Multiresolution Analysis

DS-GA 1013 / MATH-GA 2824 Optimization-based Data Analysis
http://www.cims.nyu.edu/~cfgranda/pages/OBDA_fall17/index.html

Carlos Fernandez-Granda
Frames

Short-time Fourier transform (STFT)

Wavelets

Thresholding
Definition

Let $\mathcal{V}$ be an inner-product space

A frame of $\mathcal{V}$ is a set of vectors $\mathcal{F} := \{\vec{v}_1, \vec{v}_2, \ldots\}$ such that for every $\vec{x} \in \mathcal{V}$

$$c_L \|\vec{x}\|^2 \leq \sum_{\vec{v} \in \mathcal{F}} |\langle \vec{x}, \vec{v} \rangle|^2 \leq c_U \|\vec{x}\|^2$$

for fixed positive constants $c_U \geq c_L \geq 0$

The frame is a **tight frame** if $c_L = c_U$
Frames span the whole space

Any frame $\mathcal{F} := \{\vec{v}_1, \vec{v}_2, \ldots\}$ of $\mathcal{V}$ spans $\mathcal{V}$

Proof:

Assume $\vec{y} \notin \text{span} (\vec{v}_1, \vec{v}_2, \ldots)$
Frames span the whole space

Any frame $\mathcal{F} := \{\vec{v}_1, \vec{v}_2, \ldots\}$ of $V$ spans $V$

Proof:

Assume $\vec{y} \notin \text{span}(\vec{v}_1, \vec{v}_2, \ldots)$

Then $\mathcal{P}_{\text{span}(\vec{v}_1, \vec{v}_2, \ldots)\perp} \vec{y}$ is nonzero and

$$\sum_{\vec{v} \in \mathcal{F}} \left| \left\langle \mathcal{P}_{\text{span}(\vec{v}_1, \vec{v}_2, \ldots)\perp} \vec{y}, \vec{v} \right\rangle \right|^2 =$$
Frames span the whole space

Any frame $\mathcal{F} := \{\vec{v}_1, \vec{v}_2, \ldots\}$ of $\mathcal{V}$ spans $\mathcal{V}$

Proof:

Assume $\vec{y} \not\in \text{span}(\vec{v}_1, \vec{v}_2, \ldots)$

Then $\mathcal{P}_{\text{span}(\vec{v}_1, \vec{v}_2, \ldots)^\perp} \vec{y}$ is nonzero and

$$\sum_{\vec{v} \in \mathcal{F}} \left| \langle \mathcal{P}_{\text{span}(\vec{v}_1, \vec{v}_2, \ldots)^\perp} \vec{y}, \vec{v} \rangle \right|^2 = 0$$
Orthogonal bases are tight frames

Any orthonormal basis $\mathcal{B} := \{\vec{b}_1, \vec{b}_2, \ldots\}$ is a tight frame

Proof:
Orthogonal bases are tight frames

Any orthonormal basis $\mathcal{B} := \{\vec{b}_1, \vec{b}_2, \ldots \}$ is a tight frame

Proof:

For any vector $\vec{x} \in \mathcal{V}$

$$\|\vec{x}\|^2 \langle \cdot, \cdot \rangle$$
Orthogonal bases are tight frames

Any orthonormal basis $\mathcal{B} := \{ \vec{b}_1, \vec{b}_2, \ldots \}$ is a tight frame.

Proof:

For any vector $\vec{x} \in \mathcal{V}$

$$\|\vec{x}\|_{\langle \cdot, \cdot \rangle}^2 = \left\| \sum_{\vec{b} \in \mathcal{B}} \langle \vec{x}, \vec{b} \rangle \vec{b} \right\|_{\langle \cdot, \cdot \rangle}^2$$
Orthogonal bases are tight frames

Any orthonormal basis $\mathcal{B} := \{\vec{b}_1, \vec{b}_2, \ldots\}$ is a tight frame.

