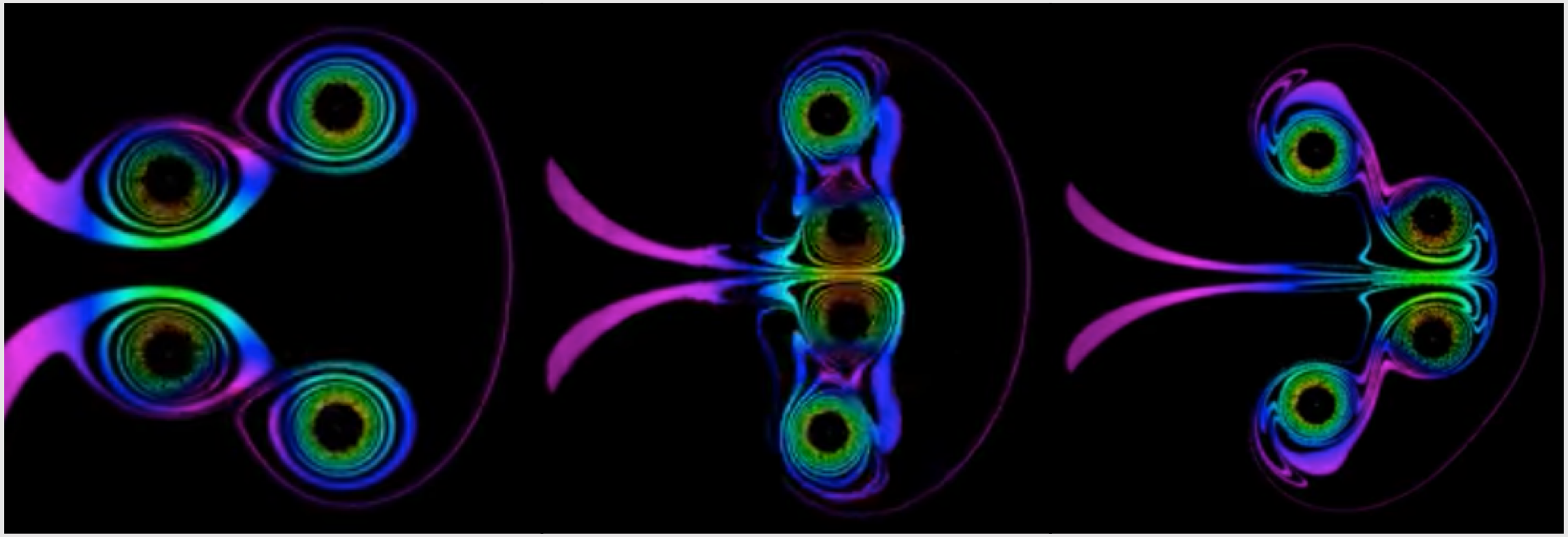
The leapfrogging of vortex pairs



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Student AIM Seminar 2018

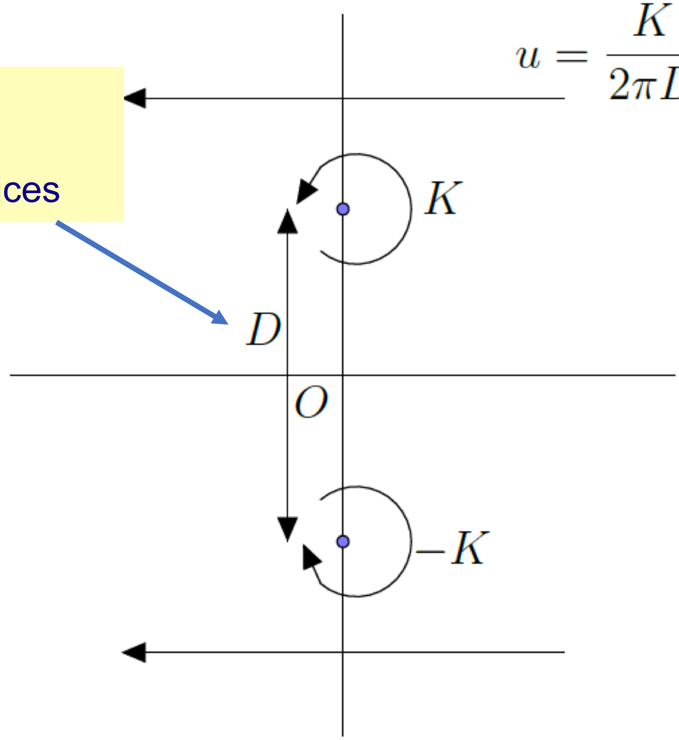
A vortex pair in a uniform flow

Counter-rotating vortex pair

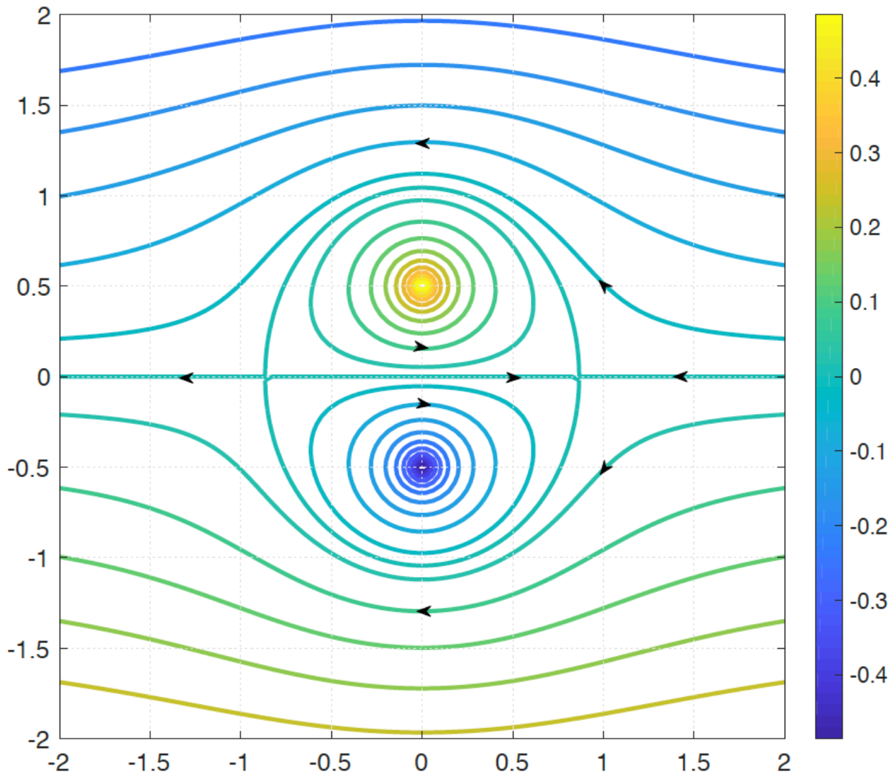
Vortex strength

$$u = \frac{K}{2\pi D}$$

Distance between point vortices



K = 1,-1



Streamline topologies

Complex potential and stream function

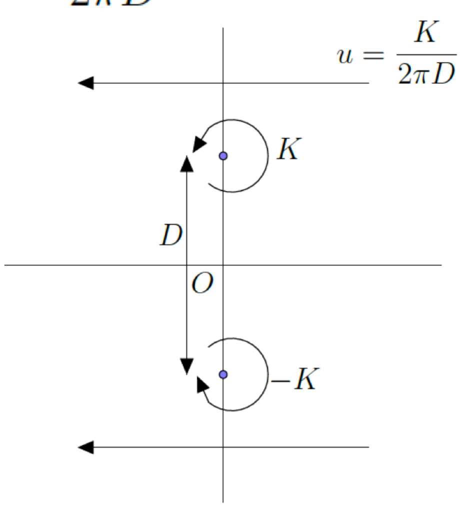
Complex potential $w(z) = \phi + i\psi$

We can show that the relative positions of the vortices are maintained.

$$w = -\frac{iK}{2\pi}\ln\left|z - \frac{iD}{2}\right| - \frac{i(-K)}{2\pi}\ln\left|z + \frac{iD}{2}\right| - \frac{K}{2\pi D}z$$
$$= -\frac{iK}{2\pi}\ln\left|x + i\left(y - \frac{D}{2}\right)\right| + \frac{iK}{2\pi}\ln\left|x + i\left(y + \frac{D}{2}\right)\right| - \frac{K}{2\pi D}(x + iy)$$

Compare the imaginary parts to get the stream function:

$$\psi = -\frac{K}{2\pi}\ln\left[x^2 + \left(y - \frac{D}{2}\right)^2\right]^{1/2} + \frac{K}{2\pi}\ln\left[x^2 + \left(y + \frac{D}{2}\right)^2\right]^{1/2} - \frac{K}{2\pi D}y$$
$$= \frac{K}{4\pi}\ln\left[\frac{x^2 + (y + D/2)^2}{x^2 + (y - D/2)^2}\right] - \frac{K}{2\pi D}y$$

