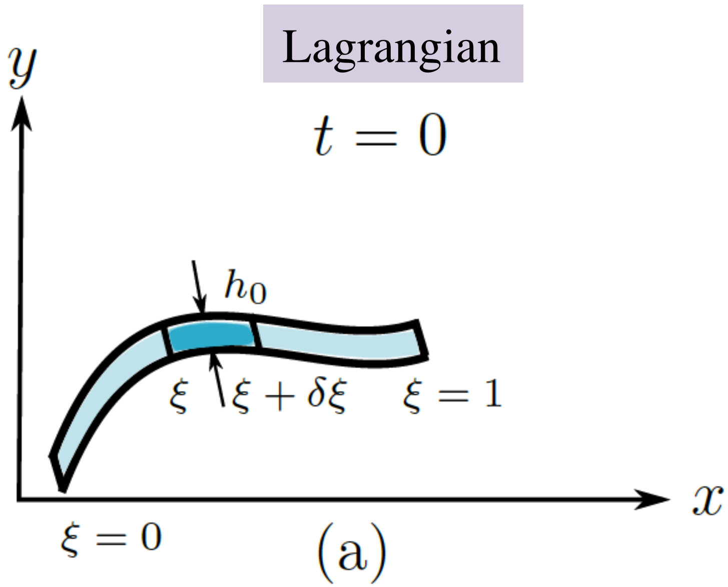
- Free viscous sheet
- Simply-supported viscous sheet
- Clamped viscous sheet
- Closed-loop viscous sheet

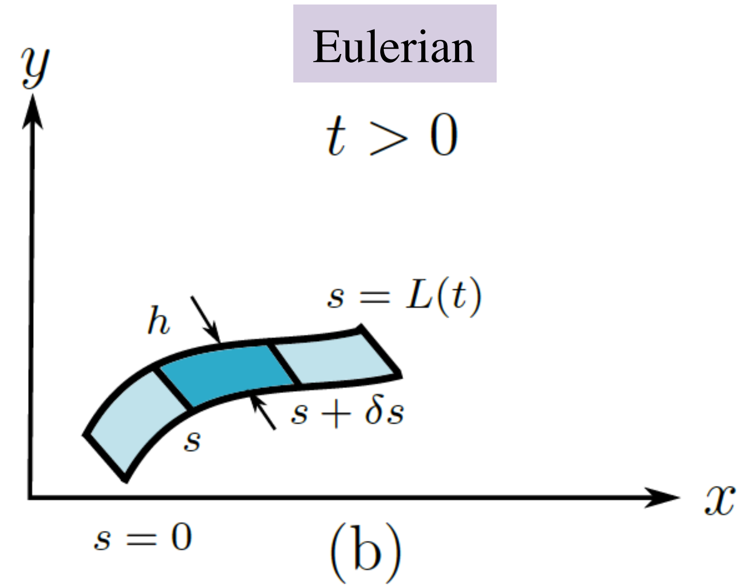


- Generalization to non-uniform thickness

Relation between *s* and ξ

- Net conservation relation: $h_0(\xi)\delta\xi = h(\xi,\tau)\delta s$





- Thus, we obtain: $\boxed{\dfrac{\partial s}{\partial \xi} = \dfrac{h_0(\xi)}{h_0(\xi) + \tau/2}}$

