The Frequency Domain

DS-GA 1013 / MATH-GA 2824 Mathematical Tools for Data Science

https://cims.nyu.edu/~cfgranda/pages/MTDS_spring19/index.html

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Motivating applications

The frequency domain

Sampling

Discrete Fourier transform

Frequency representations in multiple dimensions

Translation Invariance

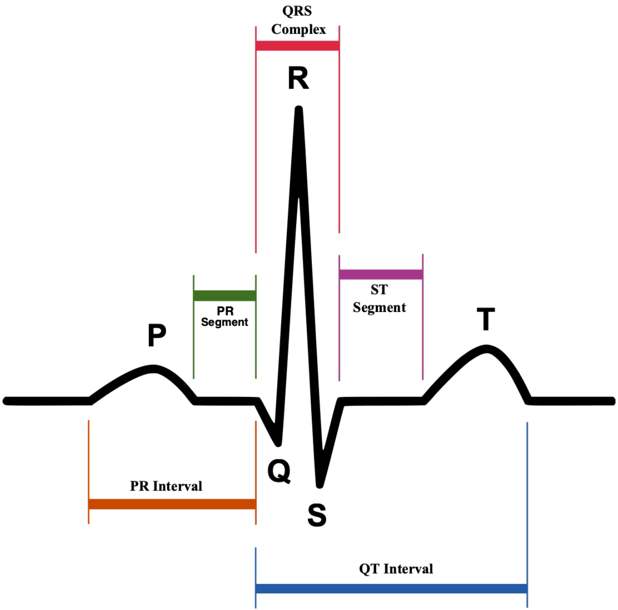
Signal processing

Signal: any structured object of interest (images, audio, video, etc.)

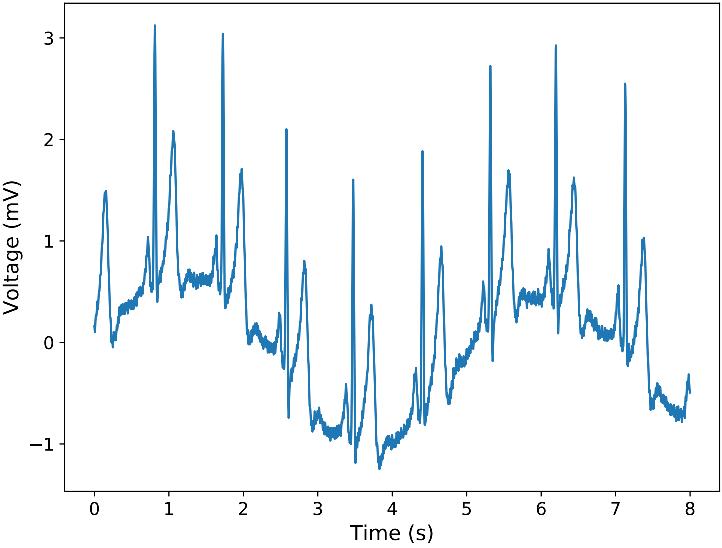
Modeled as function of space, time, etc.

Finding adequate representations is crucial to process signals effectively

Periodic signals: electrocardiogram



Periodic signals: electrocardiogram



Sampling

Signals are often modeled as continuous functions

Arbitrary continuous functions cannot be manipulated in a computer

We need finite-dimensional representation

Idea: Use samples as finite-dimensional representation

When will this work?

Denoising

Signals as functions

We model signals as square-integrable functions on an interval $[a, b] \subset \mathbb{R}$

Inner product:

$$\langle x, y \rangle := \int_a^b x(t)\, \overline{y(t)}\, \mathrm{d}t$$

Goal: Find basis functions to represent periodic signals

Sinusoids

Sinusoidal function:

$$a \cos(2\pi f t + \theta)$$

- Amplitude: a
- Frequency: f
- Time index: t (periodic with period $1/f$)
- Phase: θ

Problem

Nonlinear dependence on the phase

Infinite possible basis functions associated to each f

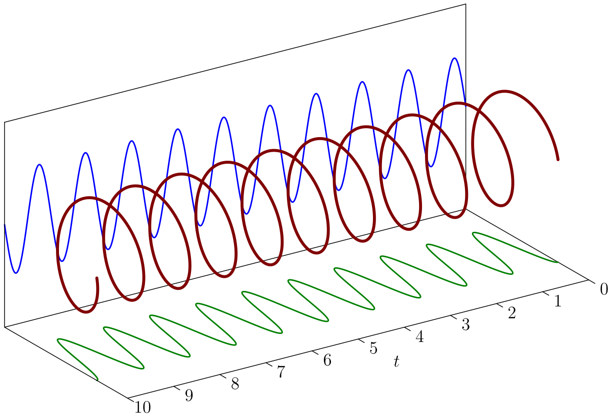
Solution: Use complex sinusoids

Complex sinusoid

The complex sinusoid with frequency $f \in \mathbb{R}$ is given by

$$\exp(i2\pi ft) := \cos(2\pi ft) + i\sin(2\pi ft)$$

Complex sinusoid



Complex sinusoid

We can express any real sinusoid in terms of complex sinusoids

$$\cos(2\pi ft + \theta) = \frac{\exp(i2\pi ft + i\theta) + \exp(-i2\pi ft - i\theta)}{2}$$

Complex sinusoid

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$$\begin{aligned}\cos(2\pi ft + \theta) &= \frac{\exp(i2\pi ft + i\theta) + \exp(-i2\pi ft - i\theta)}{2} \\ &= \frac{\exp(i\theta)}{2}\exp(i2\pi ft) + \frac{\exp(-i\theta)}{2}\exp(-i2\pi ft)\end{aligned}$$

The phase is encoded in the complex amplitude!

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The phase is encoded in the complex amplitude!

Linear subspace spanned by $\exp(i2\pi ft)$ and $\exp(-i2\pi ft)$ contains all real sinusoids with frequency f

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The phase is encoded in the complex amplitude!

Linear subspace spanned by $\exp(i2\pi ft)$ and $\exp(-i2\pi ft)$ contains all real sinusoids with frequency f

If we add two sinusoids with frequency f the result is a sinusoid with frequency f

Orthogonality of complex sinusoids

The family of complex sinusoids with integer frequencies

$$\phi_k(t) := \exp\left(\frac{i2\pi kt}{T}\right), \qquad k \in \mathbb{Z},$$

is an orthogonal set on $[a, a+T]$, where $a, T \in \mathbb{R}$ and $T > 0$

Proof

$$\langle \phi_k, \phi_j \rangle = \int_a^{a+T} \phi_k(t)\, \overline{\phi_j(t)}\, \mathrm{d}t$$

Proof

$$\begin{aligned}\langle \phi_k, \phi_j \rangle &= \int_a^{a+T} \phi_k(t)\,\overline{\phi_j(t)}\,\mathrm{d}t \\ &= \int_a^{a+T} \exp\left(\frac{i2\pi(k-j)t}{T}\right)\,\mathrm{d}t\end{aligned}$$

Proof

$$
\begin{aligned}
\langle \phi_k, \phi_j \rangle &= \int_a^{a+T} \phi_k(t)\,\overline{\phi_j(t)}\,\mathrm{d}t \\
&= \int_a^{a+T} \exp\left(\frac{i2\pi(k-j)\,t}{T}\right)\,\mathrm{d}t \\
&= \frac{T}{i2\pi(k-j)}\left(\exp\left(\frac{i2\pi(k-j)(a+T)}{T}\right) - \exp\left(\frac{i2\pi(k-j)\,a}{T}\right)\right)
\end{aligned}
$$

Proof

$$\begin{aligned}
\langle \phi_k, \phi_j \rangle &= \int_a^{a+T} \phi_k(t)\,\overline{\phi_j(t)}\,\mathrm{d}t \\
&= \int_a^{a+T} \exp\left(\frac{i2\pi(k-j)t}{T}\right)\,\mathrm{d}t \\
&= \frac{T}{i2\pi(k-j)}\left(\exp\left(\frac{i2\pi(k-j)(a+T)}{T}\right) - \exp\left(\frac{i2\pi(k-j)a}{T}\right)\right) \\
&= 0
\end{aligned}$$

Fourier series

The Fourier series coefficients of $x \in \mathcal{L}_2\,[a, a+T]$, $a, T \in \mathbb{R},\ T > 0$, are

$$\hat{x}[k] := \langle x, \phi_k \rangle = \int_a^{a+T} x(t) \exp\left(-\frac{i2\pi kt}{T}\right) \mathrm{d}t.$$

The Fourier series of order k_c is defined as

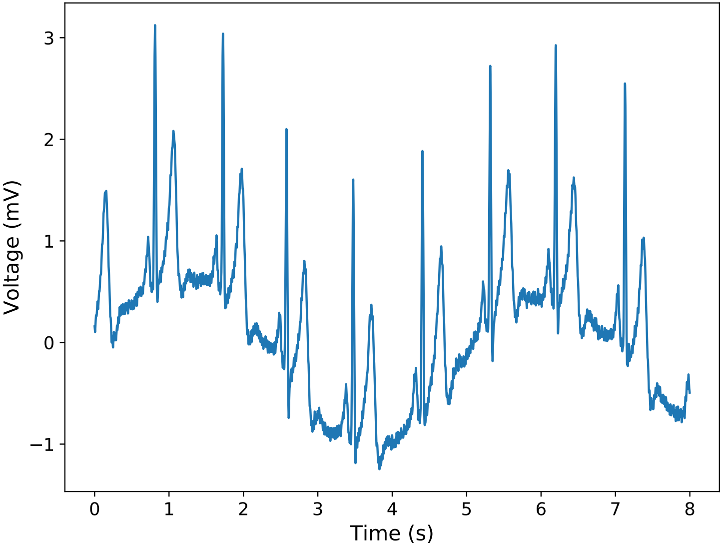
$$\mathcal{F}_{k_c}\left\{x\right\} := \frac{1}{T} \sum_{k=-k_c}^{k_c} \hat{x}[k]\phi_k$$

The Fourier series of x is $\lim_{k_c \to \infty} \mathcal{F}_{k_c}\left\{x\right\}$

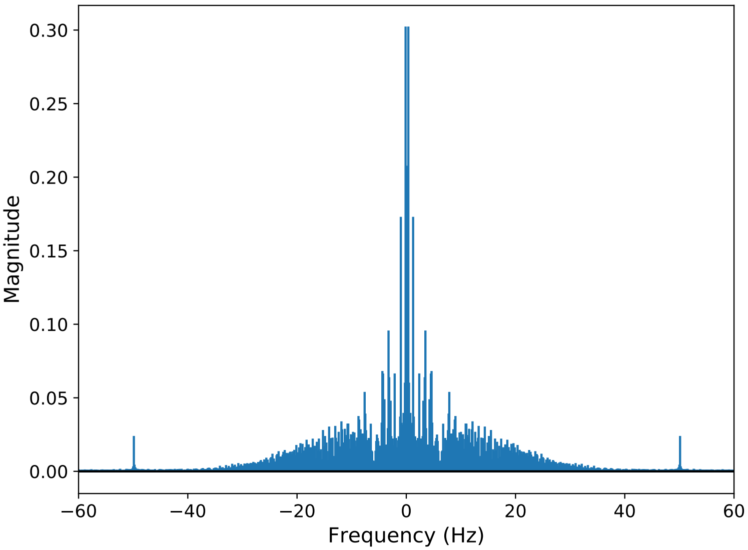
Fourier series as a projection

$$
\begin{aligned}
\mathcal{P}_{\mathrm{span}(\{\phi_{-k_c},\phi_{-k_c+1},\ldots,\phi_{k_c}\})}\, x &= \sum_{k=-k_c}^{k_c} \left\langle x, \frac{1}{\sqrt{T}}\phi_k \right\rangle \frac{1}{\sqrt{T}}\phi_k \\
&= \mathcal{F}_{k_c}\left\{x\right\}
\end{aligned}
$$

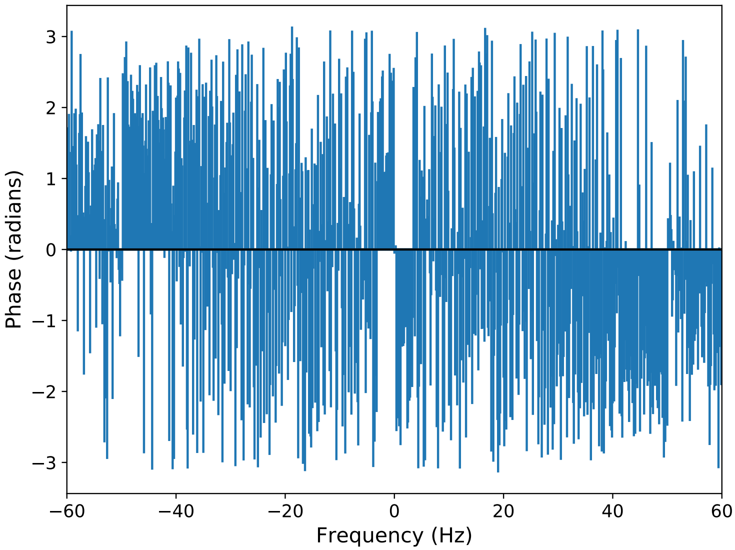
Electrocardiogram



Electrocardiogram: Fourier coefficients (magnitude)



Electrocardiogram: Fourier coefficients (phase)