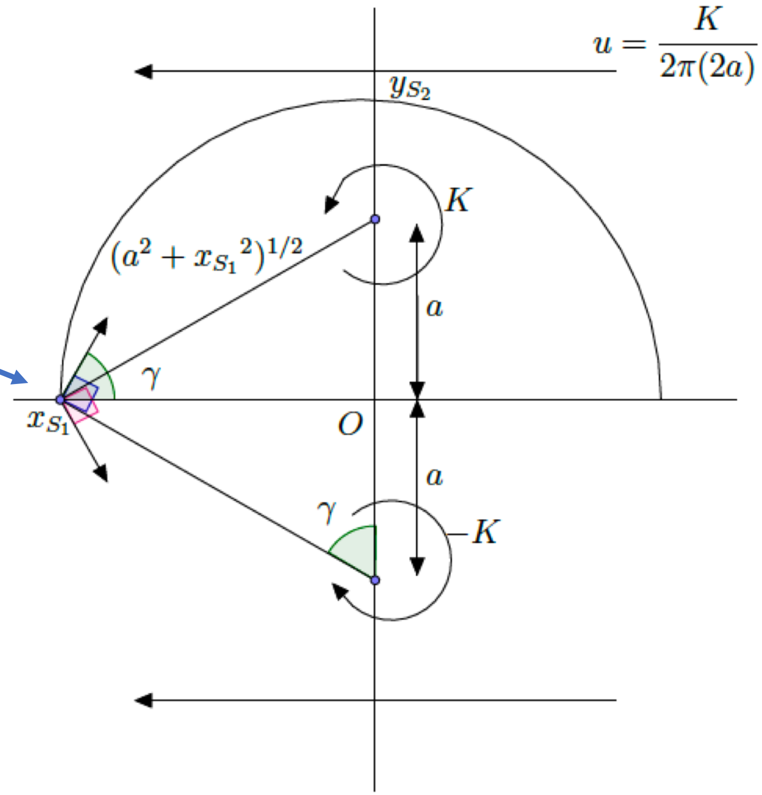


Streamlines are defined by a constant stream function.

How do we find the separating streamline?

$$\psi = -\frac{K}{2\pi}\ln\left[x^2+\left(y-\frac{D}{2}\right)^2\right]^{1/2} + \frac{K}{2\pi}\ln\left[x^2+\left(y+\frac{D}{2}\right)^2\right]^{1/2} - \frac{K}{2\pi D}y$$

$$= \frac{K}{4\pi}\ln\left[\frac{x^2+(y+D/2)^2}{x^2+(y-D/2)^2}\right] - \frac{K}{2\pi D}y$$



Find stagnation point

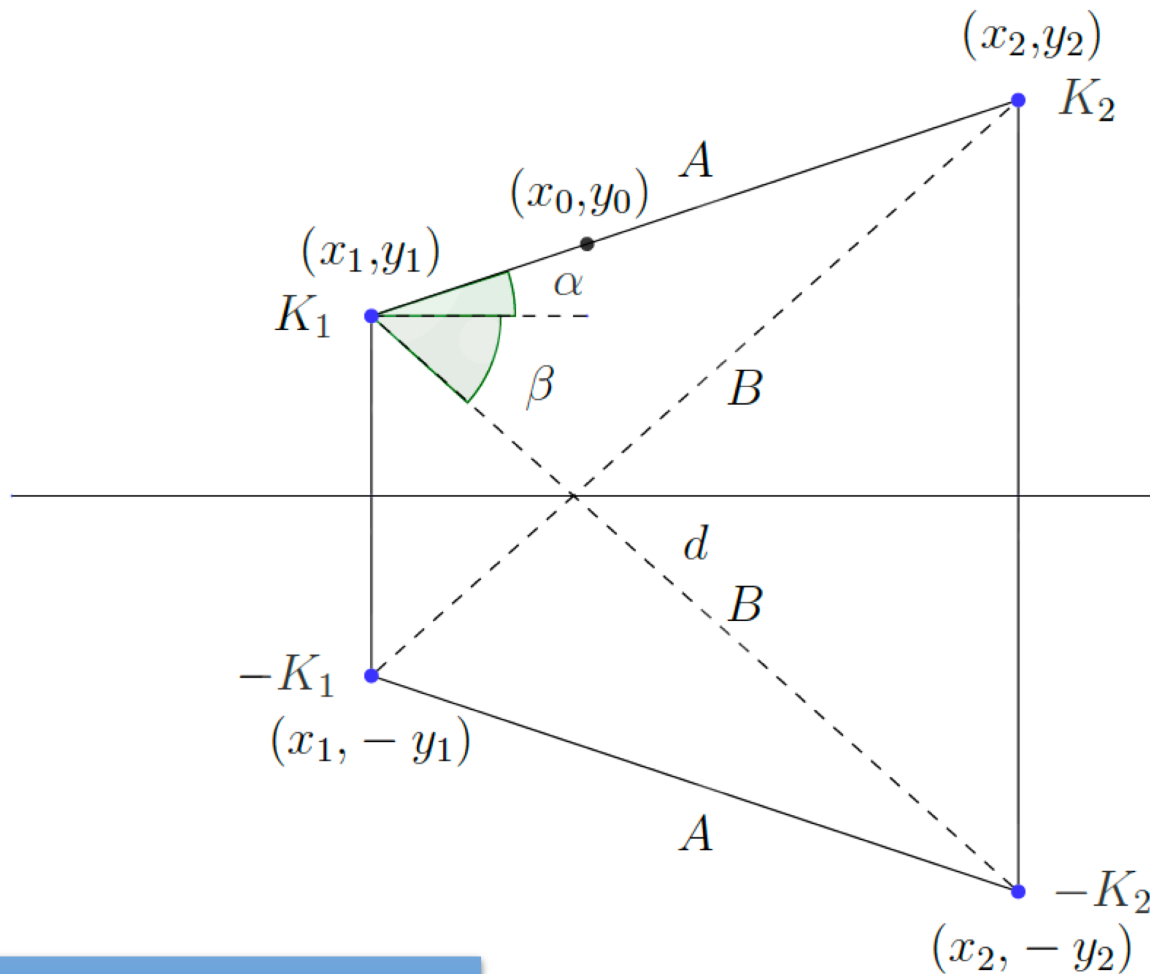
$$(x_{S_1}, y_{S_1}) = (\pm\sqrt{3}a, 0)$$