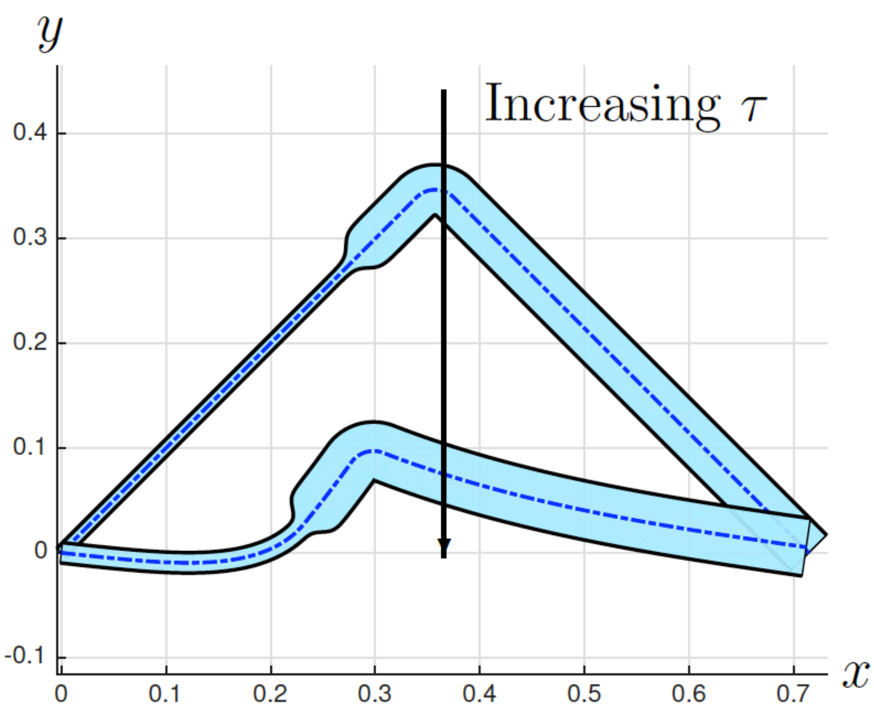
Plotting thickness profiles

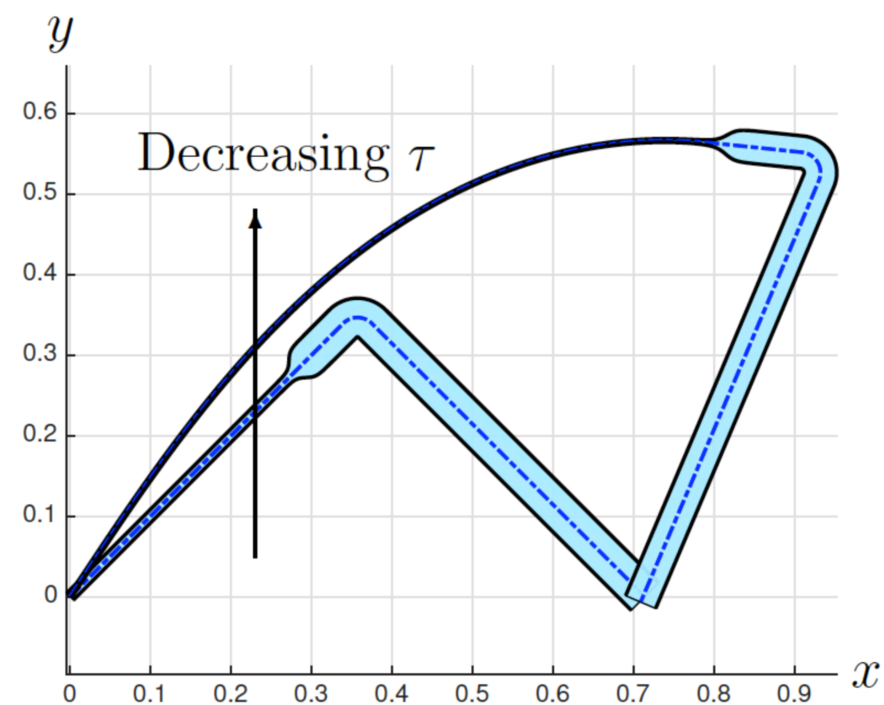
- To plot the thickness profiles we use:

$$\begin{pmatrix} x \\ y \end{pmatrix} \pm \frac{1}{2}\epsilon h\,\boldsymbol{n} = \begin{pmatrix} x \\ y \end{pmatrix} \pm \frac{1}{2}\epsilon h \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