**Proof:**

For any vector $\vec{x} \in \mathcal{V}$

$$\|\vec{x}\|^2 = \left\| \sum_{\vec{b} \in \mathcal{B}} \langle \vec{x}, \vec{b} \rangle \vec{b} \right\|_\langle \cdot, \cdot \rangle^2$$

$$= \sum_{\vec{b} \in \mathcal{B}} \left| \langle \vec{x}, \vec{b} \rangle \right|^2 \|\vec{b}\|_\langle \cdot, \cdot \rangle^2$$
Orthogonal bases are tight frames

Any orthonormal basis $\mathcal{B} := \{ \vec{b}_1, \vec{b}_2, \ldots \}$ is a tight frame.

Proof:

For any vector $\vec{x} \in \mathcal{V}$

$$\|\vec{x}\|^2_{\langle \cdot, \cdot \rangle} = \left\| \sum_{\vec{b} \in \mathcal{B}} \langle \vec{x}, \vec{b} \rangle \vec{b} \right\|^2_{\langle \cdot, \cdot \rangle}$$

$$= \sum_{\vec{b} \in \mathcal{B}} \left| \langle \vec{x}, \vec{b} \rangle \right|^2 \left\| \vec{b} \right\|^2_{\langle \cdot, \cdot \rangle}$$

$$= \sum_{\vec{b} \in \mathcal{B}} \left| \langle \vec{x}, \vec{b} \rangle \right|^2$$
The analysis operator $\Phi$ of a frame maps a vector to its coefficients

$$\Phi (\vec{x}) [k] = \langle \vec{x}, \vec{v}_k \rangle$$

For any finite frame $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ of $\mathbb{C}^n$ the analysis operator is

$$F := \begin{bmatrix} \vec{v}_1^* \\ \vec{v}_2^* \\ \vdots \\ \vec{v}_m^* \end{bmatrix}$$
Frames in finite-dimensional spaces

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are a frame of $\mathbb{C}^n$ if and only $F$ is full rank

In that case,

$$c_U = \sigma_1^2$$
$$c_L = \sigma_n^2$$

Proof:

$$\sigma_n^2 \leq ||F\vec{x}||_2^2 = \sum_{j=1}^{m} \langle \vec{x}, \vec{v}_j \rangle^2 \leq \sigma_1^2$$
Pseudoinverse

If an $n \times m$ tall matrix $A$, $m \geq n$, is full rank, then its pseudoinverse

$$A^\dagger := (A^* A)^{-1} A^*$$

is well defined, is a left inverse of $A$

$$A^\dagger A = I$$

and equals

$$A^\dagger = V S^{-1} U^*$$

where $A = U S V^*$ is the SVD of $A$
Proof

\[ A^\dagger := (A^* A)^{-1} A^* \]
Proof

\[ A^\dagger := (A^* A)^{-1} A^* \]
\[ = (VSU^* USV^* A)^{-1} VSU^* \]
Proof

\[ A^\dagger := (A^* A)^{-1} A^* \]
\[ = (VSU^* USV^* A)^{-1} VSU^* \]
\[ = (VS^2 V^*)^{-1} VSU^* \]
Proof

\[ A^\dagger := (A^* A)^{-1} A^* \]
\[ = (VSU^* USV^* A)^{-1} VSU^* \]
\[ = (VS^2 V^*)^{-1} VSU^* \]
\[ = VS^{-2} V^* VSU^* \]
Proof

\[ A^\dagger := (A^* A)^{-1} A^* \]
\[ = (VSU^* USV^* A)^{-1} VSU^* \]
\[ = (VS^2 V^*)^{-1} VSU^* \]
\[ = VS^{-2} V^* VSU^* \]
\[ = VS^{-1} U \]
Proof

\[ A^\dagger := (A^*A)^{-1}A^* \]
\[ = (VSU^*USV^*A)^{-1} VSU^* \]
\[ = (VS^2V^*)^{-1} VSU^* \]
\[ = VS^{-2}V^*VSU^* \]
\[ = VS^{-1}U \]