Two vortices and stagnation points form an equilateral triangle

Vortex rings

Leapfrogging motion of vortex pairs

Schematic diagram

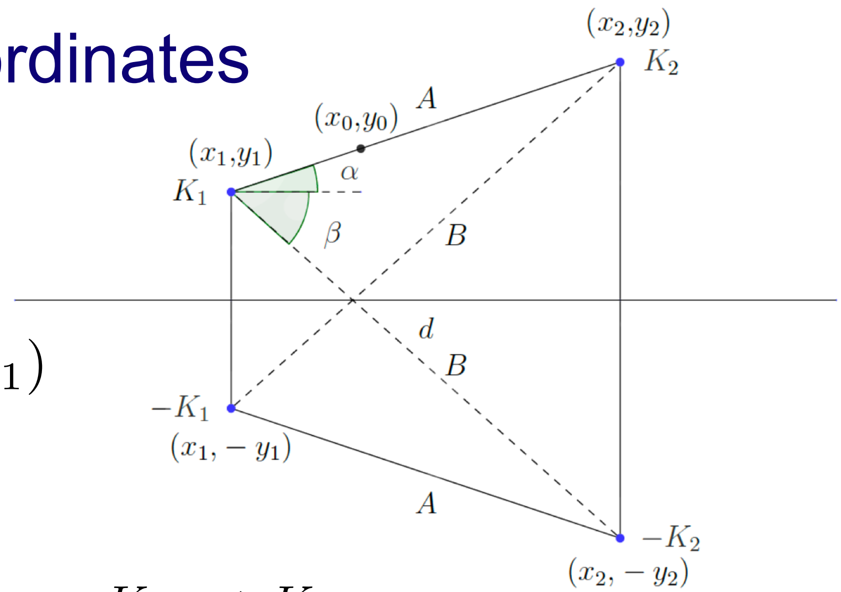


Both $K_1, K_2 > 0$

Assume that $K_2/K_1 \leq 1$

Question:
Is the periodic motion conditional on something?

Important coordinates



Relative coordinates

$$(x_r, y_r) = (x_2 - x_1, y_2 - y_1)$$

Centre-of-vorticity coordinates

$$x_0 = \frac{K_1 x_1 + K_2 x_2}{K_1 + K_2}, \qquad y_0 = \frac{K_1 y_1 + K_2 y_2}{K_1 + K_2}$$

From figure, observe that

$$A = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2} \qquad B = [(x_2 - x_1)^2 + (y_2 + y_1)^2]^{1/2}$$

$$\cos\alpha = \frac{x_2 - x_1}{A} \qquad \sin\alpha = \frac{y_2 - y_1}{A}$$

$$\cos\beta = \frac{x_2 - x_1}{B} \qquad \sin\beta = \frac{y_2 + y_1}{B}$$

Governing equations (induced velocities)