where $\epsilon \neq 0$ is small and $h(\xi,\tau) = h_0(\xi) + \tau/2.$

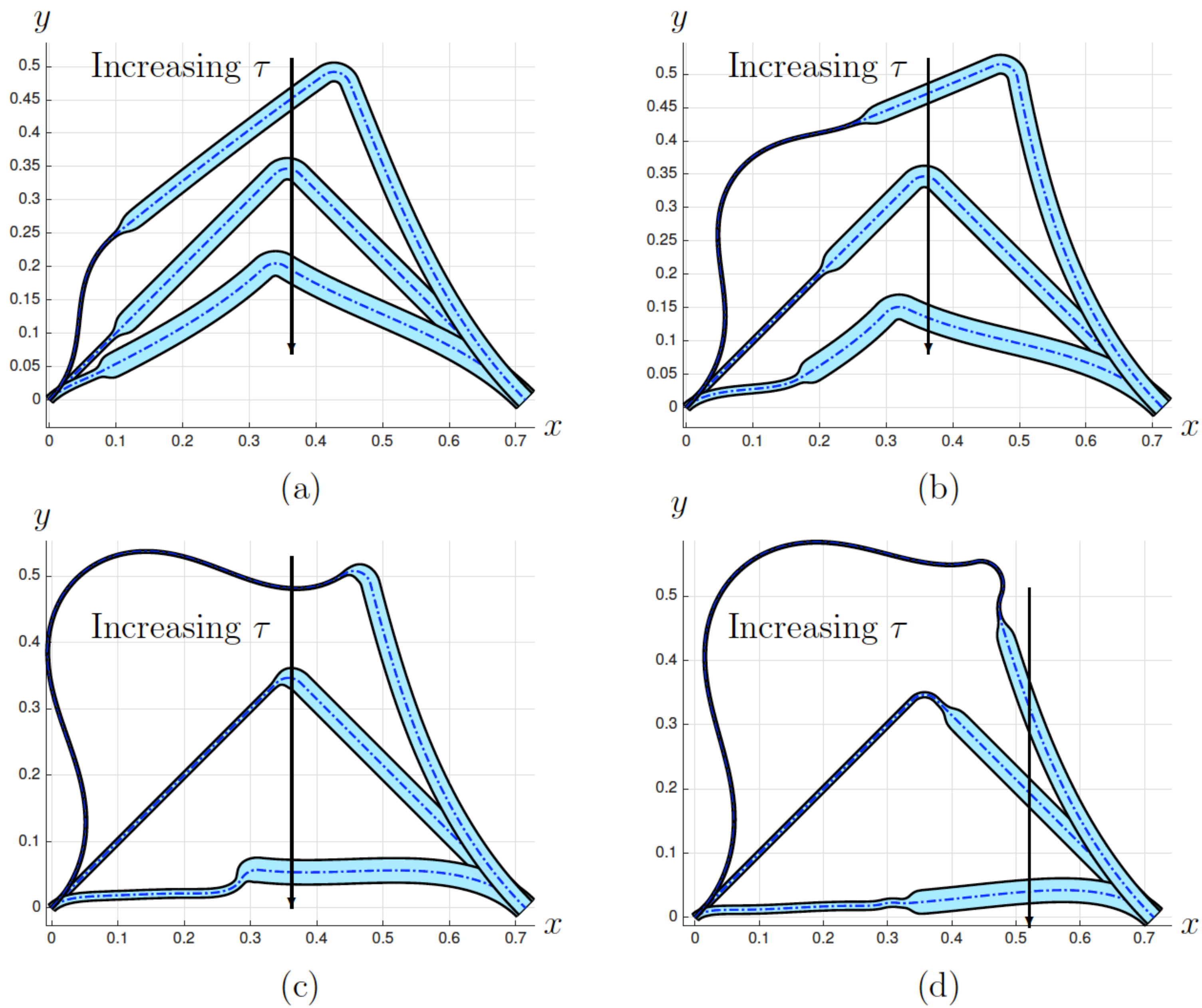
Simply-supported and non-uniform





- Thin parts of the viscous sheet evolve at a faster rate than thick parts.

Clamped and non-uniform



(a) (b) (c) (d)