\[ A^\dagger A \]
Proof

\[ A^\dagger := (A^* A)^{-1} A^* \]
\[ = (VSU^* USV^* A)^{-1} VSU^* \]
\[ = (VS^2 V^*)^{-1} VSU^* \]
\[ = VS^{-2} V^* VSU^* \]
\[ = VS^{-1} U \]

\[ A^\dagger A = VS^{-1} UV^* USV^* \]
Proof

\[ A^\dagger := (A^* A)^{-1} A^* \]
\[ = (VSU^* USV^* A)^{-1} VSU^* \]
\[ = (VS^2 V^*)^{-1} VSU^* \]
\[ = VS^{-2} V^* VSU^* \]
\[ = VS^{-1} U \]

\[ A^\dagger A = VS^{-1} UV^* USV^* = I \]
Frames

Short-time Fourier transform (STFT)

Wavelets

Thresholding
Motivation

Spectrum of speech, music, etc. varies over time

Idea: Compute frequency representation of time segments of the signal
Short-time Fourier transform

The short-time Fourier transform (STFT) of a function $f \in L_2[-1/2, 1/2]$ is

$$\text{STFT}\{f\}(k, \tau) := \int_{-1/2}^{1/2} f(t) w(t - \tau) e^{-i2\pi kt} \, dt$$

where $w \in L_2[-1/2, 1/2]$ is a window function

Frame vectors: $v_{k,\tau}(t) := w(t - \tau) e^{i2\pi kt}$
Discrete short-time Fourier transform

The STFT of a vector $\vec{x} \in \mathbb{C}^n$ is

$$\text{STFT} \{ f \} (k, l) := \langle \vec{x} \circ \vec{w}_l, \vec{h}_k \rangle$$

where $w \in \mathbb{C}^n$ is a window vector

Frame vectors: $v_{k,l}(t) := \vec{w}_l \circ \vec{h}_k$
STFT

Length of window and shifts are chosen so that shifted windows overlap.

In that case the STFT is a frame.

We can invert it using fast algorithms based on the FFT.

Window should not produce spurious high-frequency artifacts.
Rectangular window

Signal

\[
\text{Window} \times \text{Spectrum} = \text{Spectrum}
\]
Hann window

Signal \times Window = Spectrum

\begin{align*}
\text{Signal} & \quad \times \quad \text{Window} \\
\text{Spectrum} & \quad \ast \quad \text{Spectrum}
\end{align*}
Frame vector $l = 0, k = 0$

Real part

Imaginary part

Spectrum
Frame vector $l = 1/32$, $k = 0$
Frame vector $l = 0, k = 64$

Real part

Imaginary part

Spectrum
Frame vector $l = 1/32$, $k = 64$
Speech signal
Spectrum
Spectrogram (log magnitude of STFT coefficients)
Frames

Short-time Fourier transform (STFT)

Wavelets

Thresholding
Wavelet transform

Motivation: Extracting features at different scales

Idea: Frame vectors are scaled, shifted copies of a fixed function

An additional function captures low-pass component at largest scale
Wavelet transform

The wavelet transform of a function \( f \in L_2[-1/2, 1/2] \) depends on a choice of scaling function (or father wavelet) \( \phi \) and wavelet function (or mother wavelet) \( \psi \)

The scaling coefficients are

\[
W_\phi \{ f \} (\tau) := \frac{1}{\sqrt{s}} \int f(t) \phi(t - \tau) \, dt
\]

The wavelet coefficients are

\[
W_\psi \{ f \} (s, \tau) := \frac{1}{\sqrt{s}} \int_0^1 f(t) \psi \left( \frac{t - \tau}{s} \right) \, dt
\]

Wavelets can be designed to be bases or frames
Haar wavelet

Wavelets are **band-pass** filters, scaling functions are **low-pass** filters
Discrete wavelet transform

The discrete wavelet transform depends on a choice of scaling vector \( \vec{\phi} \) and wavelet \( \vec{\psi} \).

