Spectral super-resolution

DS-GA 1013 / MATH-GA 2824 Optimization-based Data Analysis

http://www.cims.nyu.edu/~cfgranda/pages/OBDA_fall17/index.html

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Spectral super-resolution

Aim: Estimate frequencies of multisinusoidal signal

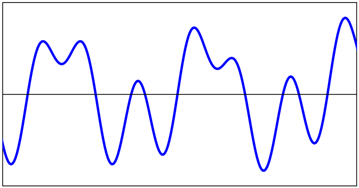
$$g(t) := \sum_{j=1}^{s} \vec{c}[j] \exp(-i2\pi f_j t)$$

where $f_1, f_2, \ldots, f_s \in [-1/2, 1/2]$ from n of samples

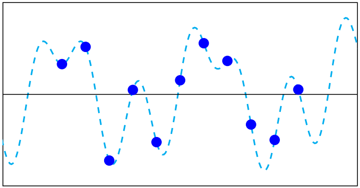
$$g(-(n-1)/2),\ g(-(n-1)/2+1),\ \ldots,\ g((n-1)/2)$$

Spectral super-resolution

Signal



Data



Dirac measure

It's a measure, not a function!

$$\int_{\mathcal{S}} \delta_{[\tau]}(\mathrm{d}u) = \begin{cases} 1 & \text{if } \tau \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases}$$

For any function h we have

$$\int_{\mathcal{S}} h(u)\, \delta_{[\tau]}(\mathrm{d}u) = \begin{cases} h(\tau) & \text{if } \tau \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases}$$

Spectral super-resolution

Aim: Estimating the measure

$$\mu_g := \sum_{j=1}^{s} \vec{c}[j]\delta_{[f_j]}$$

from n Fourier coefficients

$$\int_{-1/2}^{1/2} \overline{h_k(u)}\mu_g(\mathrm{d}u)$$

Spectral super-resolution

Aim: Estimating the measure

$$\mu_g := \sum_{j=1}^{s} \vec{c}[j]\delta_{[f_j]}$$

from n Fourier coefficients

$$\int_{-1/2}^{1/2} \overline{h_k(u)}\mu_g(\mathrm{d}u) = \sum_{j=1}^{s} \vec{c}[j] \int_{-1/2}^{1/2} \overline{h_k(u)}\delta_{[f_j]}(\mathrm{d}u)$$

Spectral super-resolution

Aim: Estimating the measure

$$\mu_g := \sum_{j=1}^{s} \vec{c}[j] \delta_{[f_j]}$$

from n Fourier coefficients

$$\int_{-1/2}^{1/2} \overline{h_k(u)} \mu_g(\mathrm{d}u) = \sum_{j=1}^{s} \vec{c}[j] \int_{-1/2}^{1/2} \overline{h_k(u)} \delta_{[f_j]}(\mathrm{d}u)$$

$$= \sum_{j=1}^{s} \vec{c}[j] \exp(-i 2\pi k f_j)$$

Spectral super-resolution

Aim: Estimating the measure

$$\mu_g := \sum_{j=1}^{s} \vec{c}[j] \delta_{[f_j]}$$

from n Fourier coefficients

$$\begin{aligned} \int_{-1/2}^{1/2} \overline{h_k(u)} \mu_g(\mathrm{d}u) &= \sum_{j=1}^{s} \vec{c}[j] \int_{-1/2}^{1/2} \overline{h_k(u)} \delta_{[f_j]}(\mathrm{d}u) \\ &= \sum_{j=1}^{s} \vec{c}[j] \exp(-i 2\pi k f_j) \\ &= g(k), \qquad -\frac{n-1}{2} \leq k \leq \frac{n-1}{2} \end{aligned}$$

The periodogram

Prony's method

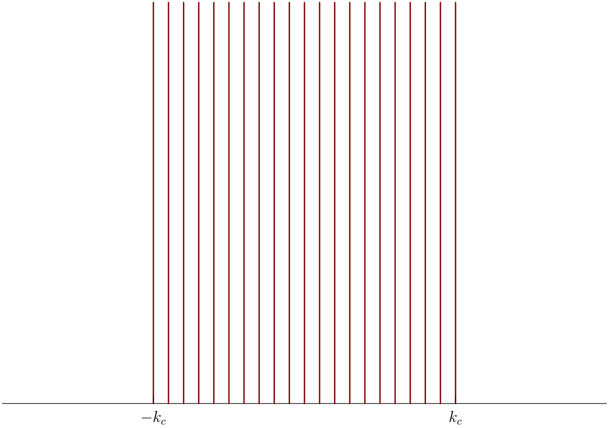
Subspace methods

Periodogram

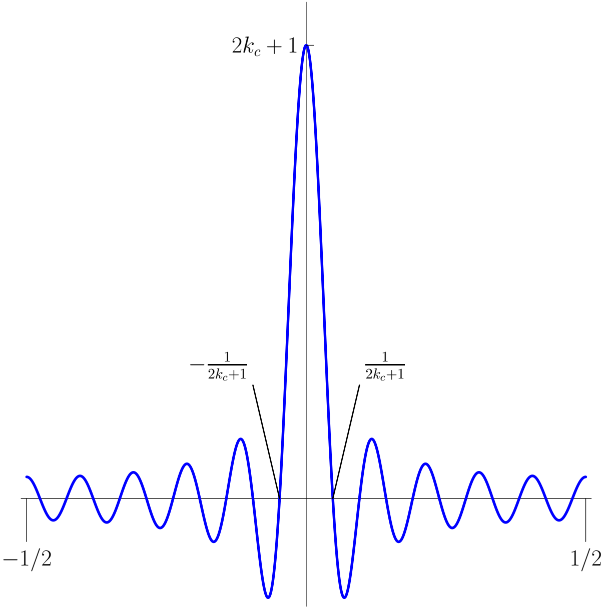
The periodogram of $\vec{y} \in \mathbb{C}^n$ is

$$P_{\vec{y}}\left(u\right) := \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \vec{y}\left[k\right] h_k\left(u\right)$$