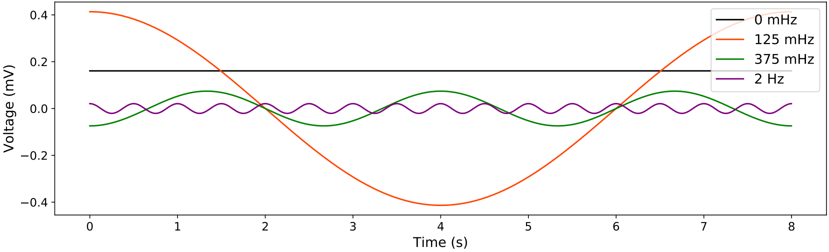


Convergence of Fourier series

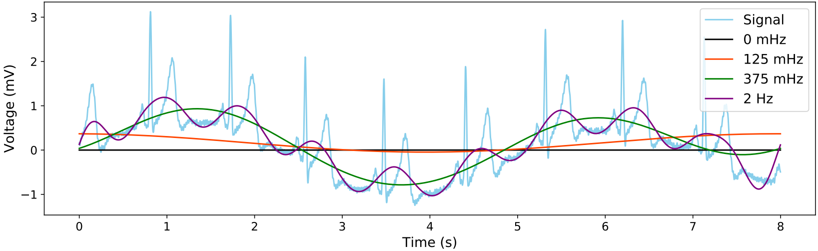
For any function $x \in \mathcal{L}_2[0, T)$, where $a, T \in \mathbb{R}$ and $T > 0$,

$$\lim_{k \to \infty} ||x - \mathcal{F}_k \{x\}||_{\mathcal{L}_2} = 0$$

Electrocardiogram: Fourier components



Electrocardiogram: Fourier series



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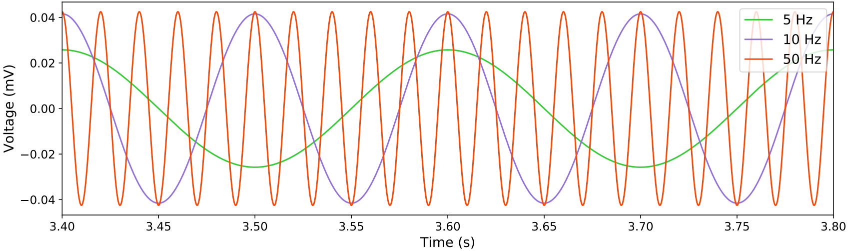
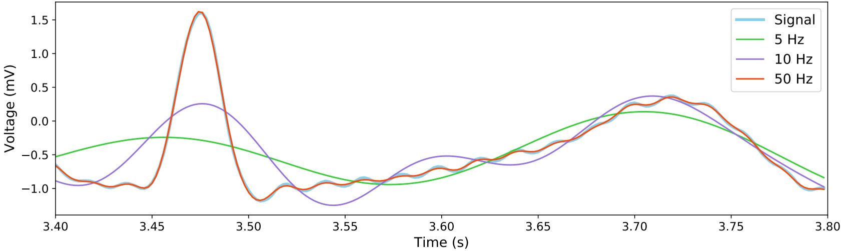
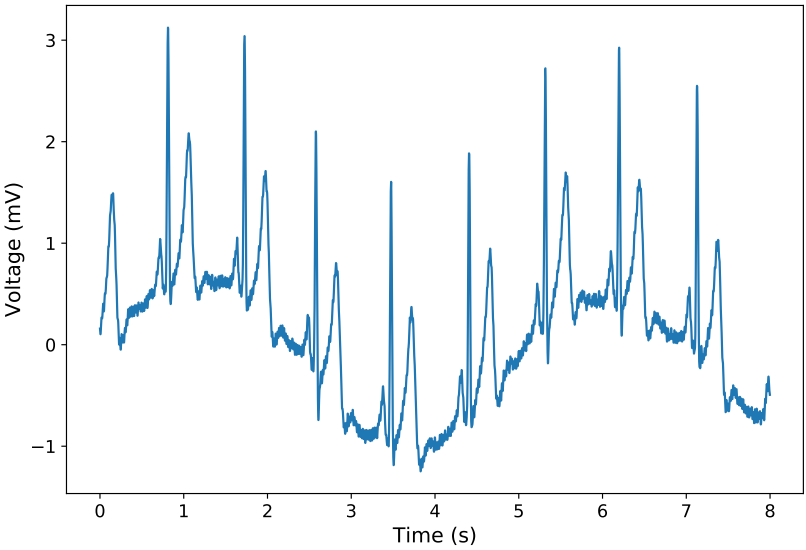


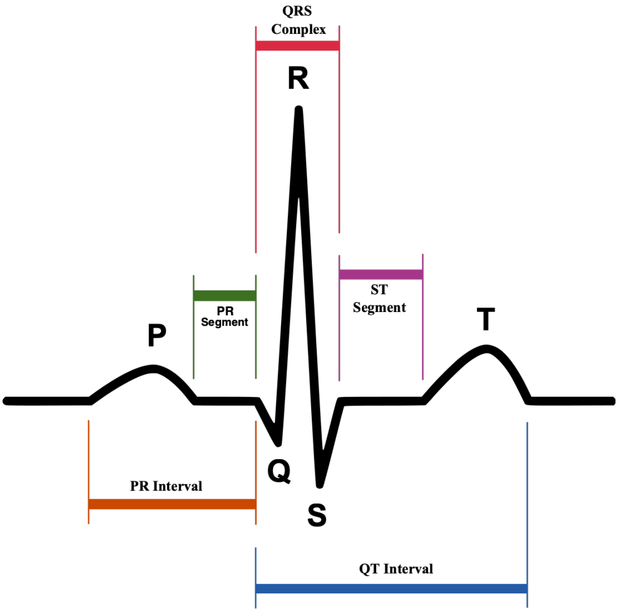
Electrocardiogram: Fourier series



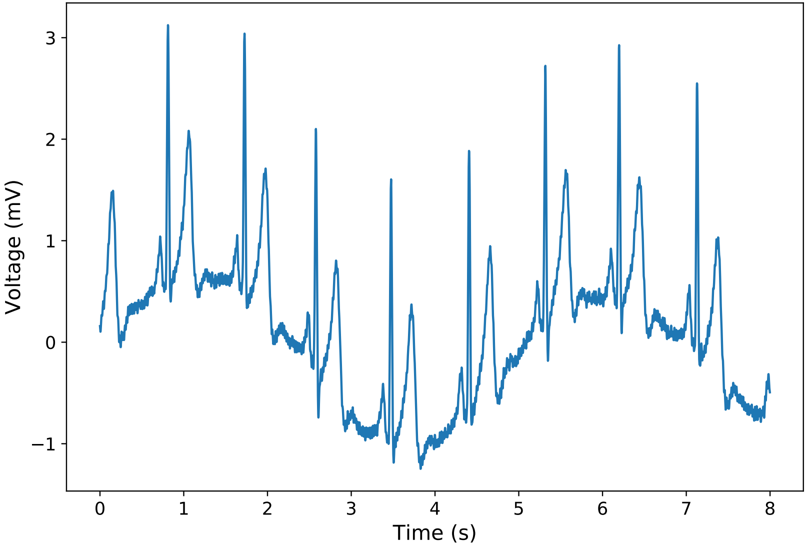
Electrocardiogram data



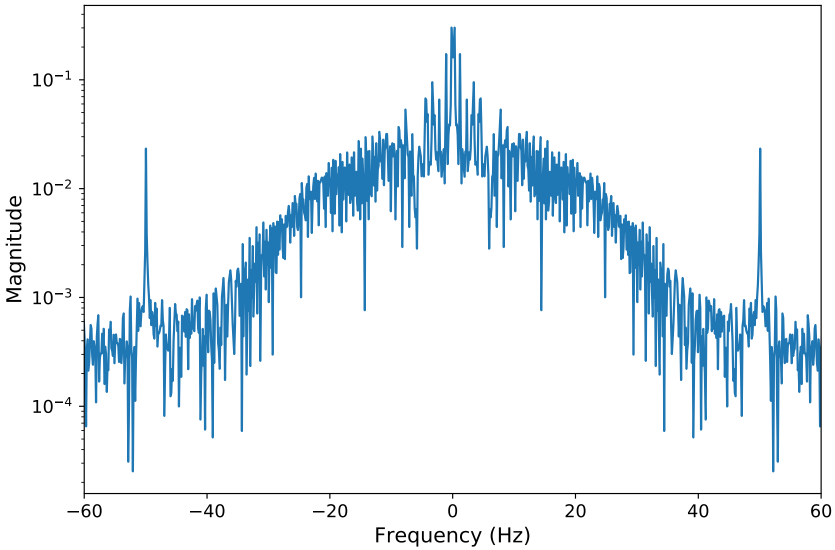
Electrocardiogram features



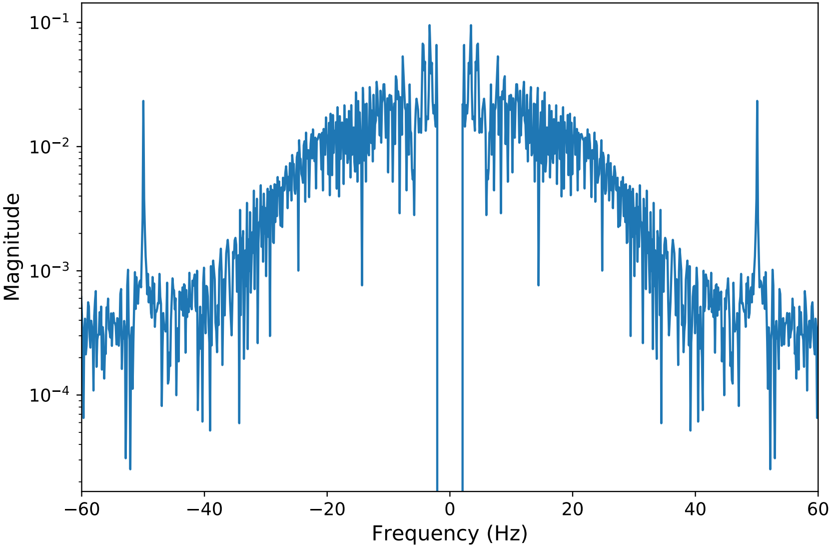
Problem: Baseline wandering



Electrocardiogram: Fourier coefficients (magnitude)



Filtered electrocardiogram



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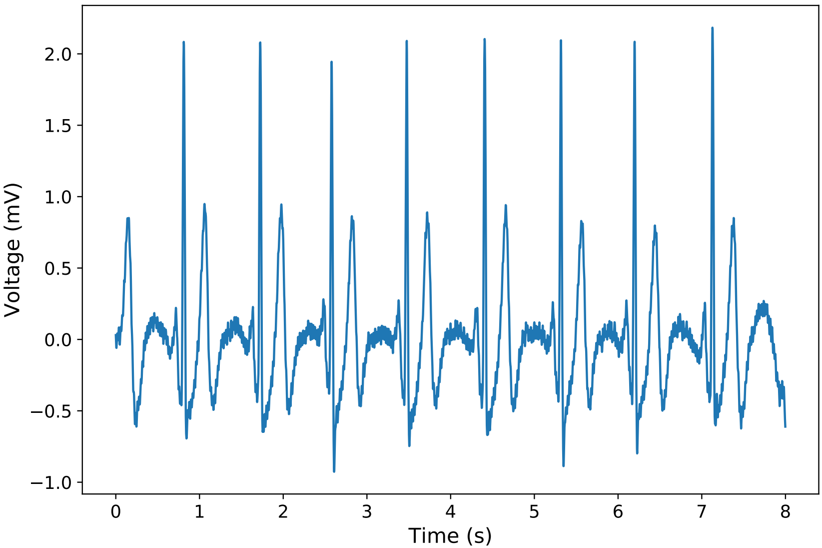
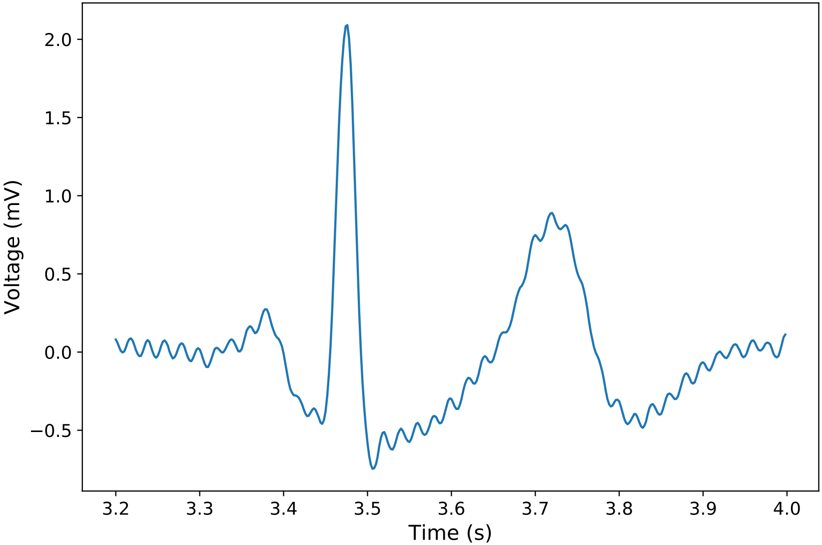
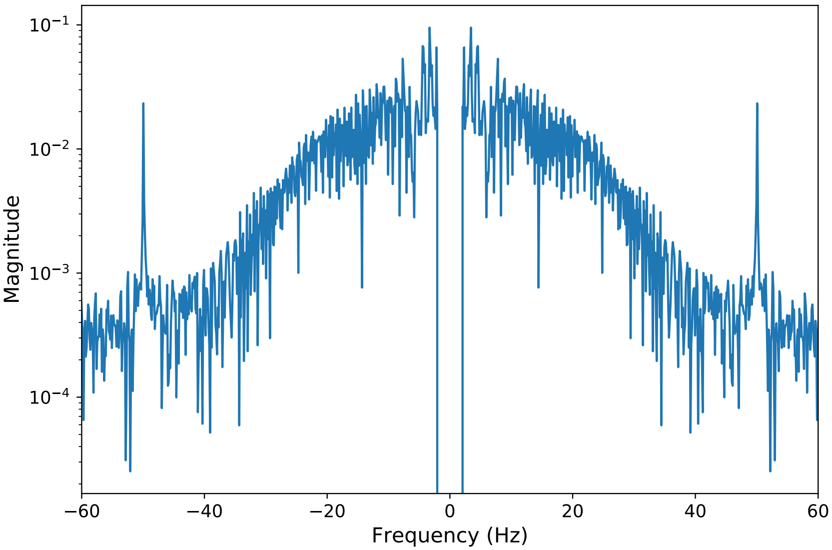