Consider the vortex with strength K_1 and find the velocities induced by the other three vortices

$$\dot{x}_1 = \frac{K_1}{2\pi(2y_1)} + \frac{K_2 \sin\alpha}{2\pi A} + \frac{K_2 \sin\beta}{2\pi B}, \qquad \dot{x}_2 = \frac{K_2}{2\pi(2y_2)} - \frac{K_1 \sin\alpha}{2\pi A} + \frac{K_1 \sin\beta}{2\pi B},$$

$$\dot{y}_1 = -\frac{K_2 \cos\alpha}{2\pi A} + \frac{K_2 \cos\beta}{2\pi B}, \qquad \dot{y}_2 = \frac{K_1 \cos\alpha}{2\pi A} - \frac{K_1 \cos\beta}{2\pi B}.$$

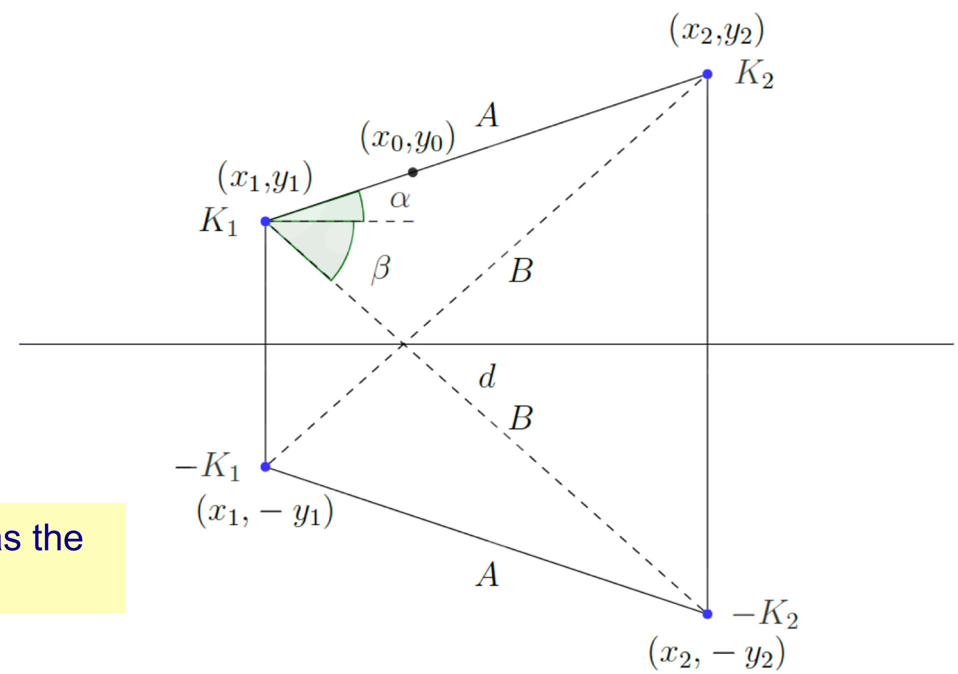
Note that

$$K_1 \dot{y}_1 + K_2 \dot{y}_2 = 0$$

which implies that

$$\dot{y}_0 = 0$$

So, y_0 a constant and can be considered as the mean width of the vortex pair.



Thinking in terms of a Hamiltonian

Main idea:
Vortex motion = finite-dimensional Hamiltonian system

Recall that the motion of individual fluid particles is given by

$$\dot{x} = \frac{\partial \psi}{\partial y} \qquad \text{where} \qquad \dot{y} = -\frac{\partial \psi}{\partial x}$$

This has a Hamiltonian structure with Hamiltonian ψ and conjugate variables x, y.
The stream function for the fluid due to N vortices is

$$\psi = \sum_{i=1}^{N} \psi_i(\boldsymbol{x}) \qquad \text{where} \qquad \psi_i = -\frac{K_i}{2\pi} \ln ||\boldsymbol{x} - \boldsymbol{x}_i||$$

H is related to ψ and physically it represents the kinetic energy of the N-vortex system.