Spectrum of Dirichlet kernel D



Dirichlet kernel d



Periodogram as a convolution

The periodogram of

$$g\left(-(n-1)/2\right),\; g\left(-(n-1)/2+1\right),\; \ldots,\; g\left((n-1)/2\right)$$

equals

$$P_g(u) = \sum_{j=1}^{s} \vec{c}\,[j]\, d_{[f_j]}(u)$$

Time shift

The τ-shifted version of a function $f \in \mathcal{L}_2[-1/2, 1/2]$ is

$$f_{[\tau]}(t) := f(t - \tau)$$

where the shift is *circular* (it wraps around)

For any shift τ

$$F_{[\tau]}[k] = \exp(-i2\pi k\tau) F[k]$$

Proof

The Fourier coefficients of the shifted Dirichlet kernel equal

$$D_{[f]}[k] := \begin{cases} \exp(-i2\pi kf) & \text{if } |k| \leq (n-1)/2 \\ 0 & \text{otherwise} \end{cases}$$

Proof

$$g\left(k\right) := \sum_{j=1}^{s} \vec{c}\,[j] \exp\left(-i 2\pi k f_j\right)$$

Proof

$$g(k) := \sum_{j=1}^{s} \vec{c}[j] \exp(-i 2\pi k f_j)$$

$$= \sum_{j=1}^{s} \vec{c}[j] D_{[f_j]}[k]$$

Proof

$$g(k) := \sum_{j=1}^{s} \vec{c}[j] \exp(-i2\pi k f_j)$$
$$= \sum_{j=1}^{s} \vec{c}[j] D_{[f_j]}[k]$$

$$P_g(u) = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} g(k) h_k(u)$$

Proof

$$
\begin{aligned}
g(k) &:= \sum_{j=1}^{s} \vec{c}[j] \exp(-i 2\pi k f_j) \\
&= \sum_{j=1}^{s} \vec{c}[j] D_{[f_j]}[k]
\end{aligned}
$$

$$
\begin{aligned}
P_g(u) &= \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} g(k)\, h_k(u) \\
&= \sum_{j=1}^{s} \vec{c}[j] \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} D_{[f_j]}[k]\, h_k(u)
\end{aligned}
$$

Proof

$$g(k) := \sum_{j=1}^{s} \vec{c}[j] \exp(-i2\pi k f_j)$$
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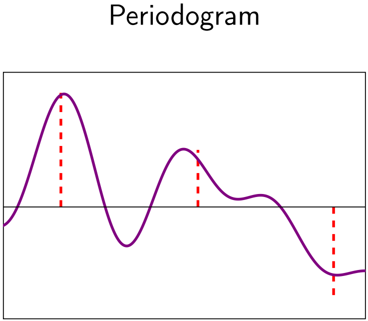
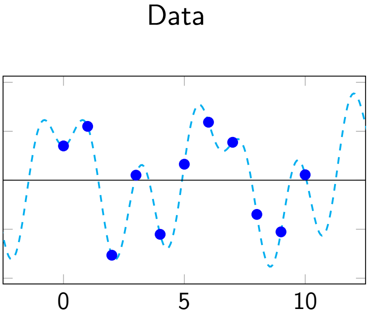
$$= \sum_{j=1}^{s} \vec{c}[j] d_{[f_j]}(u)$$