The scaling coefficients are

\[
W_{\vec{\phi}} \{ f \} (l) := \langle \vec{x}, \vec{\phi}_l \rangle
\]

The wavelet coefficients are

\[
W_{\vec{\psi}} \{ f \} (s, l) := \langle \vec{x}, \vec{\psi}_{s,l} \rangle,
\]

where

\[
\vec{\psi}_{s,l}[j] := \vec{\psi} \left[ \frac{j - l}{s} \right]
\]

Wavelets can be designed to be bases or frames.
<table>
<thead>
<tr>
<th>Scale</th>
<th>Basis functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^0$</td>
<td></td>
</tr>
</tbody>
</table>
Orthonormal wavelet basis

Scale

$2^0$
Orthonormal wavelet basis

Scale

$2^0$

Basis functions
Orthonormal wavelet basis

Scale

$2^0$

$2^1$

Basis functions
Orthonormal wavelet basis

Scale

$2^0$

$2^1$

Basis functions
Orthonormal wavelet basis

Scale

$2^0$

$2^1$

Basis functions
Orthonormal wavelet basis

Scale

Basis functions

$2^0$

$2^1$

$2^2$
Orthonormal wavelet basis

Scale

$2^0$

$2^1$

$2^2$
Orthonormal wavelet basis

Scale | Basis functions
--- | ---
$2^0$ | ![Wavelet basis functions for $2^0$ scale]
$2^1$ | ![Wavelet basis functions for $2^1$ scale]
$2^2$ | ![Wavelet basis functions for $2^2$ scale]
$2^3$ | ![Wavelet basis functions for $2^3$ scale]
Orthonormal wavelet basis

Scale | Basis functions
--- | ---
$2^0$ | ![Basis functions for $2^0$]
$2^1$ | ![Basis functions for $2^1$]
$2^2$ | ![Basis functions for $2^2$]
$2^3$ (scaling vector) | ![Basis functions for $2^3$]
Multiresolution decomposition

Sequence of subspaces \( \mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_K \) representing different scales

Fix a scaling vector \( \vec{\phi} \) and a wavelet \( \vec{\psi} \)
Multiresolution decomposition

Sequence of subspaces $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_K$ representing different scales

Fix a scaling vector $\vec{\phi}$ and a wavelet $\vec{\psi}$

$\mathcal{V}_K$ is the span of $\vec{\phi}$
Multiresolution decomposition

Sequence of subspaces $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_K$ representing different scales

Fix a scaling vector $\vec{\phi}$ and a wavelet $\vec{\psi}$

$\mathcal{V}_K$ is the span of $\vec{\phi}$
Multiresolution decomposition

\[ \mathcal{V}_k := \mathcal{W}_k \oplus \mathcal{V}_{k+1} \]

\( \mathcal{W}_k \) is the span of \( \vec{\psi} \) dilated by \( 2^k \) and shifted by multiples of \( 2^{k+1} \)
Multiresolution decomposition

\[ \mathcal{V}_k := \mathcal{W}_k \oplus \mathcal{V}_{k+1} \]

\( \mathcal{W}_k \) is the span of \( \vec{\psi} \) dilated by \( 2^k \) and shifted by multiples of \( 2^{k+1} \)

\[ \mathcal{W}_0 \]
Multiresolution decomposition

\[ \mathcal{V}_k := \mathcal{W}_k \oplus \mathcal{V}_{k+1} \]

\( \mathcal{W}_k \) is the span of \( \vec{\psi} \) dilated by \( 2^k \) and shifted by multiples of \( 2^{k+1} \)
Multiresolution decomposition

\[ \mathcal{V}_k := \mathcal{W}_k \oplus \mathcal{V}_{k+1} \]

\( \mathcal{W}_k \) is the span of \( \vec{\psi} \) dilated by \( 2^k \) and shifted by multiples of \( 2^{k+1} \)

\[ \mathcal{W}_0 \]

\[ \mathcal{W}_2 \]