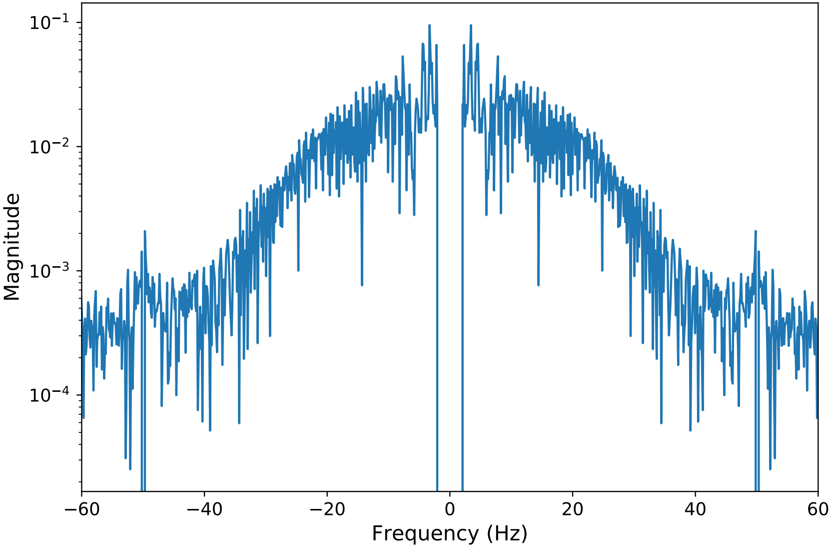
Problem: Interference



Fourier coefficients (magnitude)



Filtered electrocardiogram



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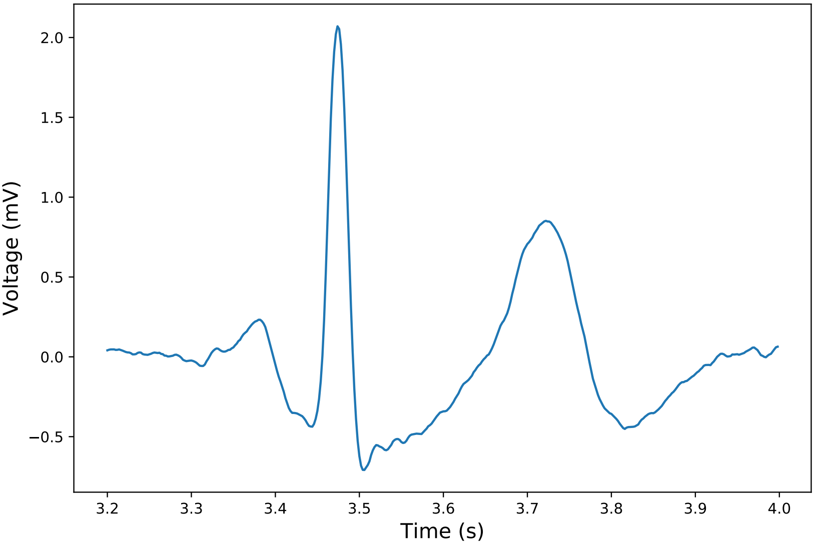
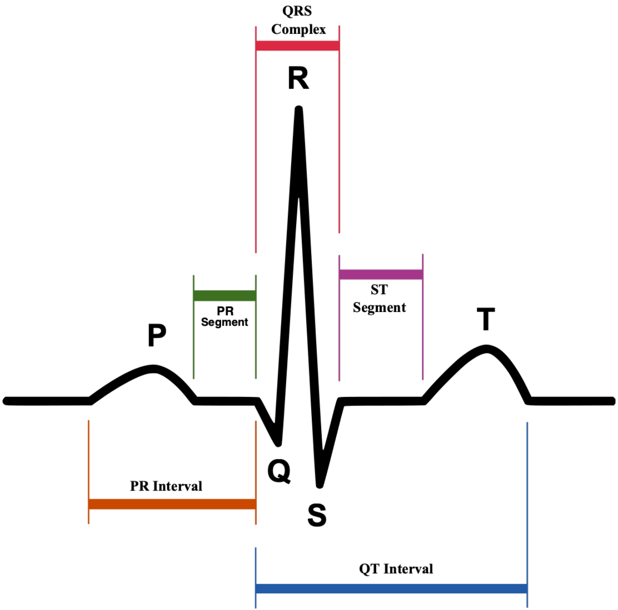


Electrocardiogram features



Bandlimited signals

A bandlimited signal cut-off frequency k_c is equal to its Fourier series of order k_c

$$x(t) = \sum_{k=-k_c}^{k_c} \hat{x}[k] \exp\left(\frac{i2\pi kt}{T}\right)$$

Bandlimited signals have a finite representation ($2k_c + 1$ coefficients)

Sampling

Evaluating signal at a finite number of fixed location

Hopefully samples preserve information

In the case of bandlimited signals they do

Sampling a bandlimited signal on a uniform grid

Bandimited signal x measured at N equispaced points in interval T

Samples: $x\left(\frac{0}{N}\right)$, $x\left(\frac{T}{N}\right)$, $x\left(\frac{2T}{N}\right)$, $\ldots$, $x\left(\frac{(N-1)T}{N}\right)$

Using Fourier series representation

$$x\left(\frac{jT}{N}\right) = \sum_{k=-k_c}^{k_c} \hat{x}_k \exp\left(\frac{i2\pi k jT}{NT}\right)$$

$$= \sum_{k=-k_c}^{k_c} \hat{x}_k \exp\left(\frac{i2\pi kj}{N}\right)$$

In matrix form

$$\begin{bmatrix} x\left(\frac{0}{N}\right) \\ x\left(\frac{T}{N}\right) \\ \cdots \\ x\left(\frac{jT}{N}\right) \\ \cdots \\ x\left(T-\frac{T}{N}\right) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \exp\left(\frac{i2\pi(-k_c)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(\frac{i2\pi(-k_c)j}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)j}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c j}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(\frac{i2\pi(-k_c)(N-1)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)(N-1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c(N-1)}{N}\right) \end{bmatrix} \begin{bmatrix} \hat{x}[-k_c] \\ \hat{x}[-k_c+1] \\ \cdots \\ \hat{x}[k_c] \end{bmatrix}$$

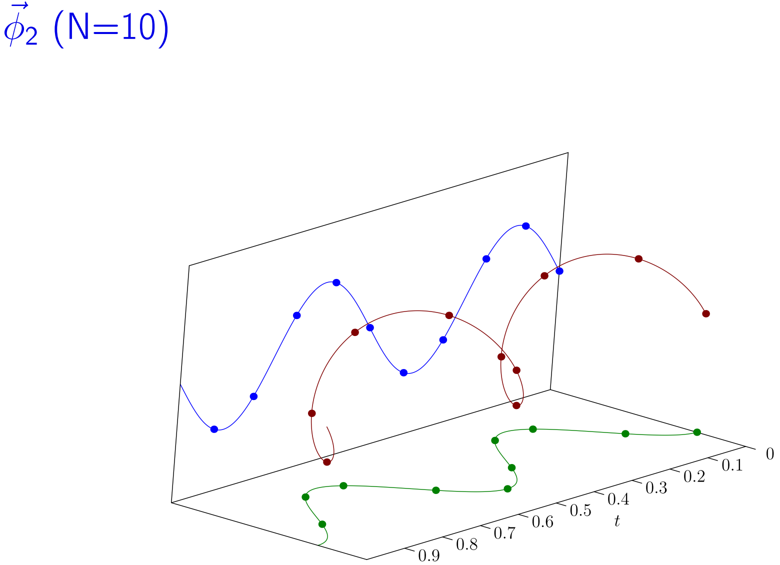
$$\vec{x}_{[N]} = \widetilde{F}_{[N]} \hat{x}_{[k_c]}$$

Discrete complex sinusoids

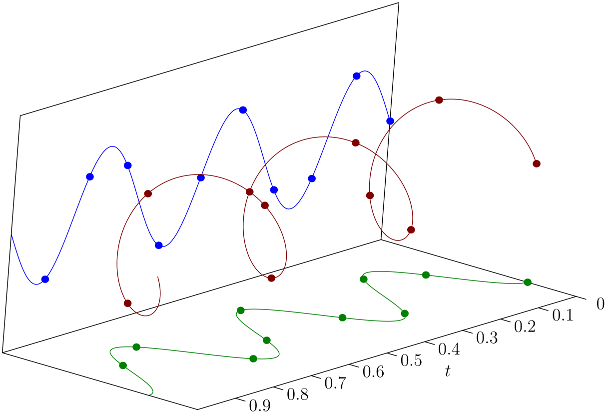
The discrete complex sinusoid $\vec{\phi}_k \in \mathbb{C}^N$ with frequency k is

$$\vec{\phi}_k[j] := \exp\left(\frac{i2\pi kj}{N}\right), \qquad 0 \leq j, k \leq N-1$$

Complex sinusoids scaled by $1/\sqrt{N}$ form an orthonormal basis of $\mathbb{C}^N$


$\vec{\phi}_2$ (N=10)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
t

$\vec{\phi}_3$ (N=10)



Orthogonality

$$\left\langle \vec{\phi}_k, \vec{\phi}_l \right\rangle = \sum_{j=0}^{N-1} \vec{\phi}_k\left[j\right] \overline{\vec{\phi}_l\left[j\right]}$$

Orthogonality

$$\left\langle \vec{\phi}_k, \vec{\phi}_l \right\rangle = \sum_{j=0}^{N-1} \vec{\phi}_k[j] \, \overline{\vec{\phi}_l[j]}$$

$$= \sum_{j=0}^{N-1} \exp\left(\frac{i2\pi (k-l) j}{N} \right)$$

Orthogonality

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\begin{aligned}
\left\langle \vec{\phi}_k, \vec{\phi}_l \right\rangle &= \sum_{j=0}^{N-1} \vec{\phi}_k\,[j]\,\overline{\vec{\phi}_l\,[j]} \\
&= \sum_{j=0}^{N-1} \exp\left(\frac{i2\pi\,(k-l)\,j}{N}\right) \\
&= \frac{1-\exp\left(\frac{i2\pi(k-l)N}{N}\right)}{1-\exp\left(\frac{i2\pi(k-l)}{N}\right)}
\end{aligned}
$$

Orthogonality

$$
\begin{aligned}
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&= \sum_{j=0}^{N-1} \exp\left(\frac{i2\pi(k-l)j}{N}\right) \\
&= \frac{1-\exp\left(\frac{i2\pi(k-l)N}{N}\right)}{1-\exp\left(\frac{i2\pi(k-l)}{N}\right)} \\
&= 0 \qquad \text{if } k \neq l
\end{aligned}
$$

Nyquist-Shannon-Kotelnikov sampling theorem

Any bandlimited signal $x \in \mathcal{L}_2[0, T)$, where $T > 0$, with cut-off frequency k_c can be recovered exactly from N uniformly spaced samples $x(0), x(T/N), \ldots, x(T - T/N)$ as long as

$$N \geq 2k_c + 1,$$

where $2k_c + 1$ is known as the Nyquist rate

Recovery

$$\hat{x}_{[k_c]} = \frac{1}{N} \widetilde{F}_{[N]}^{*} \vec{x}_{[N]}$$

$$\widetilde{F}_{[N]} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \exp\left(\frac{i2\pi(-k_c)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(\frac{i2\pi(-k_c)(N-1)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)(N-1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c(N-1)}{N}\right) \end{bmatrix}$$

Proof

For $-k_c \leq k \leq -1$ and $0 \leq j \leq N-1$,

$$\exp\left(\frac{i2\pi kj}{N}\right) = \exp\left(\frac{i2\pi\left(N+k\right)j}{N}\right)$$

$$\widetilde{F}_{[N]} = \begin{bmatrix} \vec{\phi}_{N-k_c} & \cdots & \vec{\phi}_{N-1} & \vec{\phi}_0 & \cdots & \vec{\phi}_{k_c} \end{bmatrix}$$

$\widetilde{F}_{[N]}$ is orthogonal!