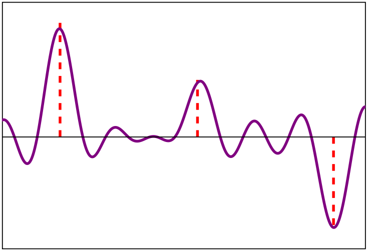
Periodogram as a convolution



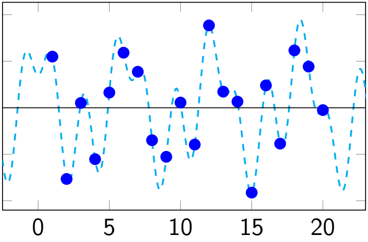


Periodogram as a convolution

Periodogram

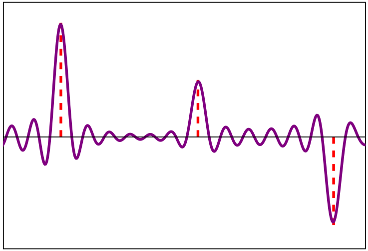


Data

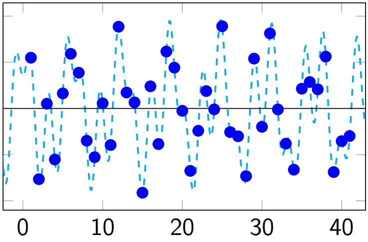


Periodogram as a convolution

Periodogram

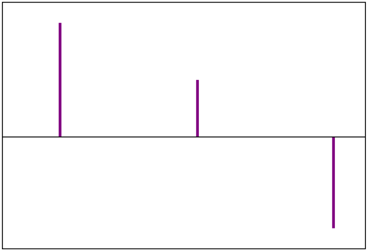


Data

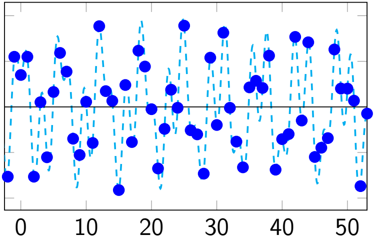


Periodogram as a convolution

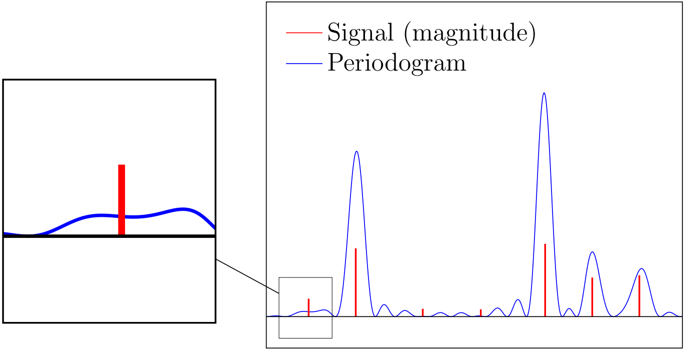
Periodogram



Data



Problem



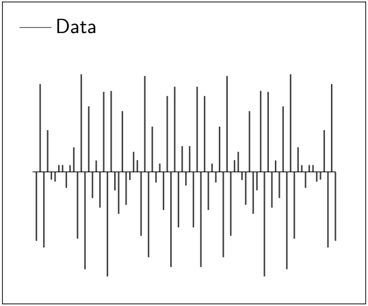
Windowed periodogram

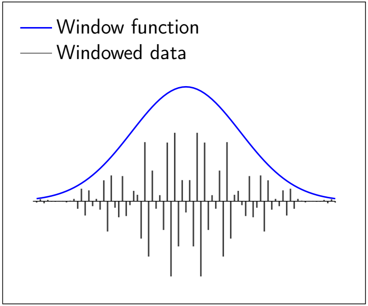
The windowed periodogram of $\vec{y} \in \mathbb{C}^n$ is

$$P_{w,\vec{y}}(u) := \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} W[k] g(k) h_k(u).$$

$W\left[-\frac{n-1}{2}\right], \ldots, W\left[\frac{n-1}{2}\right]$ are the Fourier coefficients of a window function

Windowing





Windowed periodogram

$$P_{w,\vec{y}}(u) = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} W[k] g(k) h_k(u)$$

Windowed periodogram

$$P_{w,\vec{y}}(u) = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} W[k] g(k) h_k(u)$$

$$= \sum_{j=1}^{s} \vec{c}[j] \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} W[k] \exp(-i2\pi f_j k) h_k(u)$$

Windowed periodogram

$$P_{w,\vec{y}}(u) = \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} W[k] g(k) h_k(u)$$

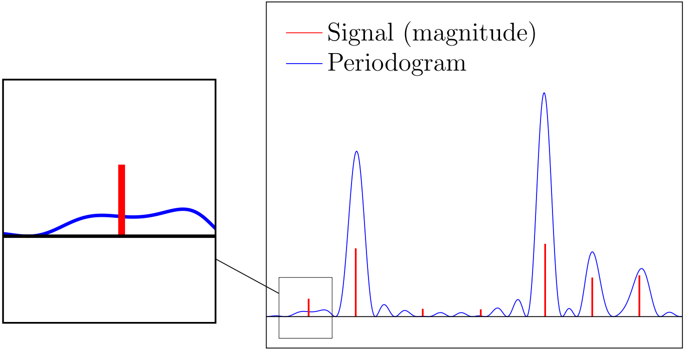
$$= \sum_{j=1}^{s} \vec{c}[j] \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} W[k] \exp(-i2\pi f_j k) h_k(u)$$

$$= \sum_{j=1}^{s} \vec{c}[j] \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} W_{[f_j]}[k] h_k(u)$$

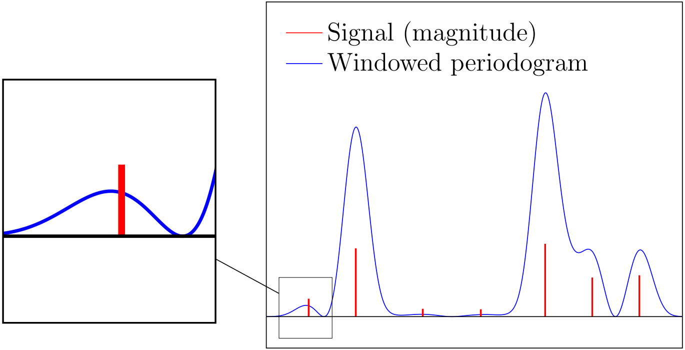
Windowed periodogram

$$
\begin{aligned}
P_{w,\vec{y}}(u) &= \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} W[k] g(k) h_k(u) \\
&= \sum_{j=1}^{s} \vec{c}[j] \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} W[k] \exp(-i2\pi f_j k) h_k(u) \\
&= \sum_{j=1}^{s} \vec{c}[j] \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} W_{[f_j]}[k] h_k(u) \\
&= \sum_{j=1}^{s} \vec{c}[j] w_{[f_j]}(u)
\end{aligned}
$$