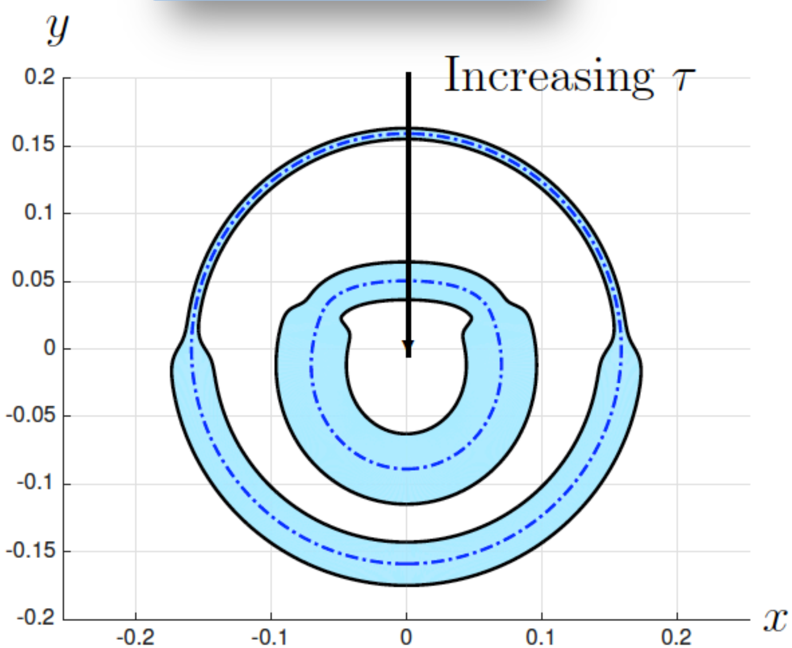
Shape predictions

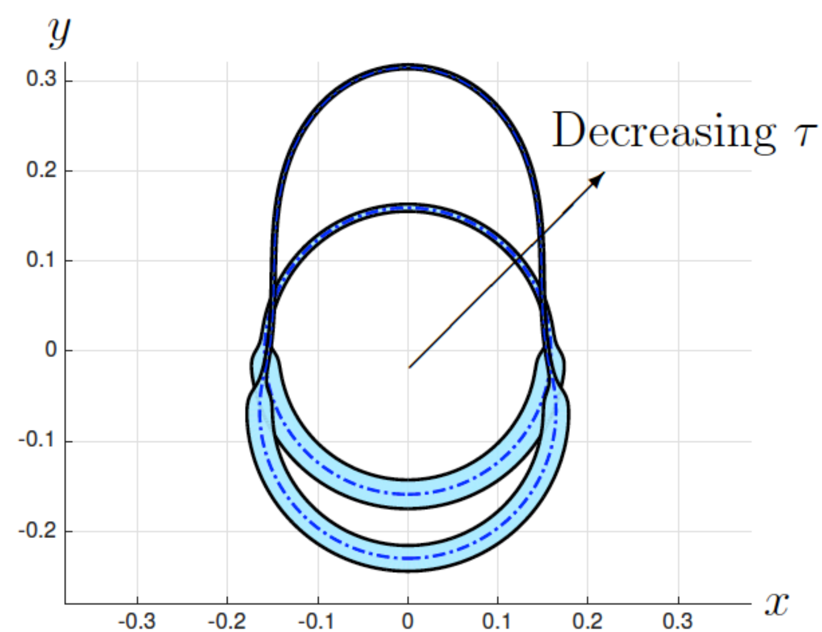
Choose initial conditions as:

$$h_0(\xi) = \begin{cases} \varepsilon \tanh\left[(\xi - \xi^\star)/\nu\right] + h^\star, & 0 \le \xi \le 1/2, \\ -\varepsilon \tanh[(\xi - \{1 - \xi^\star\})/\nu] + h^\star, & 1/2 \le \xi \le 1, \end{cases}$$

and $\theta_0(\xi) = 2\pi\xi$

This is an initially circular centerline.





Steady solutions

$$\boxed{\mathcal{A} = \mathcal{B} = 0}$$

If the centerline and the thickness profiles satisfy:

$$\theta_0\left(\xi + \frac{k}{n}\right) = \theta_0(\xi) + \frac{2\pi k}{n} \text{ and } h_0\left(\xi + \frac{k}{n}\right) = h_0(\xi),$$

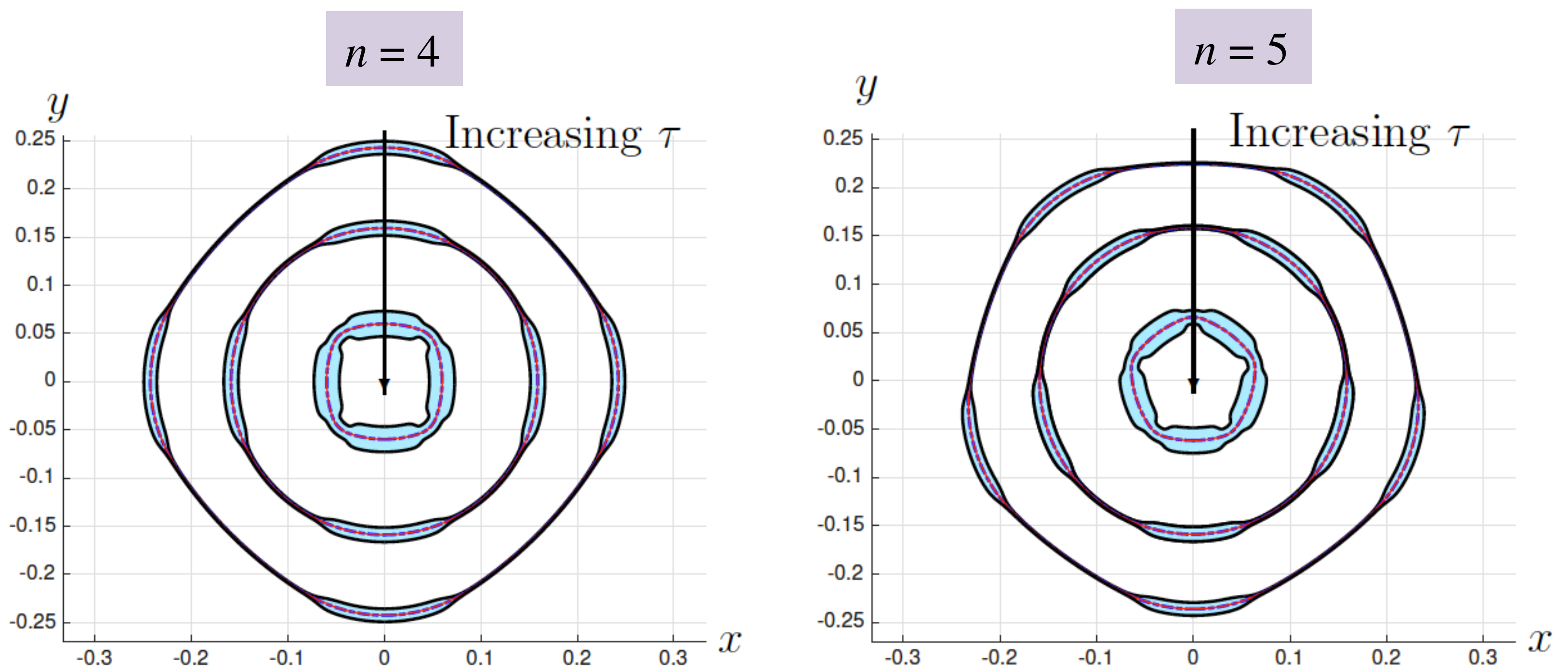
where n is a common factor of the degrees of rotational symmetry.