Audio

Range of frequencies that human beings can hear is from 20 Hz to 20 kHz

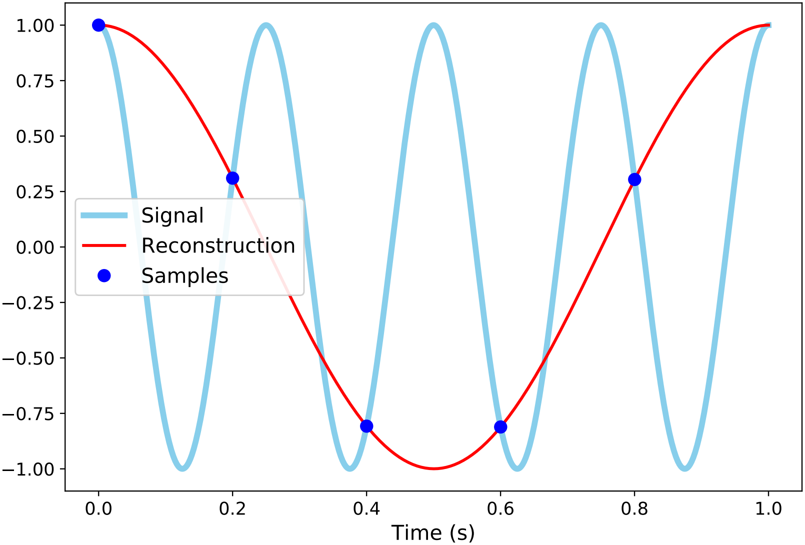
We need to sample at 40 kHz

Typical rates used in practice: 44.1 kHz (CD), 48 kHz, 88.2 kHz, 96 kHz

What happens if we sample too slowly?

Show videos

What happens if we sample too slowly?



What happens if we sample too slowly?

Let x be a signal that is with cut-off frequency k_{true}

We measure $\vec{x}_{[N]}$, N samples of x at 0, T/N, $2T/N$, $\ldots T - T/N$

What happens if we recover the signal assuming it is bandlimited with cut-off freq k_{samp}, $N = 2k_{\text{samp}} + 1$, but actually $k_{\text{true}} > k_{\text{samp}}$?

$$\begin{aligned}\hat{x}^{\text{rec}}[k] &:= \frac{1}{N}(\widetilde{F}^{*}_{[N]}\vec{x}_{[N]})[k] \\ &= \sum_{\{(m-k) \bmod N = 0\}} \hat{x}[m]\end{aligned}$$

This is called aliasing

Proof

$$\frac{1}{N}(\widetilde{F}_{[N]}^{*}\vec{x}_{[N]})[k] = \frac{1}{N}\sum_{j=0}^{N-1}\exp\left(-\frac{i2\pi kj}{N}\right)\sum_{m=-k_{\text{true}}}^{k_{\text{true}}}\hat{x}[m]\exp\left(\frac{i2\pi mj}{N}\right)$$

Proof

$$\frac{1}{N}(\widetilde{F}^*_{[N]}\vec{x}_{[N]})[k] = \frac{1}{N}\sum_{j=0}^{N-1}\exp\left(-\frac{i2\pi kj}{N}\right)\sum_{m=-k_{\text{true}}}^{k_{\text{true}}}\hat{x}[m]\exp\left(\frac{i2\pi mj}{N}\right)$$
$$= \frac{1}{N}\sum_{m=-k_{\text{true}}}^{k_{\text{true}}}\hat{x}[m]\sum_{j=0}^{N-1}\exp\left(\frac{i2\pi(m-k)j}{N}\right)$$

$$\sum_{j=0}^{N-1}\exp\left(\frac{i2\pi(m-k)j}{N}\right) = \begin{cases} N & \text{if } (m-k) \bmod N = 0 \\ \frac{1-\exp\left(\frac{i2\pi(m-k)N}{N}\right)}{1-\exp\left(\frac{i2\pi(m-k)}{N}\right)} = 0 & \text{otherwise} \end{cases}$$

Proof

$$
\begin{aligned}
\frac{1}{N}(\widetilde{F}_{[N]}^{*}\vec{x}_{[N]})[k] &= \frac{1}{N}\sum_{j=0}^{N-1}\exp\left(-\frac{i2\pi kj}{N}\right)\sum_{m=-k_{\text{true}}}^{k_{\text{true}}}\hat{x}[m]\exp\left(\frac{i2\pi mj}{N}\right) \\
&= \frac{1}{N}\sum_{m=-k_{\text{true}}}^{k_{\text{true}}}\hat{x}[m]\sum_{j=0}^{N-1}\exp\left(\frac{i2\pi(m-k)j}{N}\right) \\
&= \sum_{\{(m-k) \bmod N=0\}}\hat{x}[m]
\end{aligned}
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$$

Sampling a real sinusoid

Consider a real sinusoid with frequency equal to 4 Hz

$$\begin{aligned} x(t) &:= \cos(8\pi t) \\ &= 0.5 \exp(-i2\pi 4t) + 0.5 \exp(i2\pi 4t) \end{aligned}$$

measured over one second, i.e. $T = 1$ s

k_c?

Nyquist rate?

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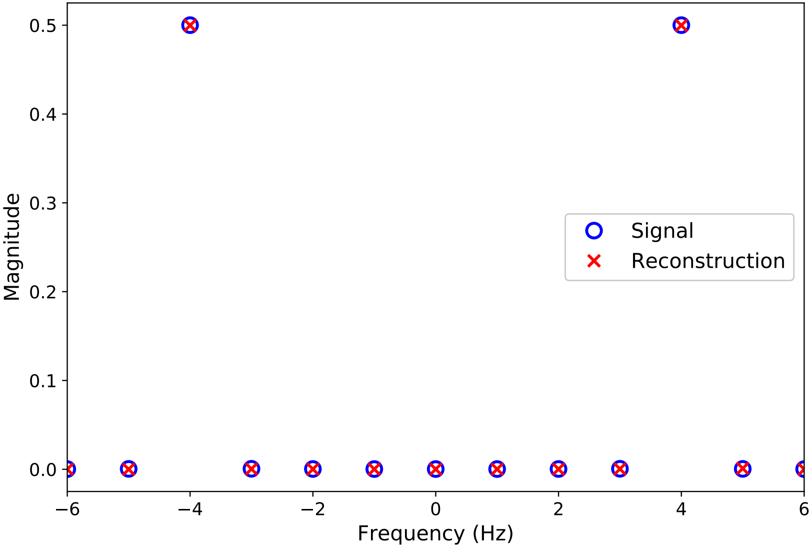
measured over one second, i.e. $T = 1$ s

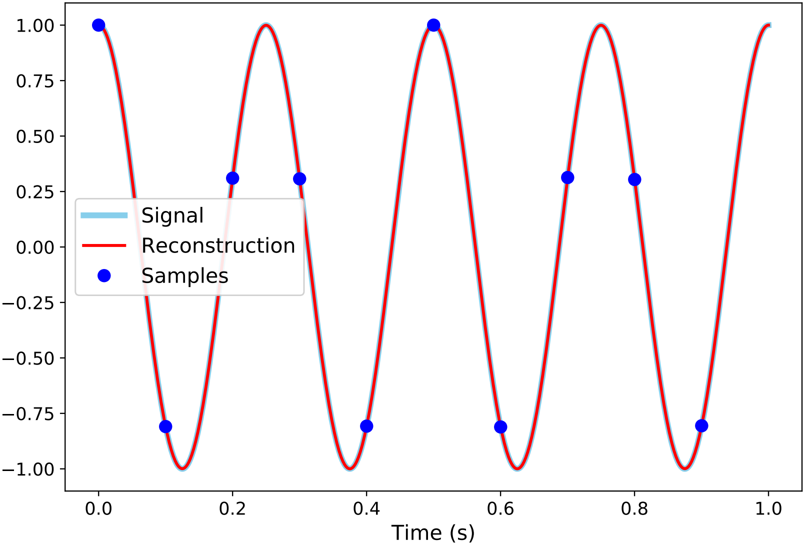
k_c? 4 Hz

Nyquist rate? 9 Hz

Recovered Fourier coefficients ($N = 10$)



Recovered signal ($N = 10$)



Sampling a real sinusoid

$$x(t) := \cos(8\pi t) = 0.5 \exp(-i 2\pi 4 t) + 0.5 \exp(i 2\pi 4 t)$$

$$N = 5 \quad (\text{as if } k_c = 2)$$

$$\begin{aligned}
\hat{x}^{\text{rec}}[k] &= \sum_{\{(m-k) \bmod 5 = 0\}} \hat{x}[m] \\
\hat{x}^{\text{rec}}[-2] &= \\
\hat{x}^{\text{rec}}[-1] &= \\
\hat{x}^{\text{rec}}[0] &= \\
\hat{x}^{\text{rec}}[1] &= \\
\hat{x}^{\text{rec}}[2] &=
\end{aligned}$$

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\hat{x}^{\text{rec}}[-1] &= \\
\hat{x}^{\text{rec}}[0] &= \\
\hat{x}^{\text{rec}}[1] &= \\
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\end{aligned}$$

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$$\hat{x}^{\text{rec}}[k] = \sum_{\{(m-k) \bmod 5 = 0\}} \hat{x}[m]$$

$$\hat{x}^{\text{rec}}[-2] = 0$$

$$\hat{x}^{\text{rec}}[-1] = \hat{x}^{\text{rec}}[4] = 0.5$$

$$\hat{x}^{\text{rec}}[0] =$$

$$\hat{x}^{\text{rec}}[1] =$$

$$\hat{x}^{\text{rec}}[2] =$$

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$$\hat{x}^{\text{rec}}[-2] = 0$$

$$\hat{x}^{\text{rec}}[-1] = \hat{x}^{\text{rec}}[4] = 0.5$$

$$\hat{x}^{\text{rec}}[0] = 0$$

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\hat{x}^{\text{rec}}[k] &= \sum_{\{(m-k) \bmod 5 = 0\}} \hat{x}[m] \\
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\hat{x}^{\text{rec}}[-1] &= \hat{x}^{\text{rec}}[4] = 0.5 \\
\hat{x}^{\text{rec}}[0] &= 0 \\
\hat{x}^{\text{rec}}[1] &= \hat{x}^{\text{rec}}[-4] = 0.5 \\
\hat{x}^{\text{rec}}[2] &=
\end{aligned}$$

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$$\hat{x}^{\text{rec}}[k] = \sum_{\{(m-k) \bmod 5 = 0\}} \hat{x}[m]$$

$$\hat{x}^{\text{rec}}[-2] = 0$$

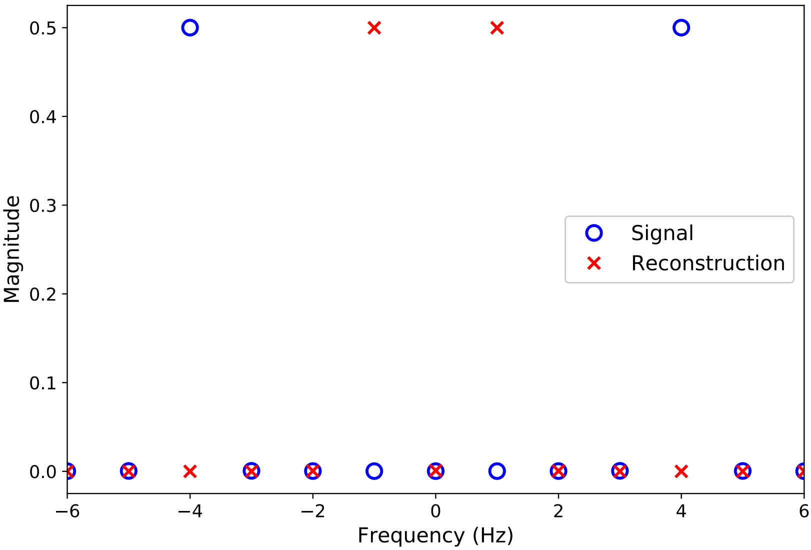
$$\hat{x}^{\text{rec}}[-1] = \hat{x}^{\text{rec}}[4] = 0.5$$

$$\hat{x}^{\text{rec}}[0] = 0$$

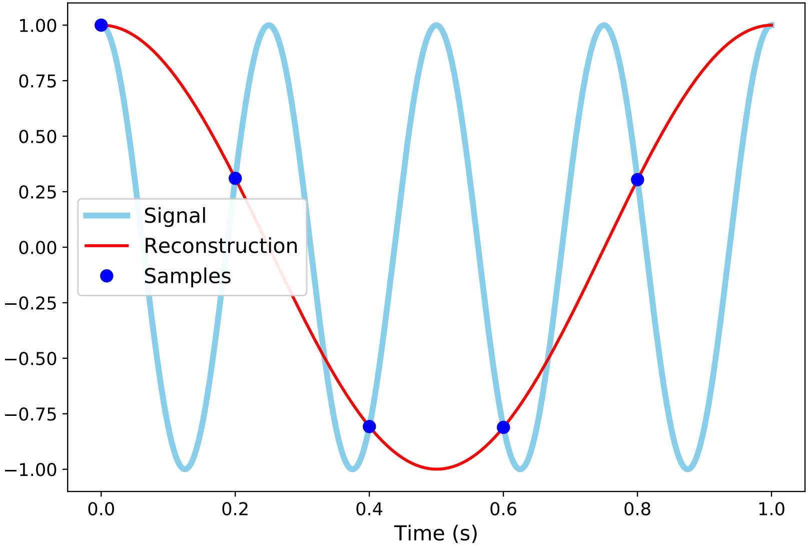
$$\hat{x}^{\text{rec}}[1] = \hat{x}^{\text{rec}}[-4] = 0.5$$

$$\hat{x}^{\text{rec}}[2] = 0$$

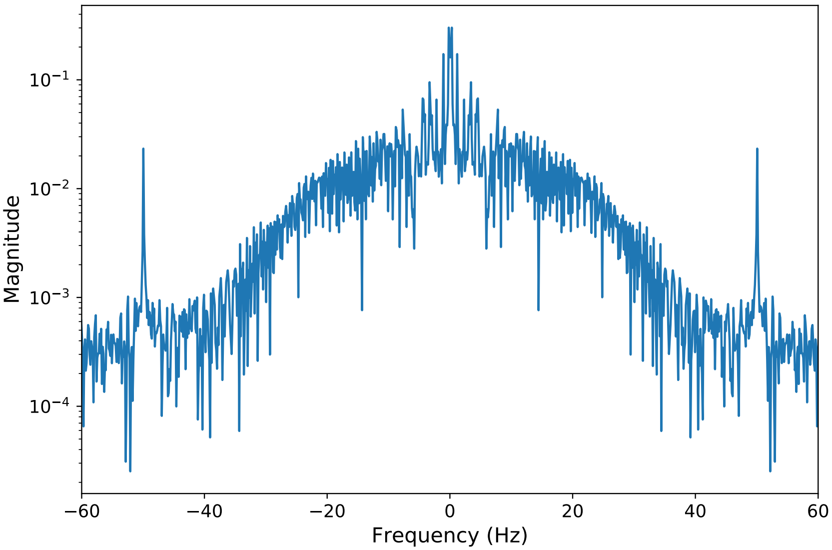
Recovered Fourier coefficients ($N = 5$)