Problem

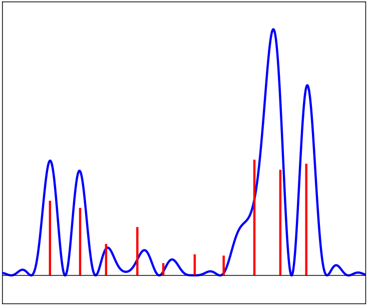


Windowed periodogram

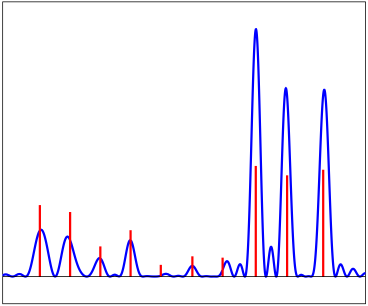


Periodogram

$\Delta = \frac{1.2}{n}$

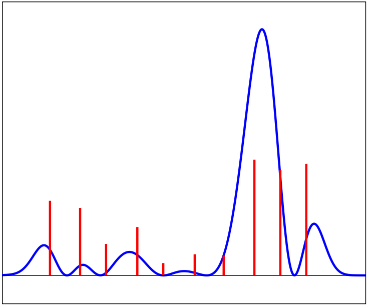


$\Delta = \frac{2.4}{n}$

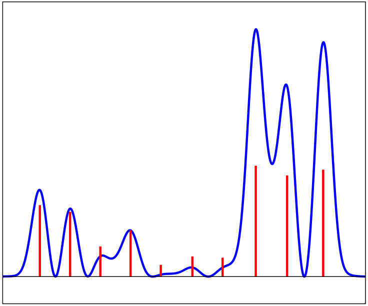


Gaussian periodogram

$\Delta = \frac{1.2}{n}$



$\Delta = \frac{2.4}{n}$

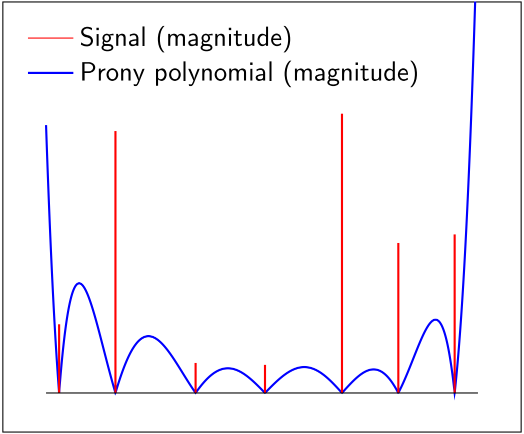


The periodogram

Prony's method

Subspace methods

Prony polynomial



Prony polynomial

Given any f_1, f_2, ..., f_s, there exists a nonzero complex polynomial of order s

$$p(z) := \sum_{k=0}^{s} P[k] z^k$$

such that its s roots are equal to $\exp(i2\pi f_1)$, $\exp(i2\pi f_2)$, ..., $\exp(i2\pi f_s)$

Proof

$$p(z) := \prod_{j=1}^{s} \left(1 - \exp\left(-i2\pi f_j\right) z\right)$$

Proof

$$p(z) := \prod_{j=1}^{s} \left(1 - \exp\left(-i2\pi f_j\right) z\right)$$

is of order s, so has at most s roots

Proof

$$p(z) := \prod_{j=1}^{s} \left(1 - \exp\left(-i2\pi f_j\right) z\right)$$

is of order s, so has at most s roots

$$p(z) = 1 + \sum_{k=1}^{s} P\left[k\right] z^k$$

Proof

$$p(z) := \prod_{j=1}^{s} \left(1 - \exp\left(-i2\pi f_j\right) z\right)$$

is of order s, so has at most s roots

$$p(z) = 1 + \sum_{k=1}^{s} P\left[k\right] z^k$$

is nonzero since $p\left(0\right) = 1$

Proof

$$p(z) := \prod_{j=1}^{s} \left(1 - \exp\left(-i2\pi f_j\right) z\right)$$

is of order s, so has at most s roots

$$p(z) = 1 + \sum_{k=1}^{s} P[k]\, z^k$$

is nonzero since $p(0) = 1$

Reveals the frequencies

$$p(\exp\left(i2\pi f_j\right)) = 0 \qquad 1 \leq j \leq s$$

Prony system

Let

$$g(k) := \sum_{j=1}^{s} \vec{c}[j] \exp(-i2\pi k f_j) \qquad -\frac{n-1}{2} \leq k \leq \frac{n-1}{2}$$

For any integer b

$$\sum_{l=0}^{s} P[l] g[l-b] = 0$$

The equation only involves the data as long as

$$-\frac{n-1}{2} \leq -b \leq s-b \leq \frac{n-1}{2}$$

Proof

$$\sum_{l=0}^{s} P[l]g[l-b]$$

Proof

$$\sum_{l=0}^{s} P[l]g[l-b] = \sum_{l=0}^{s} P[l] \int_{-1/2}^{1/2} \overline{h_{l-b}(u)} \mu_g(\mathrm{d}u)$$

Proof

$$\sum_{l=0}^{s} P[l]g[l-b] = \sum_{l=0}^{s} P[l] \int_{-1/2}^{1/2} \overline{h_{l-b}(u)} \mu_g(\mathrm{d}u)$$

$$= \int_{-1/2}^{1/2} \exp(i2\pi bu) \sum_{l=0}^{s} P[l] \exp(-i2\pi lu) \, \mu_g(\mathrm{d}u)$$

Proof

$$\begin{aligned}\sum_{l=0}^{s} P[l]g[l-b] &= \sum_{l=0}^{s} P[l] \int_{-1/2}^{1/2} \overline{h_{l-b}(u)} \mu_g(\mathrm{d}u) \\ &= \int_{-1/2}^{1/2} \exp(i2\pi bu) \sum_{l=0}^{s} P[l] \exp(-i2\pi lu) \mu_g(\mathrm{d}u) \\ &= \int_{-1/2}^{1/2} \exp(i2\pi bu)\, p(\exp(-i2\pi u)) \mu_g(\mathrm{d}u)\end{aligned}$$