Hamiltonian and velocities

Hamiltonian system:

$$H = -\sum_{i \neq j} \frac{K_i K_j}{4\pi} \ln ||\boldsymbol{x}_i - \boldsymbol{x}_j||$$

$$K_i \dot{x}_i = \frac{\partial H}{\partial y_i} \qquad \text{and} \qquad K_i \dot{y}_i = -\frac{\partial H}{\partial x_i}$$

The Hamiltonian is a conserved quantity that can be thought of as the **energy** of the vortex system.

$$H = \frac{1}{4\pi} \ln \left((2y_1)^{K_1^2} (2y_2)^{K_2^2} \left[\frac{(x_2 - x_1)^2 + (y_2 + y_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \right]^{K_1 K_2} \right) =: E$$

More on the Hamiltonian

The Hamiltonian is a conserved quantity that can be thought of as the **energy** of the vortex system.

$$H = \frac{1}{4\pi} \ln \left((2y_1)^{K_1^2} (2y_2)^{K_2^2} \left[\frac{(x_2 - x_1)^2 + (y_2 + y_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} \right]^{K_1 K_2} \right) =: E$$

From the energy conservation we get

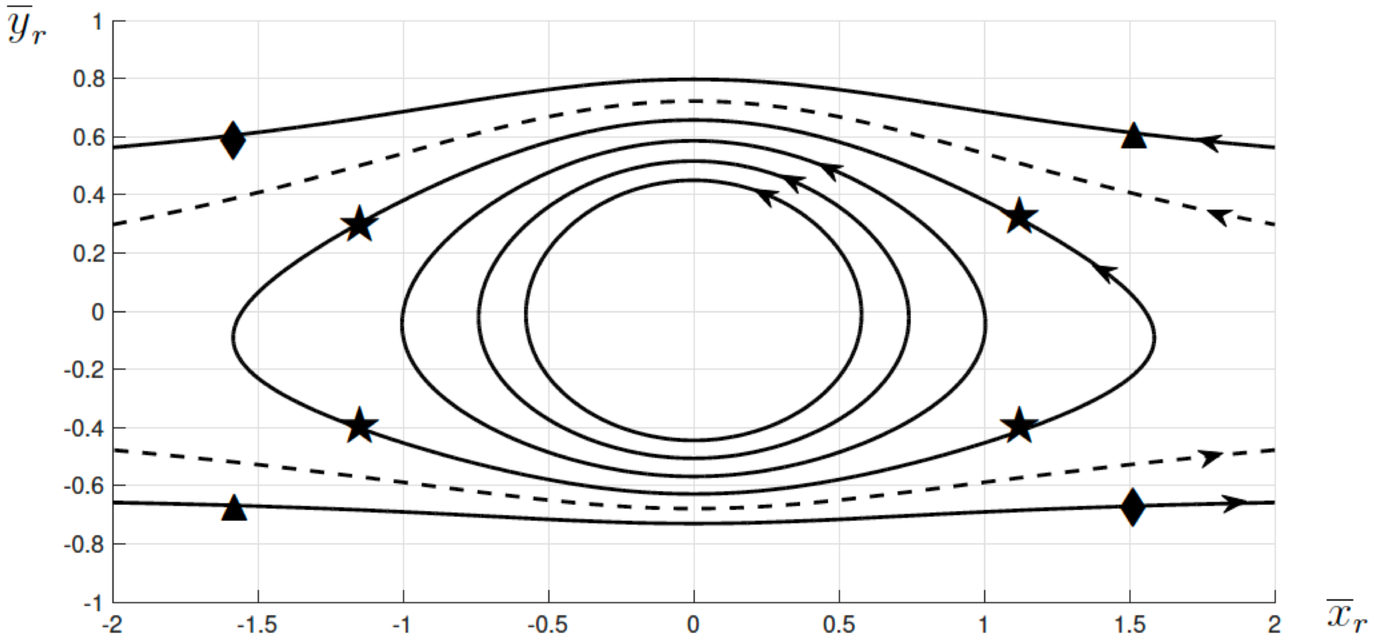
$$(1 - 2k_2 \overline{y}_r)^{-\frac{k_1}{k_2}} (1 + 2k_1 \overline{y}_r)^{-\frac{k_2}{k_1}} \left[1 - \frac{(1 - 2k_2 \overline{y}_r)(1 + 2k_1 \overline{y}_r)}{\left(\overline{x}_r^2 + [1 + (k_1 - k_2) \overline{y}_r]^2 \right)} \right] = C$$

where

$$C = \exp \left(-\frac{4\pi E}{K_1 K_2} + \ln(2) \left(\frac{K_1}{K_2} + \frac{K_2}{K_1} \right) \right)$$

Vortex trajectories in relative coordinates

Each curve corresponds to a different energy value E.



μ = 0.7

C_{CRIT} = 0.96949

How do we find the critical curve (separatrix)?