Recovered signal ($N = 5$)



Electrocardiogram: Fourier coefficients (magnitude)



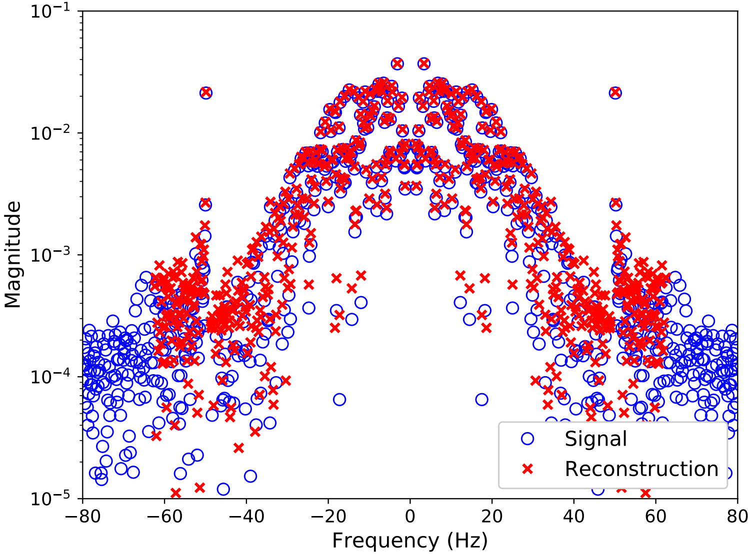
Sampling an electrocardiogram

Signal is approximately bandlimited at 50 Hz

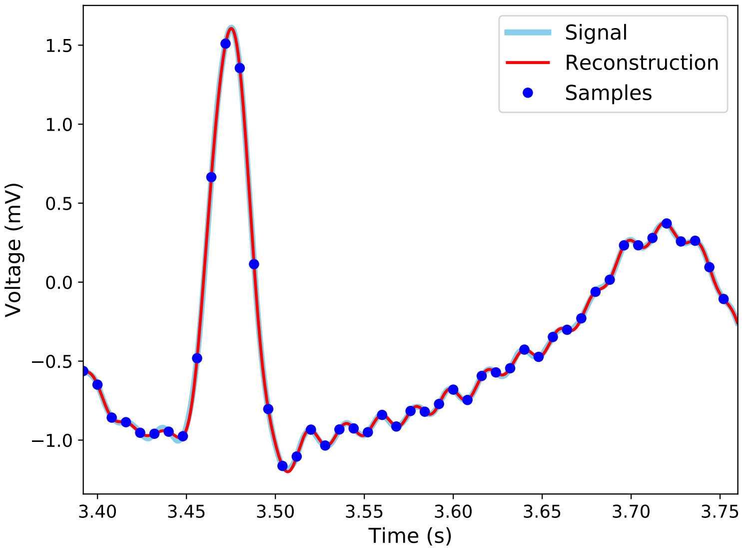
$T = 8$ s, so $k_c = 50/(1/T) = 400$

To avoid aliasing $N \geq 801$

Recovered Fourier coefficients (N=1,000)



Recovered signal (N=1,000)



Sampling an electrocardiogram

Signal is approximately bandlimited at 50 Hz

$T = 8$ s, so $k_c = 50/(1/T) = 400$

$N = 625$

$$\hat{x}^{\text{rec}}[k] = \sum_{\{(m-k) \bmod 625 = 0\}} \hat{x}[m]$$

Component at $m = \pm 400$ (50 Hz) shows up at

Sampling an electrocardiogram

Signal is approximately bandlimited at 50 Hz

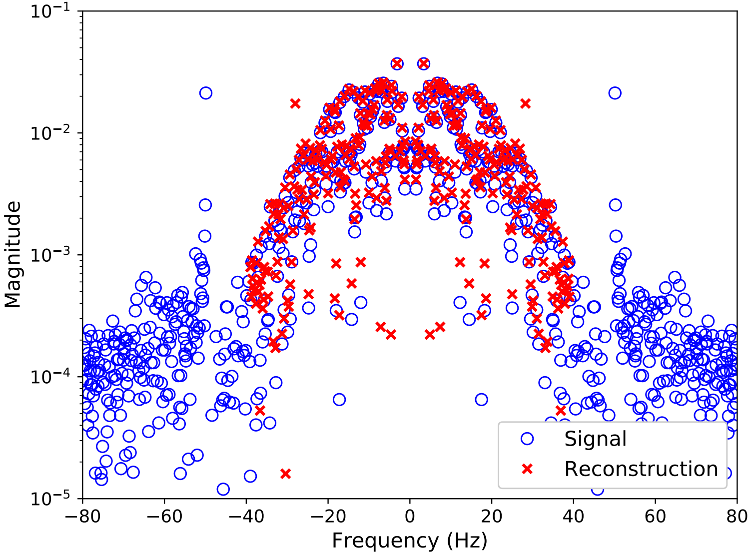
$T = 8$ s, so $k_c = 50/(1/T) = 400$

$N = 625$

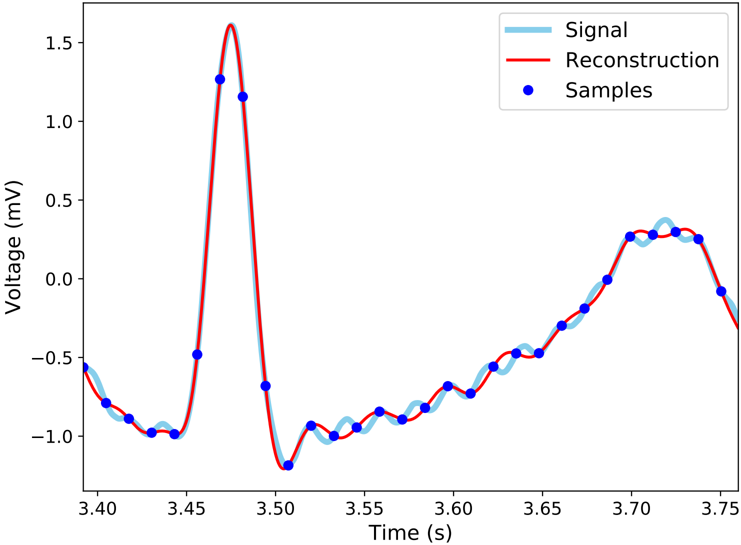
$$\hat{x}^{\text{rec}}[k] = \sum_{\{(m-k) \bmod 625 = 0\}} \hat{x}[m]$$

Component at $m = \pm 400$ (50 Hz) shows up at ± 225 (28.1 Hz)

Recovered Fourier coefficients ($N = 625$)



Recovered signal ($N = 625$)



Discrete complex sinusoids

The discrete complex sinusoid $\vec{\phi}_k \in \mathbb{C}^N$ with frequency k is

$$\vec{\phi}_k[j] := \exp\left(\frac{i2\pi kj}{N}\right), \qquad 0 \leq j, k \leq N-1$$

Discrete complex sinusoids scaled by $1/\sqrt{N}$: orthonormal basis of $\mathbb{C}^N$

Discrete Fourier transform

The discrete Fourier transform (DFT) of $\vec{x} \in \mathbb{C}^N$ is

$$\hat{x} := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \exp\left(-\frac{i2\pi}{N}\right) & \exp\left(-\frac{i2\pi 2}{N}\right) & \cdots & \exp\left(-\frac{i2\pi(N-1)}{N}\right) \\ 1 & \exp\left(-\frac{i2\pi 2}{N}\right) & \exp\left(-\frac{i2\pi 4}{N}\right) & \cdots & \exp\left(-\frac{i2\pi 2(N-1)}{N}\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \exp\left(-\frac{i2\pi(N-1)}{N}\right) & \exp\left(-\frac{i2\pi 2(N-1)}{N}\right) & \cdots & \exp\left(-\frac{i2\pi(N-1)^2}{N}\right) \end{bmatrix} \vec{x}$$

$$= F_{[N]}\vec{x}$$

$$\hat{x}[k] = \left\langle \vec{x}, \vec{\phi}_k \right\rangle, \qquad 0 \le k \le N-1$$

Inverse discrete Fourier transform

The inverse DFT of a vector $\hat{y} \in \mathbb{C}^N$ equals

$$\vec{y} = \frac{1}{N} F_{[N]}^{*} \hat{y}$$

It inverts the DFT

Interpretation in terms of bandlimited signals

If $\vec{x} \in \mathbb{C}^N$ contains samples of a bandlimited signal such that $2k_c + 1 \leq N$ the DFT contains the Fourier series coefficients of the function

$$\hat{x}_{[k_c]} = \frac{1}{N} \widetilde{F}_{[N]}^{*} \vec{x}_{[N]}$$

$$\widetilde{F}_{[N]} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \exp\left(\frac{i2\pi(-k_c)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \exp\left(\frac{i2\pi(-k_c)(N-1)}{N}\right) & \exp\left(\frac{i2\pi(-k_c+1)(N-1)}{N}\right) & \cdots & \exp\left(\frac{i2\pi k_c(N-1)}{N}\right) \end{bmatrix}$$

Rows of $\widetilde{F}_{[N]}$ equal rows of $F_{[N]}$ in a different order!

Complexity of computing the DFT

Complexity of multiplying $N \times N$ matrix with N-dim. vector is

Complexity of computing the DFT

Complexity of multiplying $N \times N$ matrix with N-dim. vector is N^2

Very slow!

We can exploit the structure of the matrix to do much better

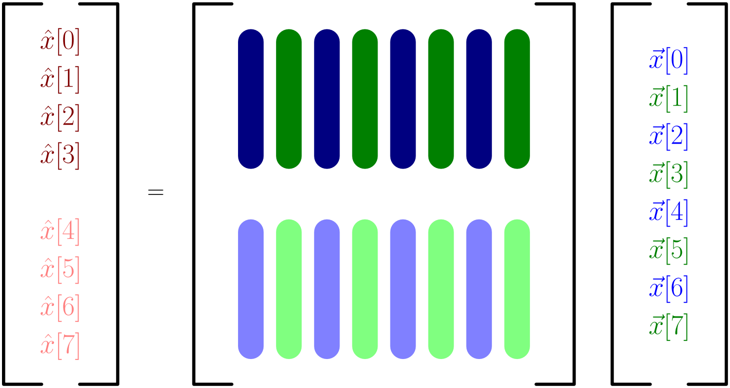
Fast Fourier transform

The most important numerical algorithm of our lifetime (G. Strang)

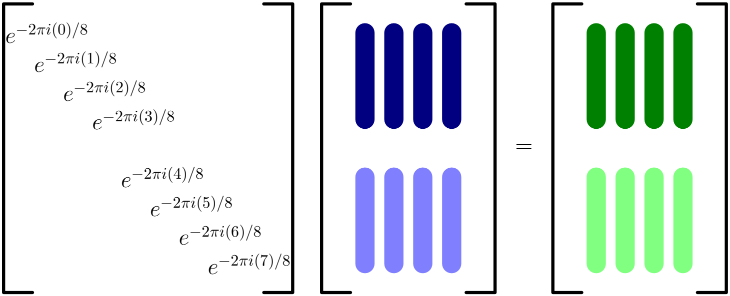
Main insight:

Action of N-order DFT matrix on vector can be decomposed into action of $N/2$-order DFT submatrices on subvectors

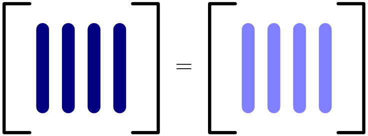
DFT


$\hat{x}[0]$
$\hat{x}[1]$
$\hat{x}[2]$
$\hat{x}[3]$
$\hat{x}[4]$
$\hat{x}[5]$
$\hat{x}[6]$
$\hat{x}[7]$
$=$
$\vec{x}[0]$
$\vec{x}[1]$
$\vec{x}[2]$
$\vec{x}[3]$
$\vec{x}[4]$
$\vec{x}[5]$
$\vec{x}[6]$
$\vec{x}[7]$