\( \mathcal{P}_{\mathcal{V}_k} \vec{x} \) is an approximation of \( \vec{x} \) at scale \( 2^k \)
Multiresolution decomposition

Properties

- $V_0 = \mathbb{C}^n$ (approximation at scale $2^0$ is perfect)
- $V_k$ is invariant to translations of scale $2^k$
- Dilating vectors in $V_j$ by 2 yields vectors in $V_{j+1}$
Electrocardiogram

Signal

Haar transform
Scale $2^9$

$\mathcal{P}_{\mathcal{W}_9} \vec{x}$

$\mathcal{P}_{\mathcal{V}_9} \vec{x}$
Scale $2^8$

$\mathcal{P}_{W_8} \vec{x}$

$\mathcal{P}_{V_8} \vec{x}$
Scale $2^7$

$\mathcal{P}_{W_7} \vec{x}$

$\mathcal{P}_{V_7} \vec{x}$
$\mathcal{P}_{\mathcal{W}_6} \bar{x}$

$\mathcal{P}_{\mathcal{V}_6} \bar{x}$
Scale $2^5$

$\mathcal{P}_{W_5} \vec{x}$

$\mathcal{P}_{V_5} \vec{x}$
Scale $2^4$

$\mathcal{P}_{W_4} \vec{x}$

$\mathcal{P}_{V_4} \vec{x}$
Scale $2^3$

$\mathcal{P}_{W_3} \vec{x}$

$\mathcal{P}_{V_3} \vec{x}$
Scale $2^2$

$\mathcal{P}_{W_2} \vec{x}$

$\mathcal{P}_{V_2} \vec{\bar{x}}$
$\mathcal{P}_{W_1} \tilde{x}$  

$\mathcal{P}_{V_1} \tilde{x}$
Scale $2^0$

$\mathcal{P}_{\mathcal{W}_0} \tilde{x}$

$\mathcal{P}_{\mathcal{V}_0} \tilde{x}$
2D Wavelets

Extension to 2D by using outer products of 1D atoms

\[ \xi_{s_1,s_2,k_1,k_2}^{2D} := \xi_{s_1,k_1}^{1D} \left( \xi_{s_2,k_2}^{1D} \right)^T \]

The JPEG 2000 compression standard is based on 2D wavelets

Many extensions:

Steerable pyramid, ridgelets, curvelets, bandlets, ...
2D Haar transform
2D wavelet transform
2D wavelet transform
Frames

Short-time Fourier transform (STFT)

Wavelets

Thresholding
Denoising

**Aim:** Extracting information *(signal)* from data in the presence of uninformative perturbations *(noise)*

Additive noise model

\[
\vec{y} = \vec{x} + \vec{z}
\]

Prior knowledge about structure of signal vs structure of noise is required
Assumption

- Signal is a sparse superposition of basis/frame vectors
- Noise is not
Assumption

- Signal is a sparse superposition of basis/frame vectors
- Noise is not

Example:

Gaussian noise $\tilde{z}$ with covariance matrix $\sigma^2 I$, distribution of $F \tilde{z}$?
Example
Thresholding

Hard-thresholding operator

\[ \mathcal{H}_\eta (\vec{v}) [j] := \begin{cases} \vec{v} [j] & \text{if } |\vec{v} [j]| > \eta \\ 0 & \text{otherwise} \end{cases} \]
Denoising via hard thresholding

![Graph showing denoising via hard thresholding]
Multisinusoidal signal

\[ \vec{y} \]

\[ F\vec{y} \]
Denoising via hard thresholding

Data: $\tilde{y} = \tilde{x} + \tilde{z}$

Assumption: $F\tilde{x}$ is sparse, $F\tilde{z}$ is not

1. Apply the hard-thresholding operator $\mathcal{H}_\eta$ to $F\tilde{y}$

2. If $F$ is a basis, then

$$\tilde{x}_{est} := F^{-1}\mathcal{H}_\eta (F\tilde{y})$$

If $F$ is a frame,

$$\tilde{x}_{est} := F^\dagger \mathcal{H}_\eta (F\tilde{y}),$$

where $F^\dagger$ is the pseudoinverse of $F$ (other left inverses of $F$ also work)
Denoising via hard thresholding in Fourier basis