Proof

$$
\begin{aligned}
\sum_{l=0}^{s} P[l]g[l-b] &= \sum_{l=0}^{s} P[l] \int_{-1/2}^{1/2} \overline{h_{l-b}(u)} \mu_g(\mathrm{d}u) \\
&= \int_{-1/2}^{1/2} \exp(i2\pi bu) \sum_{l=0}^{s} P[l] \exp(-i2\pi lu) \, \mu_g(\mathrm{d}u) \\
&= \int_{-1/2}^{1/2} \exp(i2\pi bu) \, p(\exp(-i2\pi u)) \, \mu_g(\mathrm{d}u) \\
&= \sum_{j=1}^{s} \vec{c}[j] \exp(i2\pi bf_j) \, p(\exp(-i2\pi f_j))
\end{aligned}
$$

Proof

$$
\begin{aligned}
\sum_{l=0}^{s} P[l]g[l-b] &= \sum_{l=0}^{s} P[l] \int_{-1/2}^{1/2} \overline{h_{l-b}(u)} \mu_g(\mathrm{d}u) \\
&= \int_{-1/2}^{1/2} \exp(i2\pi bu) \sum_{l=0}^{s} P[l] \exp(-i2\pi lu) \mu_g(\mathrm{d}u) \\
&= \int_{-1/2}^{1/2} \exp(i2\pi bu)\, p(\exp(-i2\pi u)) \mu_g(\mathrm{d}u) \\
&= \sum_{j=1}^{s} \vec{c}[j] \exp(i2\pi b f_j)\, p(\exp(-i2\pi f_j)) \\
&= 0
\end{aligned}
$$

Prony's method

1. Solve the system of equations

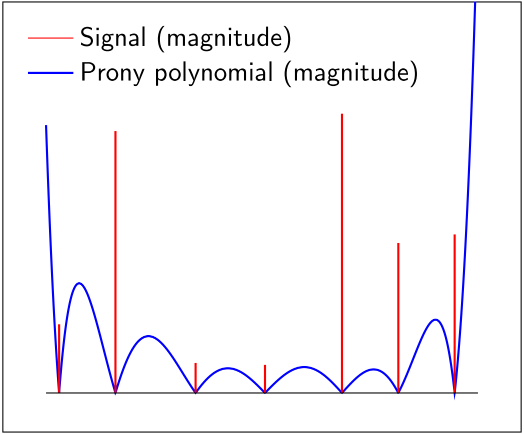
$$\begin{bmatrix} g(1) & g(2) & \cdots & g(s) \\ g(0) & g(1) & \cdots & g(s-1) \\ \cdots & \cdots & \cdots & \cdots \\ g(-s+2) & g(-s+3) & \cdots & g(1) \end{bmatrix} \vec{P} = - \begin{bmatrix} g(0) \\ g(-1) \\ \cdots \\ g(-s+1) \end{bmatrix}$$

2. Compute the roots $z_1, \ldots, z_s$ of the polynomial

$$p(z) := 1 + \sum_{k=1}^{s} \vec{P}[k] z^k$$

3. For every root on the unit circle $z_j = \exp(i2\pi\tau)$ include τ in the set of estimated frequencies

Without noise it works!



Proof

$$\begin{bmatrix} g(1) & g(2) & \cdots & g(s) \\ g(0) & g(1) & \cdots & g(s-1) \\ & & \cdots & \\ g(-s+2) & g(-s+3) & \cdots & g(1) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i2\pi f_1} & e^{-i2\pi f_2} & \cdots & e^{-i2\pi f_s} \\ 1 & 1 & \cdots & 1 \\ & & \cdots & \\ e^{-i2\pi(2-s)f_1} & e^{-i2\pi(2-s)f_2} & \cdots & e^{-i2\pi(2-s)f_s} \end{bmatrix}$$

$$\begin{bmatrix} \vec{c}[1] & 0 & \cdots & 0 \\ 0 & \vec{c}[2] & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \vec{c}[s] \end{bmatrix}$$

$$\begin{bmatrix} 1 & e^{-i2\pi f_1} & \cdots & e^{-i2\pi(s-1)f_1} \\ 1 & e^{-i2\pi f_2} & \cdots & e^{-i2\pi(s-1)f_2} \\ & & \cdots & \\ 1 & e^{-i2\pi f_s} & \cdots & e^{-i2\pi(s-1)f_s} \end{bmatrix}.$$

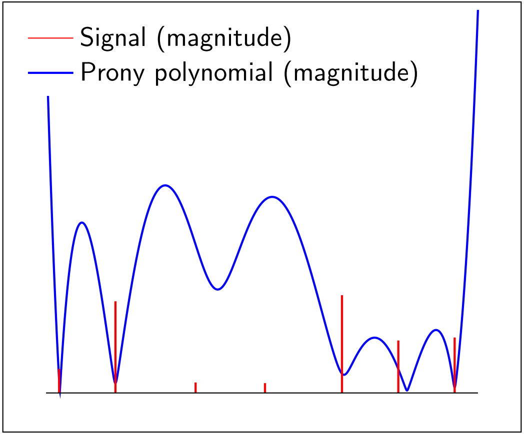
Vandermonde matrix

For any distinct s nonzero $z_1, z_2, \ldots, z_s \in \mathbb{C}$ and any m_1, m_2, s such that $m_2 - m_1 + 1 \geq s$

$$\begin{bmatrix} z_1^{m_1} & z_2^{m_1} & \cdots & z_s^{m_1} \\ z_1^{m_1+1} & z_2^{m_1+1} & \cdots & z_s^{m_1+1} \\ z_1^{m_1+2} & z_2^{m_1+2} & \cdots & z_s^{m_1+2} \\ & & \cdots & \\ z_1^{m_2} & z_2^{m_2} & \cdots & z_s^{m_2} \end{bmatrix}$$

is full rank

SNR = 140 dB (relative ℓ_2 norm of noise $= 10^{-8}$)