Steps for finding the leapfrogging criterion

We let the relative coordinate in the x-direction go to +/- infinity

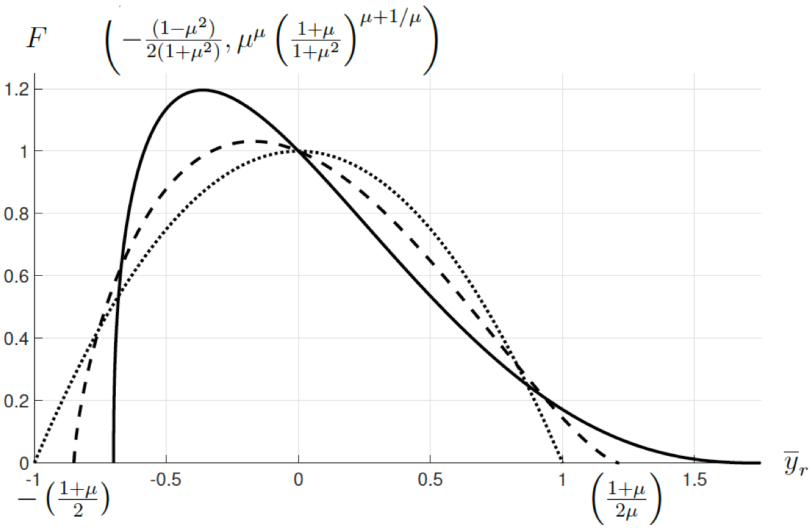
$$F \equiv \left(1 - \frac{2\mu}{1+\mu}\overline{y}_r\right)^{1/\mu} \left(1 + \frac{2}{1+\mu}\overline{y}_r\right)^{\mu} = \frac{1}{C}$$

Now take the derivative wrt $\overline{y}_r$ and set it equal to 0. We obtain

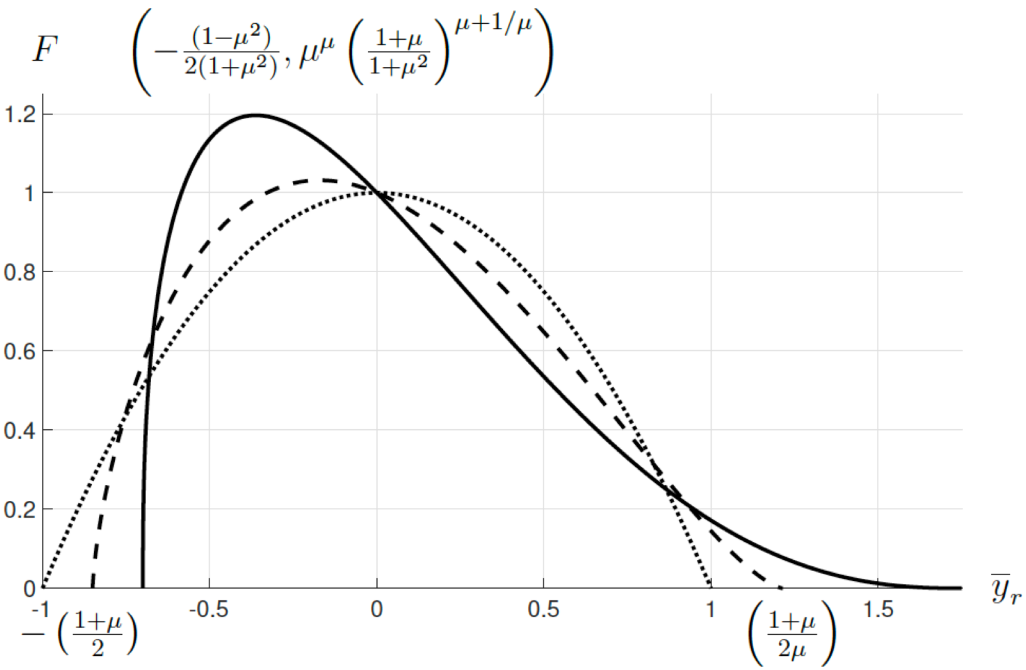
$$-\frac{2}{1+\mu}\left(1 - \frac{2\mu}{1+\mu}\overline{y}_r\right)^{-1} + \frac{2\mu}{1+\mu}\left(1 + \frac{2}{1+\mu}\overline{y}_r\right)^{-1} = 0$$