Steady solutions

- Example: $$\theta_0(\xi) = 2\pi\xi$$

$$h_0(\xi) = \sum_j (-1)^j \varepsilon \tanh\left(\frac{\xi - (2j-1)\xi^\star}{\nu}\right) + h^\star$$

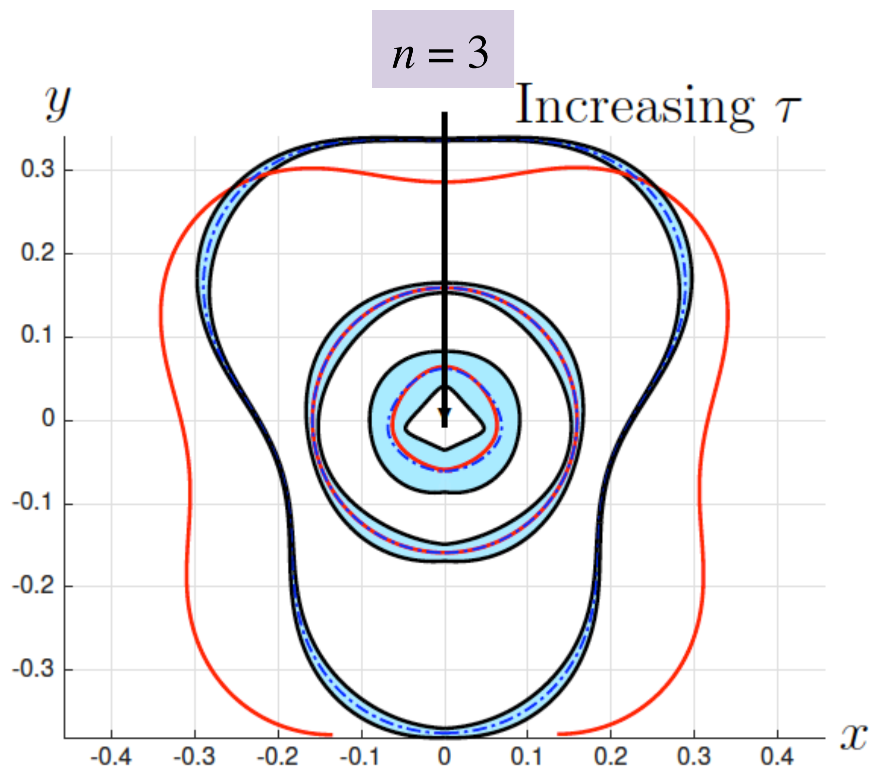
with $j \in \{1, 2, 3, \ldots, 2n\}$, $\xi^\star = 1/(4n)$ and $n \geq 2$.