The periodogram

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Subspace methods

Alternative interpretation of Prony's method

Prony's method finds nonzero vector in the null space of $Y\left(s+1\right)^{T}$

$$Y\left(m\right) := \begin{bmatrix} \vec{y}\left[0\right] & \vec{y}\left[1\right] & \cdots & \vec{y}\left[n-m\right] \\ \vec{y}\left[1\right] & \vec{y}\left[2\right] & \cdots & \vec{y}\left[n-m+1\right] \\ \cdots & \cdots & \cdots & \cdots \\ \vec{y}\left[m-1\right] & \vec{y}\left[m\right] & \cdots & \vec{y}\left[n-1\right] \end{bmatrix}$$

$$\vec{y}[k] = g(-k+1) \qquad 0 \leq k \leq n$$

The vector corresponds to the coefficients of the Prony polynomial

Notation

For $k > 0$

$$\vec{a}_{0:k}(u) := \begin{bmatrix} 1 & \exp(-i2\pi u) & \exp(-i2\pi 2u) & \cdots & \exp(-i2\pi k u) \end{bmatrix}^T$$

$$A_{0:k} := \begin{bmatrix} \vec{a}_{0:k}(f_1) & \vec{a}_{0:k}(f_2) & \cdots & \vec{a}_{0:k}(f_s) \end{bmatrix}$$

Decomposition

$$Y(m) = \begin{bmatrix} \vec{a}_{0:m-1}(f_1) & \vec{a}_{0:m-1}(f_2) & \cdots & \vec{a}_{0:m-1}(f_s) \end{bmatrix}$$

$$\begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & c_s \end{bmatrix} \begin{bmatrix} \vec{a}_{0:n-m}(f_1)^T \\ \vec{a}_{0:n-m}(f_2)^T \\ \cdots \\ \vec{a}_{0:n-m}(f_s)^T \end{bmatrix}$$

$$= A_{0:m-1}\, C\, A_{0:m}^T$$

Idea: Find $\vec{a}_{0:m-1}(f)$ in column space of $Y(m)$

Pseudospectrum

To find vectors that are *close* to the column space of $Y(m)$

- Compute orthogonal complement $\mathcal{N}$ of column space of $Y(m)$
- Locate local maxima of pseudospectrum

$$P_{\mathcal{N}}(u) = \log \frac{1}{\left|\mathcal{P}_{\mathcal{N}}\left(\vec{a}_{0:m-1}(u)\right)\right|^2}$$

Pseudospectrum

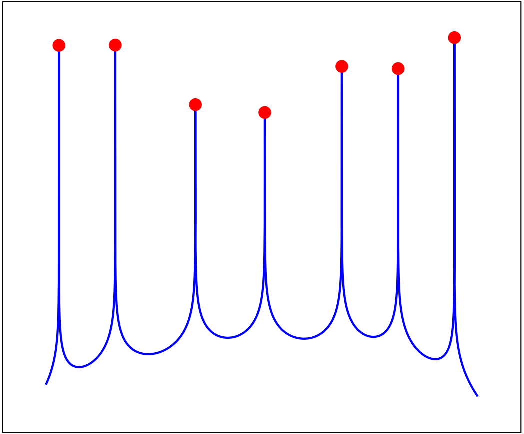
$$Y(m) = A_{0:m-1}\, C\, A_{0:m}^T$$

implies

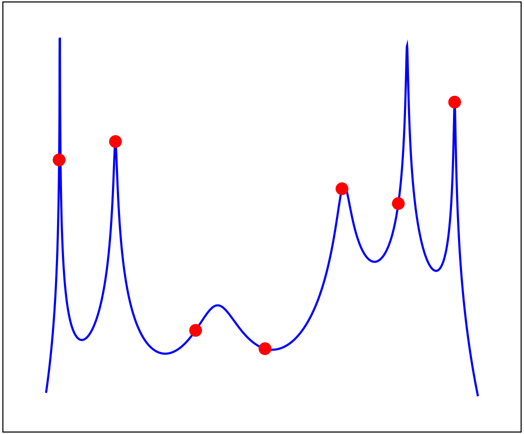
$$P_{\mathcal{N}}(f_j) = \infty, \qquad 1 \leq j \leq s$$

$$P_{\mathcal{N}}(u) < \infty, \qquad \text{for } u \notin \{f_1, \ldots, f_s\}$$

Pseudospectrum: No noise



Pseudospectrum: SNR = 140 dB, $n = 2s$



Empirical covariance matrix

$\mathcal{N}$ is the null space of the empirical covariance matrix

$$\Sigma(m) = \frac{1}{n-m+1} Y Y^*$$

$$= \frac{1}{n-m+1} \sum_{j=0}^{n-m} \begin{bmatrix} \vec{y}[j] \\ \vec{y}[j+1] \\ \cdots \\ \vec{y}[j+m-1] \end{bmatrix} \begin{bmatrix} \overline{\vec{y}[j]} \\ \overline{\vec{y}[j+1]} \\ \cdots \\ \overline{\vec{y}[j+m-1]} \end{bmatrix}^T$$