\[ \vec{y} \]

\[ F\vec{y} \]
Denoising via hard thresholding in Fourier basis

\[ F^{-1} \mathcal{H}_\eta (F \tilde{y}) \]  

\[ \mathcal{H}_\eta (F \tilde{y}) \]
Image denoising

$\tilde{x}$

$F\tilde{x}$
Image denoising

\[ \tilde{z} \]

\[ F\tilde{z} \]
Data (SNR=2.5)
$\vec{F}_y$
$F^{-1} \mathcal{H}_\eta (F \tilde{y})$
Data (SNR=1)

\[ \vec{y} \]

\[ F \vec{y} \]
\vec{F}_y
$\mathcal{H}_\eta (F\tilde{y})$
\( F^{-1} \mathcal{H}_\eta (F \bar{Y}) \)
Image denoising

\[ \bar{y} \quad F^{-1} \mathcal{H}_{\eta}(F \bar{y}) \quad \bar{x} \]
Denoising via thresholding

\[ \hat{y} \quad F^{-1}H_\eta(F\hat{y}) \quad \tilde{x} \]
Speech denoising
Time thresholding
Spectrum
Frequency thresholding
Frequency thresholding

- Data
- DFT thresholding
Spectrogram (STFT)
STFT thresholding
STFT thresholding

Data

STFT thresholding
Coefficients are structured
Coefficients are structured
Assumption: Coefficients are *group sparse*, nonzero coefficients *cluster* together

Partition coefficients into blocks \( \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k \) and threshold whole blocks

\[
B_\eta (\vec{v}) [j] := \begin{cases} 
\vec{v} [j] & \text{if } j \in \mathcal{I}_j \text{ such that } \| \vec{v}_{\mathcal{I}_j} \|_2 > \eta, \\
0 & \text{otherwise},
\end{cases}
\]
Denoising via block thresholding

1. Apply the hard-thresholding operator \( B_\eta \) to \( F \hat{y} \)

2. If \( F \) is a basis, then

\[
\tilde{x}_{est} := F^{-1} B_\eta (F \hat{y})
\]

If \( F \) is a frame,

\[
\tilde{x}_{est} := F^\dagger B_\eta (F \hat{y}) ,
\]

where \( F^\dagger \) is the pseudoinverse of \( F \) (other left inverses of \( F \) also work)
Image denoising (SNR=2.5)
\text{Fy}
$\mathcal{H}_\eta (F \tilde{y})$
$\mathcal{B}_\eta (F\tilde{y})$
$F^{-1} \mathcal{H}_\eta \left( F \tilde{y} \right)$
\( F^{-1} B_\eta (F \tilde{y}) \)
Image denoising (SNR=1)

\[ \tilde{y} \hspace{1cm} F\tilde{y} \]
$F \vec{y}$
$\mathcal{H}_\eta (F\vec{y})$
$\beta_\eta (F \tilde{y})$
$F^{-1}\mathcal{H}_\eta (Fy)$
\( F^{-1} \mathcal{B}_\eta (F\vec{y}) \)
Denoising via thresholding

\[ \tilde{y} \quad F^{-1} \mathcal{H}_\eta (F \tilde{y}) \quad \tilde{x} \]
Denoising via thresholding

\[ \tilde{y} \quad F^{-1}B_\eta (F \tilde{y}) \quad \tilde{x} \]
Denoising via thresholding

\[ \tilde{y} \quad F^{-1}H_\eta (F\tilde{y}) \quad \tilde{x} \]
Denoising via thresholding

\[ \tilde{y} \quad F^{-1}B_\eta(\hat{y}) \quad \tilde{x} \]
Speech denoising
STFT thresholding
STFT thresholding
STFT block thresholding
STFT block thresholding