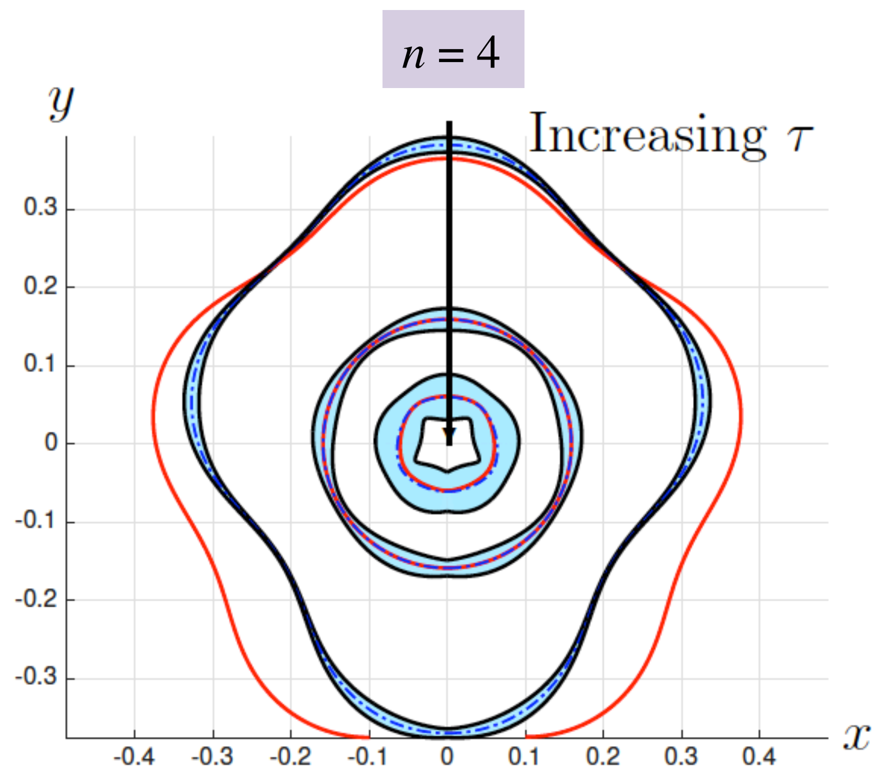


No rotational symmetry

- Ends do not join up if we assume $\mathcal{A} = \mathcal{B} = 0$ for not rotationally symmetric profiles.

$$h_0(\xi) = \varepsilon \left[\sin\left\{(2n+1)\pi\xi\right\} + 1\right] + h^{\star}$$

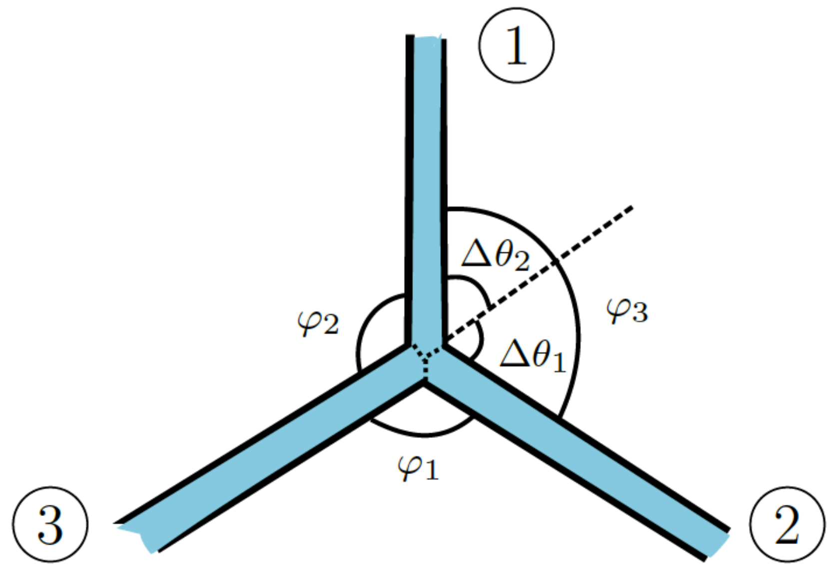




Junction of *n* viscous sheets

- Balancing moments at the n-viscous-sheets junction:

$$\frac{\partial}{\partial \tau}\left[\left(1+\frac{\tau}{2}\right)(\varphi_i-\varphi_j)\right]=0,$$



- Identity: $\sum_{i=1}^{n}\varphi_i=2\pi,$

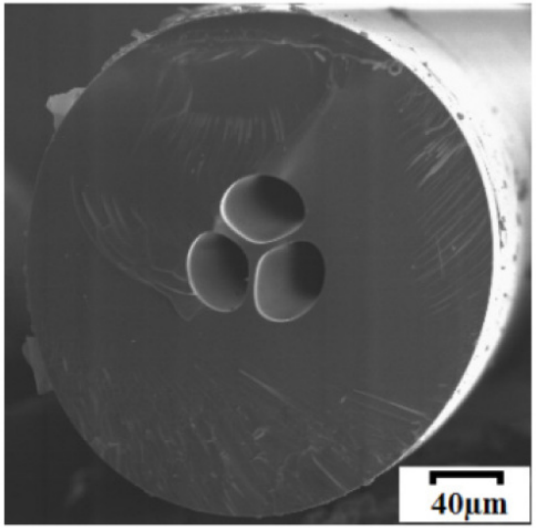
- All interior angles tend to same value as time increases.

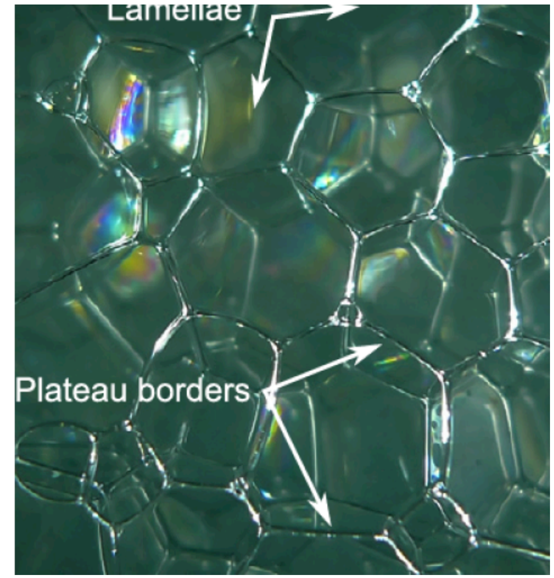
$$\varphi_i(\tau)=\frac{2\pi}{n}+\frac{\varphi_i(0)-2\pi/n}{1+\tau/2}$$