So the separating $\overline{y}_r$ is

$$\overline{y}_r = -\frac{(1-\mu^2)}{2(1+\mu^2)} \leq 0$$



Leapfrogging criterion in terms of energies

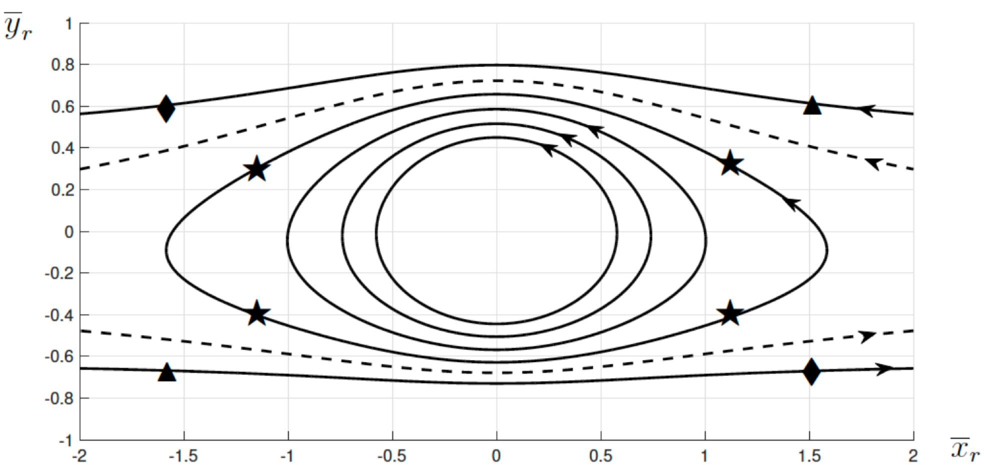


- $\frac{1}{C} < F_{\max}$: two distinct y_r values
- $\frac{1}{C} = F_{\max}$: two equal y_r values
- $\frac{1}{C} > F_{\max}$: no y_r values

Leapfrogging occurs **only** when:

$$\frac{1}{C} \geq \mu^\mu \left(\frac{1+\mu}{1+\mu^2}\right)^{\mu+1/\mu} = \frac{1}{C_{\text{CRIT}}}$$

In the case of equality the period of the leapfrogging is infinite.



Leapfrogging criterion in terms of vortex pair separation

There is an upper bound for the distance, d, between two given vortex pairs for which leapfrogging occurs.

$$d^2 < \frac{4y_1y_2}{1 - \left(\frac{K_1^2 + K_2^2}{K_1y_1 + K_2y_2}\right)^{\frac{K_1}{K_2}+\frac{K_2}{K_1}} \left(\frac{y_1}{K_1}\right)^{\frac{K_1}{K_2}} \left(\frac{y_2}{K_2}\right)^{\frac{K_2}{K_1}}} - (y_1 + y_2)^2$$

Too complicated to look at…

Main idea:

- If the vortices start a distance greater than d apart, no leapfrogging occurs
- The separation of vortices increases to infinity with or without a pass by

Brief derivation of the criterion

How did we derive this upper bound for d?

Recall from before:

$$\left(1-2k_2\overline{y}_r\right)^{-\frac{k_1}{k_2}}\left(1+2k_1\overline{y}_r\right)^{-\frac{k_2}{k_1}}\left[1-\frac{\left(1-2k_2\overline{y}_r\right)\left(1+2k_1\overline{y}_r\right)}{\left(\overline{x}_r^2+\left[1+\left(k_1-k_2\right)\overline{y}_r\right]^2\right)}\right]=C$$

Initial separation coordinates:

$$\overline{x}_{r0}=\frac{d(K_1+K_2)}{2(K_1y_1+K_2y_2)},\ \overline{y}_{r0}=\frac{(y_2-y_1)(K_1+K_2)}{2(K_1y_1+K_2y_2)}$$

- Replace all the y_r coordinates with the initial y separation (constant of motion)
- Solve for the ${x_r}^2$ term
- Get a condition that relates the ${x_r}^2$ to the critical energy curve

$$\overline{x}_{r_m}^2=\frac{\left(1-\frac{2\mu}{1+\mu}\overline{y}_{r0}\right)\left(1+\frac{2}{1+\mu}\overline{y}_{r0}\right)}{\left[1-C_{\text{CRIT}}\left(1-\frac{2\mu}{1+\mu}\overline{y}_{r0}\right)^{1/\mu}\left(1+\frac{2}{1+\mu}\overline{y}_{r0}\right)^{\mu}\right]}-\left(1+\left(\frac{1-\mu}{1+\mu}\right)\overline{y}_{r0}\right)^2$$

Finally use $\overline{x}_{r_0}^2<\overline{x}_{r_m}^2$


THANK YOU FOR YOUR ATTENTION
PLEASE CLAP AND DO NOT
MAKE TOUGH QUESTIONS