If the data are noisy

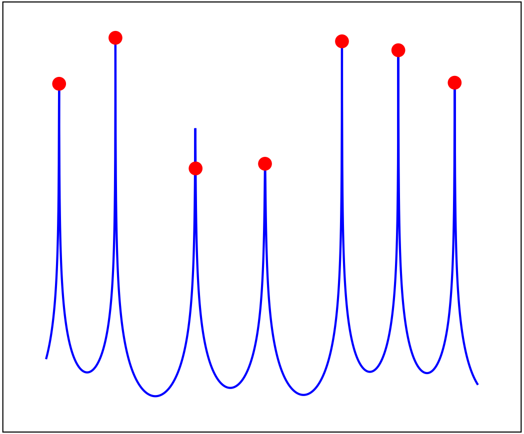
$$\vec{y}[k] = g(-k+1) + \vec{z}[k], \qquad 0 \le k \le n$$

we can average over more data to cancel out noise

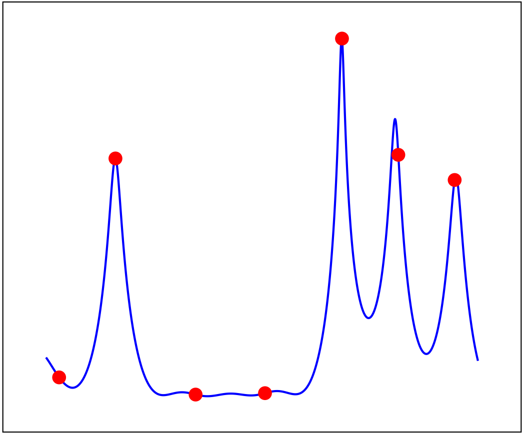
Multiple-signal classification (MUSIC)

1. Build the empirical covariance matrix $\Sigma(m)$
2. Compute the eigendecomposition of $\Sigma(m)$
3. Select $U_{\mathcal{N}}$ corresponding to $m - s$ smallest singular values
4. Estimate frequencies by computing the pseudospectrum

Pseudospectrum: SNR = 40 dB, $n = 81$, $m = 30$



Pseudospectrum: SNR = 1 dB, $n = 81$, $m = 30$



Probabilistic model: Signal

$$\mu_g = \sum_{t_j \in T} \vec{c}[j] \delta_{f_j} = \sum_{t_j \in T} \alpha_j e^{i \boldsymbol{\phi}_{\mathbf{j}}} \delta_{f_j},$$

$\boldsymbol{\phi}_{\mathbf{1}}, \ldots, \boldsymbol{\phi}_{\mathbf{s}}$ are independent and uniformly distributed in $[0, 2\pi]$

$$\mathrm{E}(\vec{\mathbf{c}}) = \vec{0}$$

$$\mathrm{E}[\vec{\mathbf{c}}\vec{\mathbf{c}}^*] = S_{\vec{c}} := \begin{bmatrix} \alpha_1^2 & 0 & \cdots & 0 \\ 0 & \alpha_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha_s^2 \end{bmatrix}$$

Probabilistic model: Noise

Noise $\vec{\mathbf{z}}$ is a zero-mean Gaussian vector with covariance $\sigma^2 I$

$$\vec{\mathbf{y}}[k] := \int_0^1 e^{-i2\pi kt} \mu_g(\mathrm{d}t) + \vec{\mathbf{z}}_k$$
$$= \sum_{j=1}^{s} \vec{\mathbf{c}}_j e^{-i2\pi k f_j} + \vec{\mathbf{z}}_k$$

$$\vec{y} = A_{0:m-1}\,\vec{\mathbf{c}} + \vec{z}$$

Covariance matrix of the data

$$\mathrm{E}\left[\vec{\mathbf{y}}\vec{\mathbf{y}}^*\right] = A_{1:m} S_{\vec{c}} A_{1:m}^* + \sigma^2 I$$

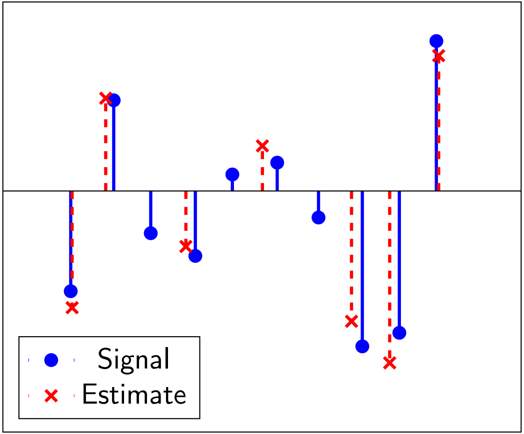
SVD of covariance matrix

SVD of $\mathrm{E}[\vec{y}\vec{y}^*]$

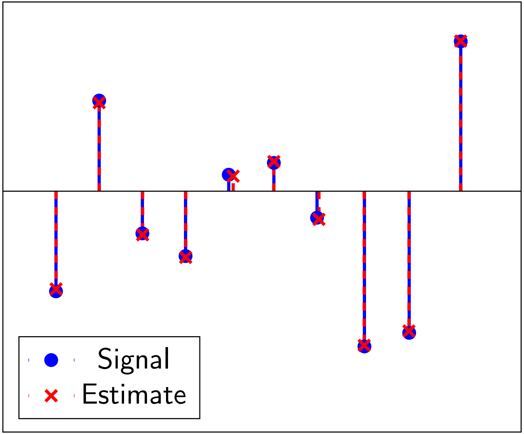
$$\mathrm{E}[\vec{\mathbf{y}}\vec{\mathbf{y}}^*] = \begin{bmatrix} U_{\mathcal{S}} & U_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} \Lambda + \sigma^2 I_s & 0 \\ 0 & \sigma^2 I_{n-s} \end{bmatrix} \begin{bmatrix} U_{\mathcal{S}}^* \\ U_{\mathcal{N}}^* \end{bmatrix},$$

- $U_{\mathcal{S}} \in \mathbb{C}^{m \times s}$: unitary matrix that spans column space of $A_{1:m}$
- $U_{\mathcal{N}} \in \mathbb{C}^{m \times (m-s)}$: unitary matrix spanning the orthogonal complement
- $\Lambda \in \mathbb{C}^{k \times k}$ is a diagonal matrix with positive entries