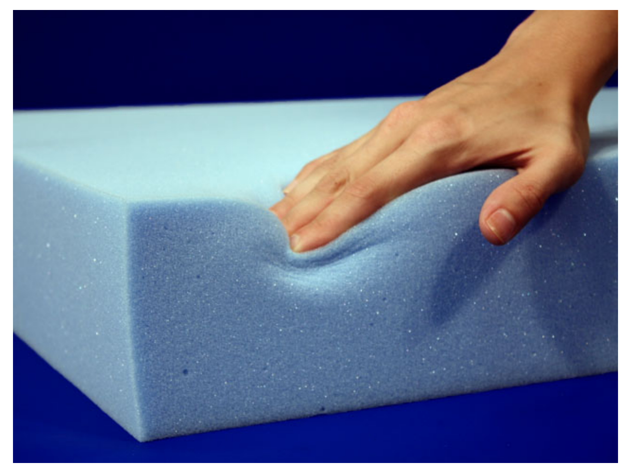
Impact of our model

- Inverse problem is easy to solve.
- We overcome the need for expensive computational methods.
- We get useful insights not available through experiments.
- The cost of trial and error is reduced, saving lots of money!

Applications

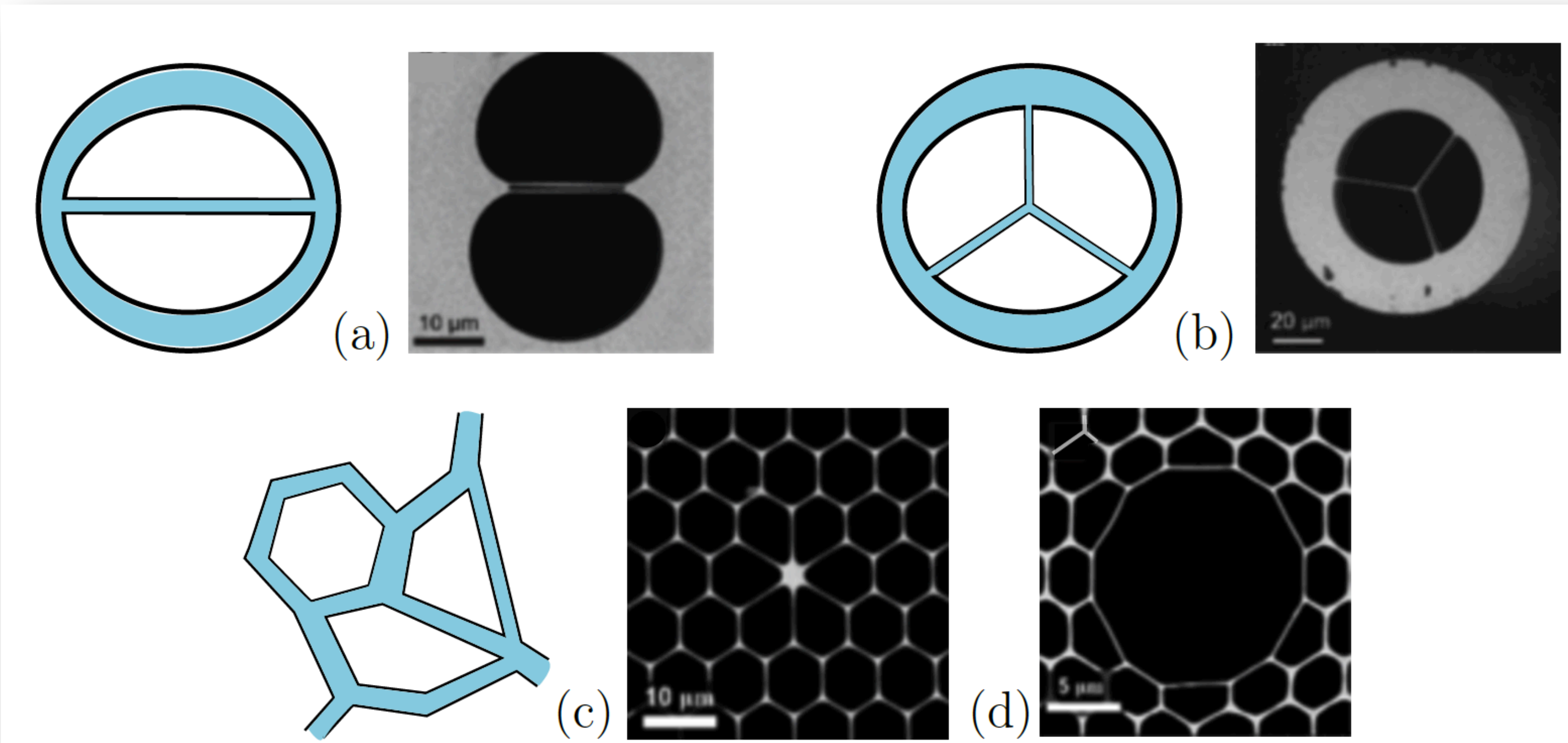


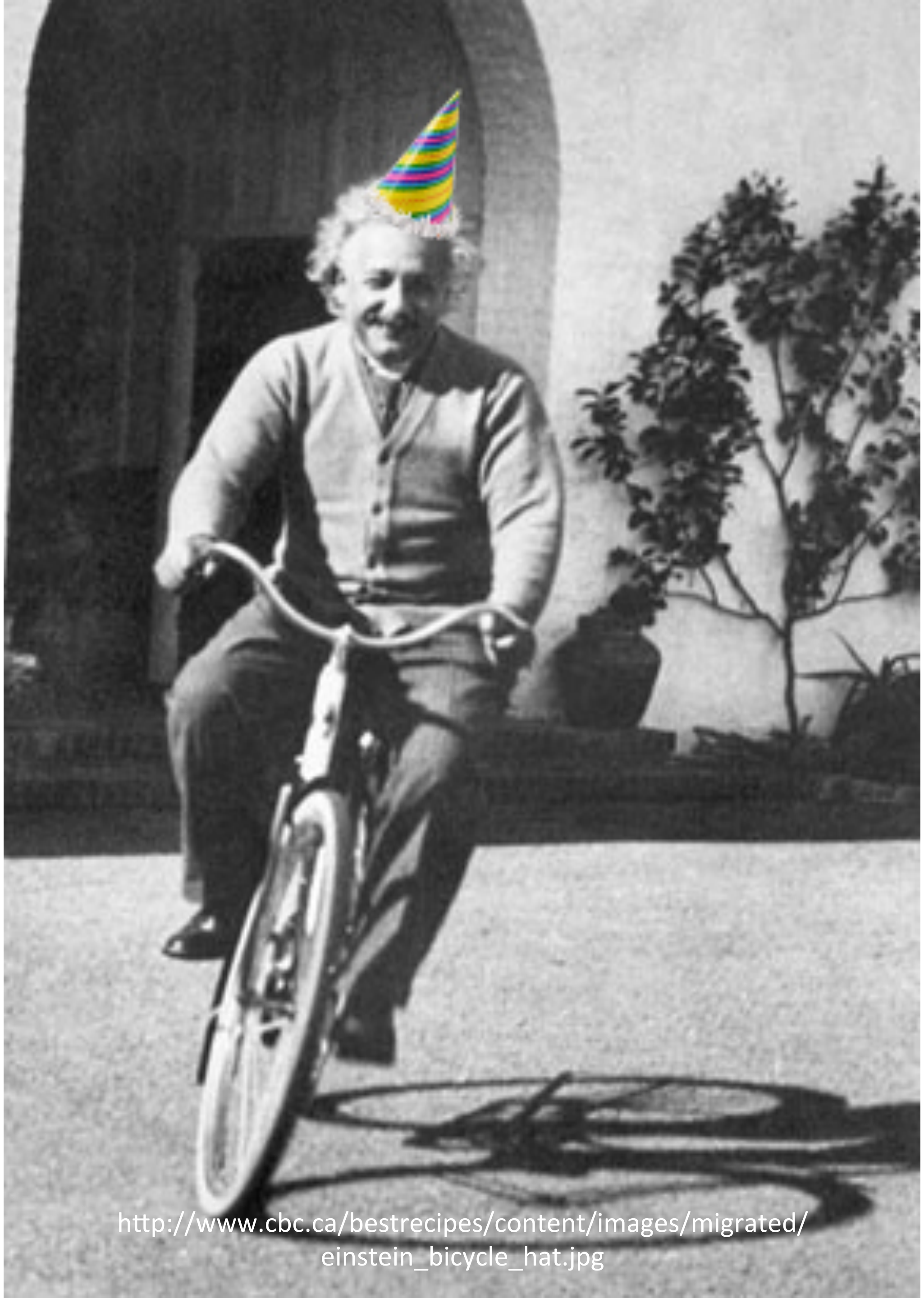




Future work

- Connect edges with non-uniform thickness.




http://www.cbc.ca/bestrecipes/content/images/migrated/
einstein_bicycle_hat.jpg