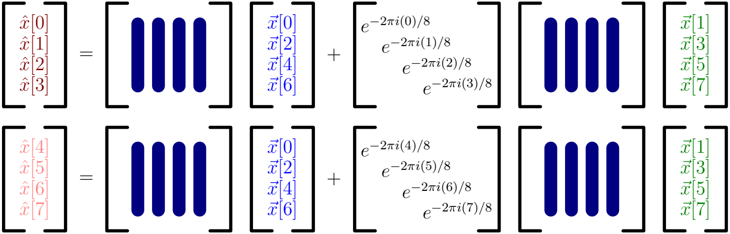
DFT


$e^{-2\pi i(0)/8}$
$e^{-2\pi i(1)/8}$
$e^{-2\pi i(2)/8}$
$e^{-2\pi i(3)/8}$
$e^{-2\pi i(4)/8}$
$e^{-2\pi i(5)/8}$
$e^{-2\pi i(6)/8}$
$e^{-2\pi i(7)/8}$
=

DFT



DFT


$\hat{x}[0]$ $\hat{x}[1]$ $\hat{x}[2]$ $\hat{x}[3]$
$=$
$\vec{x}[0]$ $\vec{x}[2]$ $\vec{x}[4]$ $\vec{x}[6]$
$+$
$e^{-2\pi i(0)/8}$ $e^{-2\pi i(1)/8}$ $e^{-2\pi i(2)/8}$ $e^{-2\pi i(3)/8}$
$\vec{x}[1]$ $\vec{x}[3]$ $\vec{x}[5]$ $\vec{x}[7]$
$\hat{x}[4]$ $\hat{x}[5]$ $\hat{x}[6]$ $\hat{x}[7]$
$=$
$\vec{x}[0]$ $\vec{x}[2]$ $\vec{x}[4]$ $\vec{x}[6]$
$+$
$e^{-2\pi i(4)/8}$ $e^{-2\pi i(5)/8}$ $e^{-2\pi i(6)/8}$ $e^{-2\pi i(7)/8}$
$\vec{x}[1]$ $\vec{x}[3]$ $\vec{x}[5]$ $\vec{x}[7]$

DFT

Let $F_{[N]}$ denote the $N \times N$ DFT matrix, where N is even.
For $k = 0, 1, \ldots, N/2 - 1$, and any vector $\vec{x} \in \mathbb{C}^N$

$$F_{[N]}\vec{x}\,[k] = F_{[N/2]}\vec{x}_{\text{even}}\,[k] + \exp\left(-\frac{i2\pi k}{N}\right) F_{[N/2]}\vec{x}_{\text{odd}}\,[k]\,,$$

$$F_{[N]}\vec{x}\,[k + N/2] = F_{[N/2]}\vec{x}_{\text{even}}\,[k] - \exp\left(-\frac{i2\pi k}{N}\right) F_{[N/2]}\vec{x}_{\text{odd}}\,[k]\,,$$

where $\vec{x}_{\text{even}}$ and $\vec{x}_{\text{odd}}$ contain the even and odd entries of $\vec{x}$ respectively.

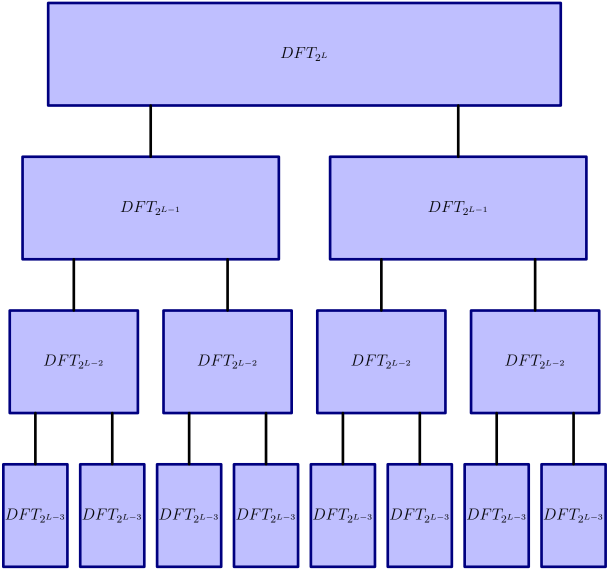
Cooley-Tukey Fast Fourier transform

1. Compute $F_{[N/2]}\vec{x}_{\text{even}}$.
2. Compute $F_{[N/2]}\vec{x}_{\text{odd}}$.
3. For $k = 0, 1, \ldots, N/2 - 1$ set

$$F_{[N]}\vec{x}\,[k] := F_{[N/2]}\vec{x}_{\text{even}}\,[k] + \exp\left(-\frac{i2\pi k}{N}\right) F_{[N/2]}\vec{x}_{\text{odd}}\,[k]\,,$$

$$F_{[N]}\vec{x}\,[k + N/2] := F_{[N/2]}\vec{x}_{\text{even}}\,[k] - \exp\left(-\frac{i2\pi k}{N}\right) F_{[N/2]}\vec{x}_{\text{odd}}\,[k]\,.$$

Complexity


DFT_{2^L}
$DFT_{2^{L-1}}$
$DFT_{2^{L-1}}$
$DFT_{2^{L-2}}$
$DFT_{2^{L-2}}$
$DFT_{2^{L-2}}$
$DFT_{2^{L-2}}$
$DFT_{2^{L-3}}$
$DFT_{2^{L-3}}$
$DFT_{2^{L-3}}$
$DFT_{2^{L-3}}$
$DFT_{2^{L-3}}$
$DFT_{2^{L-3}}$
$DFT_{2^{L-3}}$
$DFT_{2^{L-3}}$

Complexity

Assume $N = 2^L$

$L = \log_2 N$ levels

At level $l \in \{1, \ldots, L\}$ there are 2^l nodes

At each node, scale a vector of dim 2^{L-l} and add to another vector

Complexity at each node:

Complexity at each level:

Complexity

Assume $N = 2^L$

$L = \log_2 N$ levels

At level $l \in \{1, \ldots, L\}$ there are 2^l nodes

At each node, scale a vector of dim 2^{L-l} and add to another vector

Complexity at each node: 2^{L-l}

Complexity at each level:

Complexity

Assume $N = 2^L$

$L = \log_2 N$ levels

At level $l \in \{1, \dots, L\}$ there are 2^l nodes

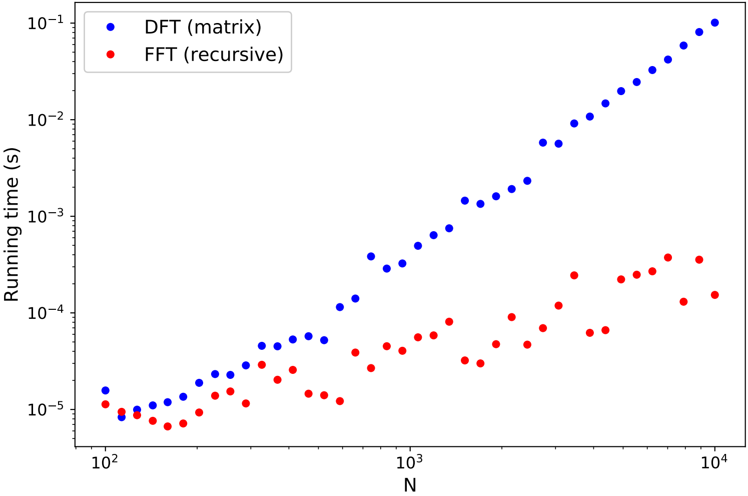
At each node, scale a vector of dim 2^{L-l} and add to another vector

Complexity at each node: 2^{L-l}

Complexity at each level: $2^{L-l}2^l = 2^L = N$

Complexity is $O(N \log N)$!

In practice



DFT (matrix)
FFT (recursive)
Running time (s)
N
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{2}
10^{3}
10^{4}

Multidimensional signals

Square-integrable functions defined on a hyperrectangle
$\mathcal{I} := [a_1, b_1] \times \ldots \times [a_p, b_p] \subset \mathbb{R}^p$

Inner product:

$$\langle x, y \rangle := \int_{\mathcal{I}} x\left(\vec{t}\right) \overline{y\left(\vec{t}\right)} \, \mathrm{d}\vec{t}.$$

Goal: Extension of frequency representations to multidimensional signals

Multidimensional sinusoid

$$a \cos\left(2\pi\langle \vec{f}, \vec{t}\,\rangle + \theta\right).$$

The frequency and time indices are now d-dimensional

Periodic with period $1/\left|\left|f\right|\right|_2$ in direction of $\vec{f}$

For any integer m

$$a \cos\left(i2\pi\left\langle \vec{f}, \vec{t} + \frac{m}{\left|\left|f\right|\right|_2}\frac{\vec{f}}{\left|\left|f\right|\right|_2}\right\rangle + \theta\right)$$

Multidimensional sinusoid

$$a \cos \left(2\pi \langle \vec{f}, \vec{t} \rangle + \theta \right).$$

The frequency and time indices are now d-dimensional

Periodic with period $1 / \left|\left|f\right|\right|_2$ in direction of $\vec{f}$

For any integer m

$$a \cos \left(i2\pi \left\langle \vec{f}, \vec{t} + \frac{m}{\left|\left|f\right|\right|_2} \frac{\vec{f}}{\left|\left|f\right|\right|_2} \right\rangle + \theta \right) = a \cos \left(i2\pi \langle \vec{f}, \vec{t} \rangle + i2\pi m + \theta \right)$$

Multidimensional sinusoid

$$a \cos\left(2\pi \langle \vec{f}, \vec{t}\,\rangle + \theta\right).$$

The frequency and time indices are now d-dimensional

Periodic with period $1/\left|\left|f\right|\right|_2$ in direction of $\vec{f}$

For any integer m

$$\begin{aligned} a \cos\left(i2\pi \left\langle \vec{f}, \vec{t} + \frac{m}{\left|\left|f\right|\right|_2} \frac{\vec{f}}{\left|\left|f\right|\right|_2} \right\rangle + \theta\right) &= a \cos\left(i2\pi \langle \vec{f}, \vec{t}\rangle + i2\pi m + \theta\right) \\ &= a \cos\left(i2\pi \langle \vec{f}, \vec{t}\,\rangle + \theta\right) \end{aligned}$$

Multidimensional complex sinusoids

Complex sinusoid with frequency $\vec{f} \in R^d$:

$$\exp(i2\pi\langle\vec{f}, \vec{t}\,\rangle) := \cos(2\pi\langle\vec{f}, \vec{t}\,\rangle) + i\sin(2\pi\langle\vec{f}, \vec{t}\,\rangle).$$

Multidimensional complex sinusoids

Complex sinusoid with frequency $\vec{f} \in R^d$:

$$\exp(i2\pi\langle \vec{f}, \vec{t}\,\rangle) := \cos(2\pi\langle \vec{f}, \vec{t}\,\rangle) + i\sin(2\pi\langle \vec{f}, \vec{t}\,\rangle).$$

$$\cos\left(i2\pi\langle \vec{f}, \vec{t}\,\rangle + \theta\right) = \frac{\exp(i\theta)}{2}\exp(i2\pi\langle \vec{f}, \vec{t}\,\rangle) + \frac{\exp(-i\theta)}{2}\exp(-i2\pi\langle \vec{f}, \vec{t}\,\rangle)$$

Multidimensional complex sinusoids

Can be expressed as product of 1D complex sinusoids

$$\exp(i2\pi\langle \vec{f}, \vec{t}\,\rangle) := \exp\left(i2\pi \sum_{j=1}^{d} \vec{f}[j]\vec{t}[j]\right)$$

$$= \prod_{j=1}^{d} \exp(i2\pi \vec{f}\,[j]\vec{t}\,[j])$$

From now on $d = 2$: $\vec{t}\,[1] = t_1$, $\vec{t}\,[2] = t_2$

Orthogonality of multidimensional complex sinusoids

The family of complex sinusoids with integer frequencies

$$\phi^{2D}_{k_1,k_2}(t_1,t_2) := \exp\left(\frac{i2\pi k_1 t_1}{T}\right)\exp\left(\frac{i2\pi k_2 t_2}{T}\right), \qquad k_1, k_2 \in \mathbb{Z},$$

is an orthogonal set of functions on any interval of the form $[a, a+T] \times [b, b+T]$, $a, b, T \in \mathbb{R}$ and $T > 0$

Proof

We have

$$\phi^{2D}_{k_1,k_2}(t_1, t_2) = \phi_{k_1}(t_1)\,\phi_{k_2}(t_2)\,,$$

so that

$$\left\langle \phi^{2D}_{k_1,k_2}, \phi^{2D}_{j_1,j_2} \right\rangle = \int_{t_1=a}^{a+T} \int_{t_2=b}^{b+T} \phi_{k_1}(t_1)\,\phi_{k_2}(t_2)\,\overline{\phi_{j_1}(t_1)\,\phi_{j_2}(t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2$$

Proof

We have

$$\phi^{\text{2D}}_{k_1,k_2}\left(t_1,t_2\right)=\phi_{k_1}\left(t_1\right)\phi_{k_2}\left(t_2\right),$$

so that

$$\begin{aligned}\left\langle \phi^{\text{2D}}_{k_1,k_2},\phi^{\text{2D}}_{j_1,j_2}\right\rangle &= \int_{t_1=a}^{a+T}\int_{t_2=b}^{b+T}\phi_{k_1}\left(t_1\right)\phi_{k_2}\left(t_2\right)\overline{\phi_{j_1}\left(t_1\right)\phi_{j_2}\left(t_2\right)}\,\mathrm{d}t_1\,\mathrm{d}t_2\\ &= \left\langle \phi_{k_1},\phi_{j_1}\right\rangle\left\langle \phi_{k_2},\phi_{j_2}\right\rangle\end{aligned}$$