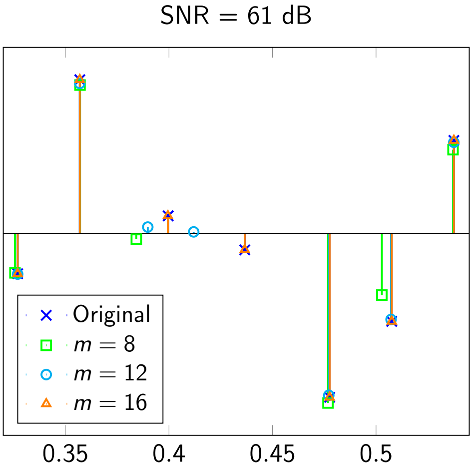
$\Delta = \frac{1.2}{n}$, SNR = 20 dB, $n = 81$, $m = 40$



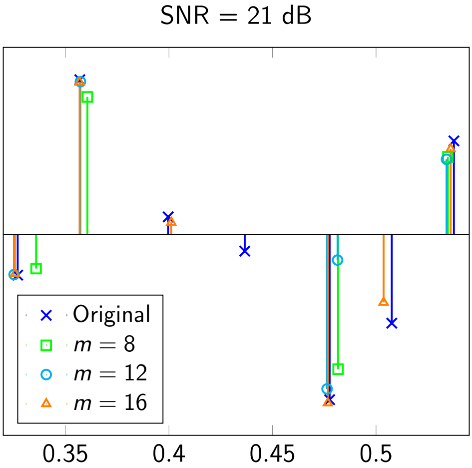
$\Delta = \frac{2.4}{n}$, SNR = 20 dB, $n = 81$, $m = 40$



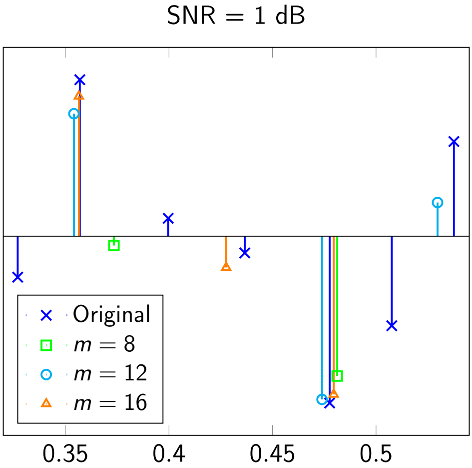
Different values of m



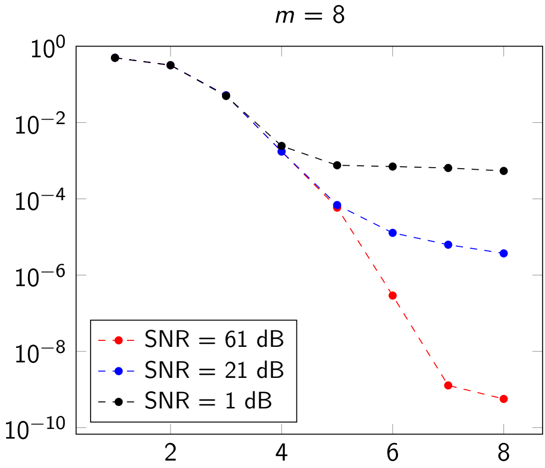
Different values of m



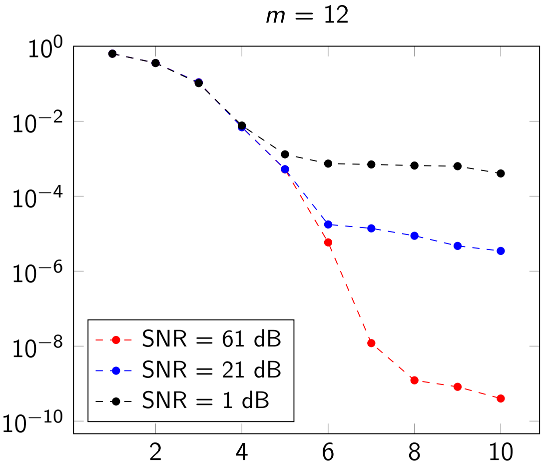
Different values of m



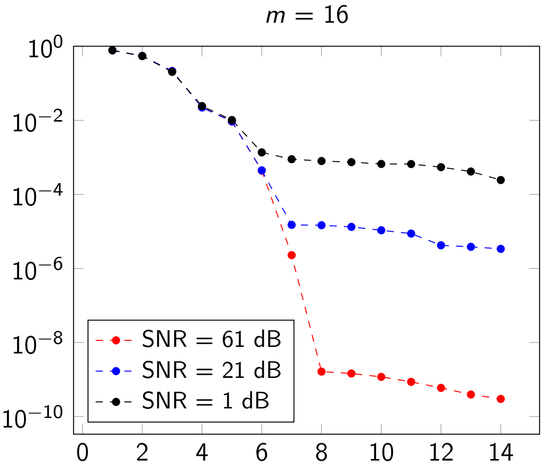
Singular values



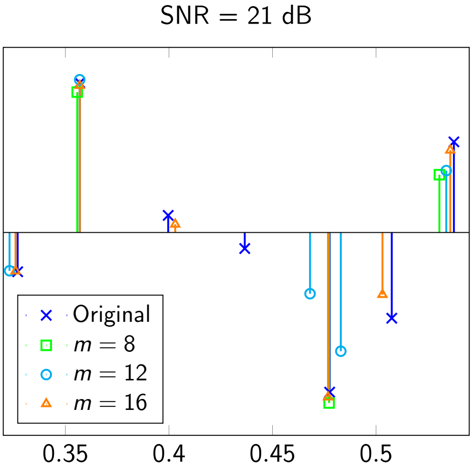
Singular values



Singular values



Wrong s $(s-1)$



Wrong s $(s+1)$