Proof

We have

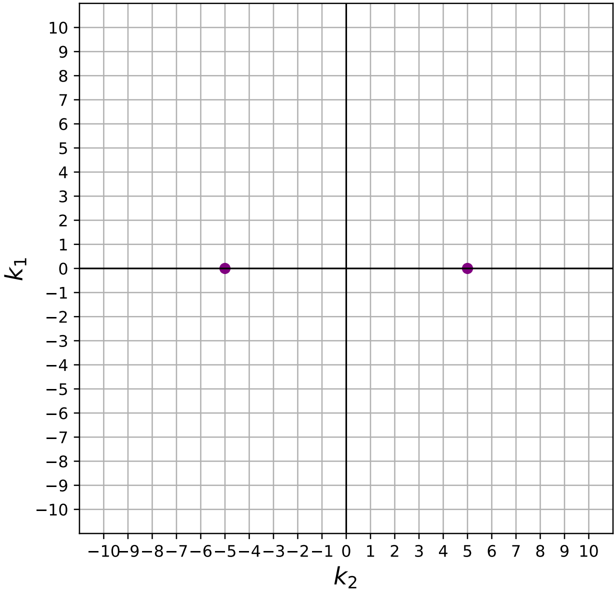
$$\phi^{2D}_{k_1,k_2}(t_1,t_2) = \phi_{k_1}(t_1)\,\phi_{k_2}(t_2)\,,$$

so that

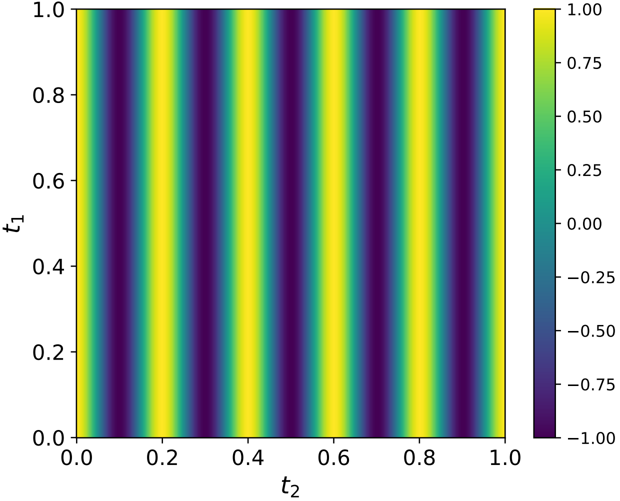
$$\begin{aligned}\left\langle \phi^{2D}_{k_1,k_2}, \phi^{2D}_{j_1,j_2} \right\rangle &= \int_{t_1=a}^{a+T} \int_{t_2=b}^{b+T} \phi_{k_1}(t_1)\,\phi_{k_2}(t_2)\,\overline{\phi_{j_1}(t_1)\,\phi_{j_2}(t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &= \langle \phi_{k_1}, \phi_{j_1} \rangle \langle \phi_{k_2}, \phi_{j_2} \rangle \\ &= 0\end{aligned}$$

as long as $j_1 \neq k_1$ or $j_2 \neq k_2$

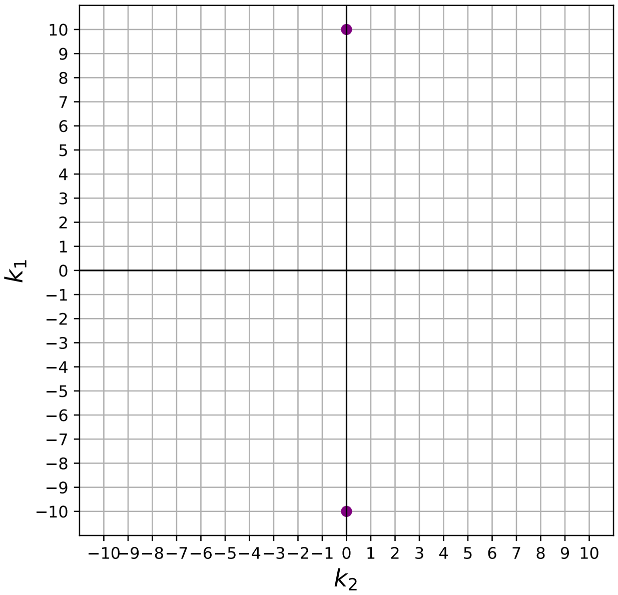
$\phi^{\mathrm{2D}}_{0,5} + \phi^{\mathrm{2D}}_{0,-5}$



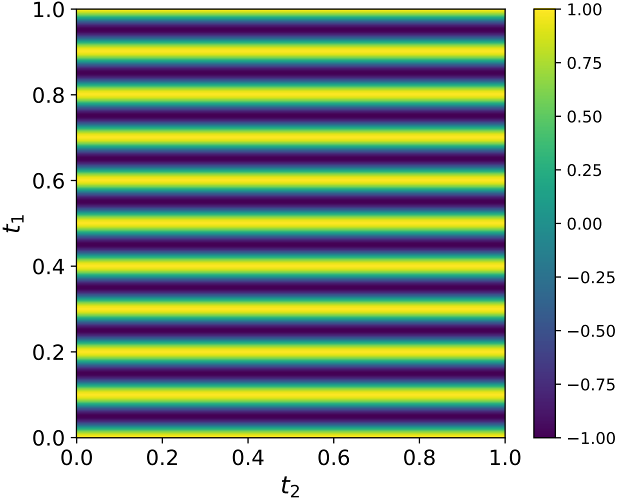
$\phi_{0,5}^{2D} + \phi_{0,-5}^{2D}$



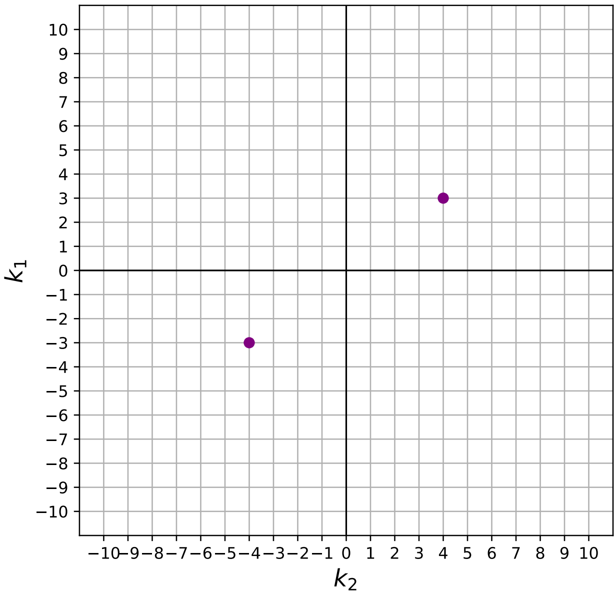
$\phi^{2D}_{10,0} + \phi^{2D}_{-10,0}$



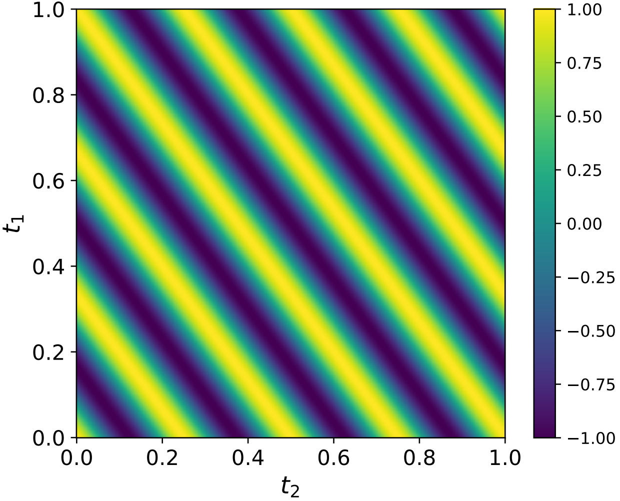
$\phi^{\mathrm{2D}}_{10,0} + \phi^{\mathrm{2D}}_{-10,0}$



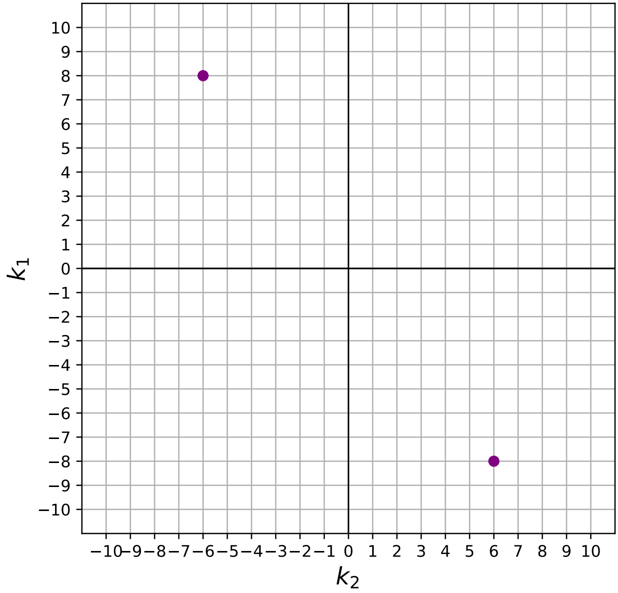
$\phi^{2\text{D}}_{3,4} + \phi^{2\text{D}}_{-3,-4}$



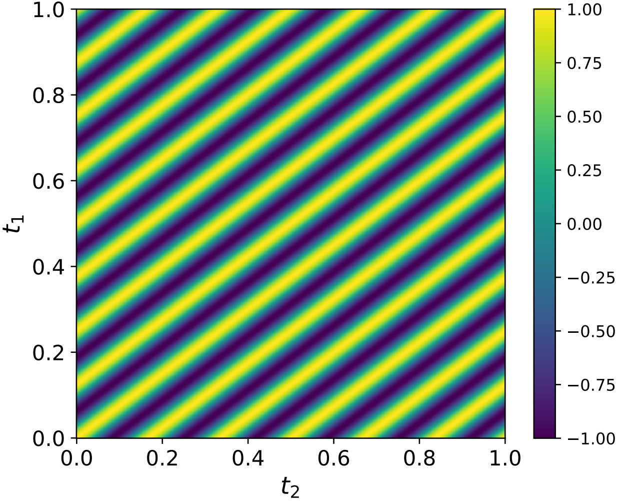
$\phi^{\mathrm{2D}}_{3,4} + \phi^{\mathrm{2D}}_{-3,-4}$



$\phi^{2\mathrm{D}}_{8,-6} + \phi^{2\mathrm{D}}_{-8,6}$



$\phi^{\mathrm{2D}}_{8,-6} + \phi^{\mathrm{2D}}_{-8,6}$



2D Fourier series

The Fourier series coefficients of a function $x \in \mathcal{L}_2\left[a, a+T\right]$ for any $a, T \in \mathbb{R},\ T > 0$, are given by

$$\begin{aligned}\hat{x}[k_1, k_2] &:= \left\langle x, \phi^{\mathrm{2D}}_{k_1,k_2} \right\rangle \\ &= \int_{t_1=a}^{a+T} \int_{t_2=b}^{b+T} x(t_1, t_2) \exp\left(-\frac{i2\pi k_1 t_1}{T}\right) \exp\left(-\frac{i2\pi k_2 t_2}{T}\right) \mathrm{d}t_1 \, \mathrm{d}t_2\end{aligned}$$

The Fourier series of order $k_{c,1},\ k_{c,2}$ is defined as

$$\mathcal{F}_{k_{c,1},k_{c,2}}\left\{x\right\} := \frac{1}{T} \sum_{k_1=-k_{c,1}}^{k_{c,1}} \sum_{k_2=-k_{c,2}}^{k_{c,2}} \hat{x}[k_1, k_2] \phi^{\mathrm{2D}}_{k_1,k_2}.$$

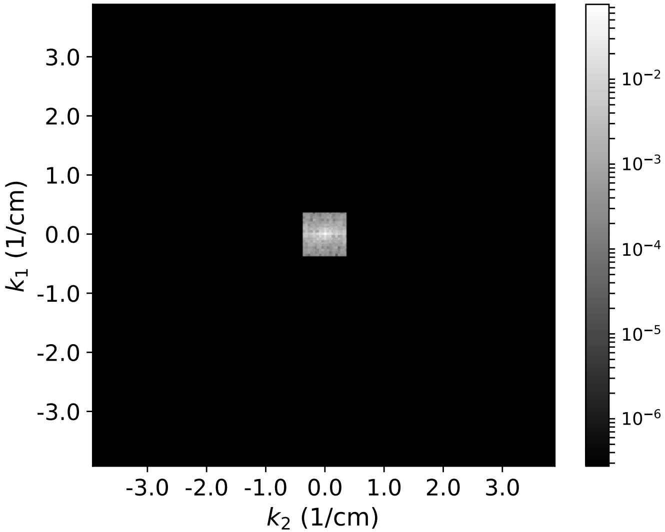
Magnetic resonance imaging

Non-invasive medical-imaging technique

Measures response of atomic nuclei in biological tissues to high-frequency radio waves when placed in a strong magnetic field

Radio waves adjusted so that each measurement equals 2D Fourier coefficients of proton density of hydrogen atoms in a region of interest

Data



3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
k_1 (1/cm)
-3.0
-2.0
-1.0
0.0
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k_2 (1/cm)
10^{-2}
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10^{-5}
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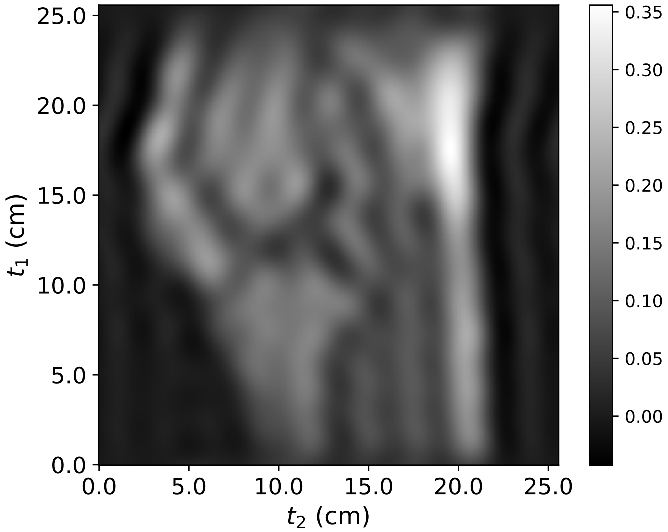
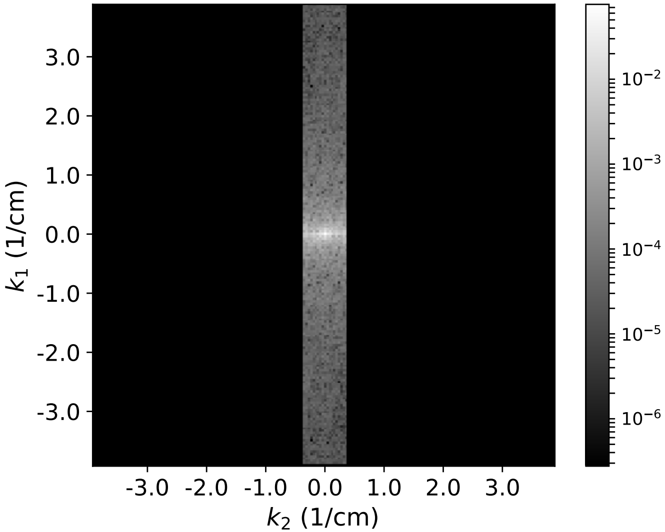
Recovered image



Data



3.0
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k_1 (1/cm)
-3.0 -2.0 -1.0 0.0 1.0 2.0 3.0
k_2 (1/cm)
10^{-2}
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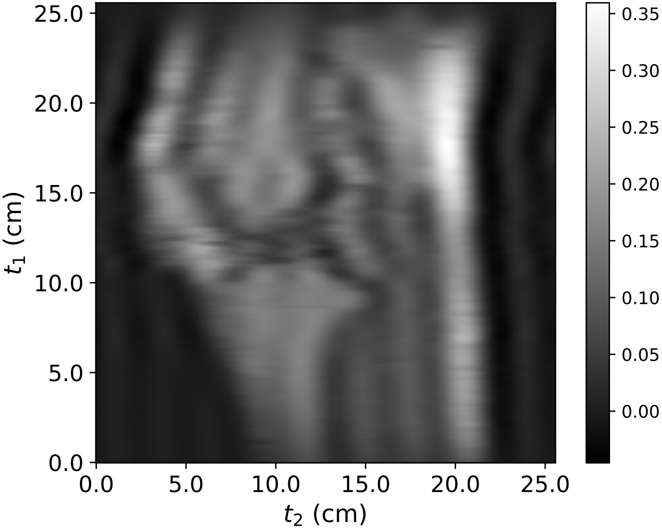
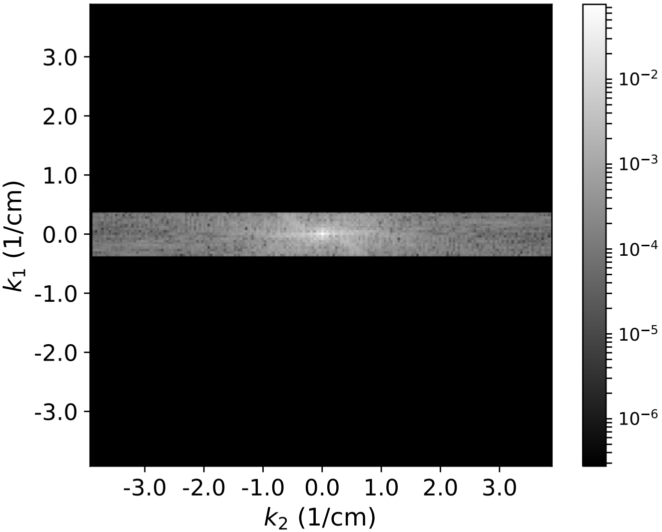
Recovered image



Data



3.0
2.0
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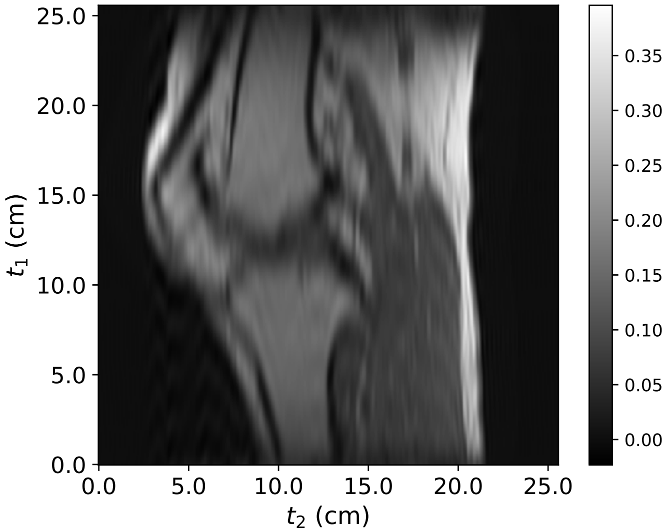
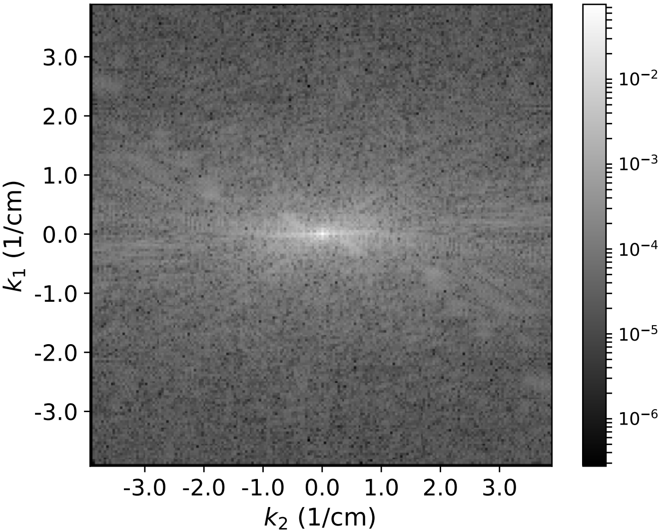
-3.0
k_1 (1/cm)
-3.0 -2.0 -1.0 0.0 1.0 2.0 3.0
k_2 (1/cm)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}

Recovered image

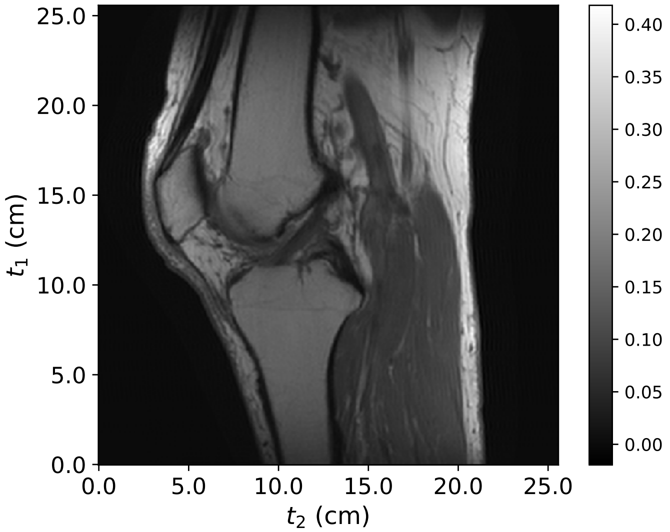


Data



3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
k_1 (1/cm)
-3.0 -2.0 -1.0 0.0 1.0 2.0 3.0
k_2 (1/cm)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}

Recovered image



Bandlimited signal

A signal defined on the 2D rectangle $[a, a+T] \times [b, b+T]$, where $a, b, T \in \mathbb{R}$ and $T > 0$ is bandlimited with a cut-off frequency k_c if it is equal to its Fourier series representation of order k_c, i.e.

$$x(t_1, t_2) = \sum_{k_1=-k_c}^{k_c} \sum_{k_2=-k_c}^{k_c} \hat{x}[k_1, k_2] \exp\left(\frac{i2\pi k_1 t_1}{T}\right) \exp\left(\frac{i2\pi k_2 t_2}{T}\right)$$

Equispaced grid

$$X_{[N]} := \begin{bmatrix} x\left(\frac{0}{N}, \frac{0}{N}\right) & x\left(\frac{0}{N}, \frac{T}{N}\right) & \cdots & x\left(\frac{0}{N}, T - \frac{T}{N}\right) \\ x\left(\frac{T}{N}, \frac{0}{N}\right) & x\left(\frac{T}{N}, \frac{T}{N}\right) & \cdots & x\left(\frac{T}{N}, T - \frac{T}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ x\left(T - \frac{T}{N}, \frac{0}{N}\right) & x\left(T - \frac{T}{N}, \frac{T}{N}\right) & \cdots & x\left(T - \frac{T}{N}, T - \frac{T}{N}\right) \end{bmatrix}.$$

Nyquist-Shannon-Kotelnikov sampling theorem

Any bandlimited signal $x \in \mathcal{L}_2[0, T)^2$, where $T > 0$, with cut-off frequency k_c can be recovered from N^2 uniformly spaced samples if

$$N \geq 2k_c + 1,$$

where $2k_c + 1$ is known as the *Nyquist rate*

Nyquist-Shannon-Kotelnikov sampling theorem

We have

$$X_{[N]} = \widetilde{F}_{[N]} \widehat{X}_{[k_c]} \widetilde{F}_{[N]}^T,$$

$$\widehat{X}_{[k_c]} := \begin{bmatrix} \hat{x}_{-k_c,-k_c} & \hat{x}_{-k_c,-k_c+1} & \cdots & \hat{x}_{-k_c,k_c} \\ \hat{x}_{-k_c+1,-k_c} & \hat{x}_{-k_c+1,-k_c+1} & \cdots & \hat{x}_{-k_c+1,k_c} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{x}_{k_c,-k_c} & \hat{x}_{k_c,-k_c+1} & \cdots & \hat{x}_{k_c,k_c} \end{bmatrix}$$

So that

$$\widehat{X}_{[k_c]} = \frac{1}{N^2} \widetilde{F}_{[N]}^* X_{[N]} \left(\widetilde{F}_{[N]}^* \right)^T$$

2D discrete signals

We represent 2D signals as matrices belonging to the vector space of $\mathbb{C}^{N\times N}$ matrices endowed with the standard inner product

$$\langle A, B\rangle := \operatorname{tr}\left(A^* B\right), \quad A, B \in \mathbb{C}^{N\times N}.$$

Equivalent to dot product between vectorized matrices

Discrete complex sinusoids

The discrete complex sinusoid $\Phi_{k_1,k_2} \in \mathbb{C}^{N \times N}$ with integer frequencies k_1 and k_2 is defined as

$$\Phi_{k_1,k_2}[j_1, j_2] := \exp\left(\frac{i2\pi k_1 j_1}{N}\right)\exp\left(\frac{i2\pi k_2 j_2}{N}\right), \qquad 0 \leq j_1, j_2 \leq N-1,$$

Equivalently

$$\Phi_{k_1,k_2} = \vec{\phi}_{k_1}\vec{\phi}_{k_2}^{\,T}.$$

The discrete complex exponentials $\frac{1}{N}\Phi_{k_1,k_2}$, $0 \leq k_1, k_2 \leq N-1$, form an orthonormal basis of $\mathbb{C}^{N \times N}$

Proof

$$
\begin{aligned}
\langle \Phi_{k_1,k_2}, \Phi_{l_1,l_2} \rangle &= \operatorname{tr}\left((\Phi_{l_1,l_2})^* \, \Phi_{k_1,k_2} \right) \\
&= (\vec{\phi}_{k_1})^* \vec{\phi}_{l_1} (\vec{\phi}_{k_2})^* \vec{\phi}_{l_2}
\end{aligned}
$$

2D discrete Fourier transform

The discrete Fourier transform (DFT) of a 2D array $X \in \mathbb{C}^{N \times N}$ is

$$\widehat{X}[k_1, k_2] := \langle X, \Phi_{k_1,k_2} \rangle, \qquad 0 \le k_1, k_2 \le N-1,$$

or equivalently

$$\widehat{X} := F_{[N]} X F_{[N]},$$

where $F_{[N]}$ is the 1D DFT matrix

Inverse 2D discrete Fourier transform

The inverse DFT of a 2D array $\widehat{Y} \in \mathbb{C}^{N \times N}$ equals

$$Y = \frac{1}{N^2} F_{[N]}^* \widehat{Y} F_{[N]}^*$$

It inverts the 2D DFT

2D discrete Fourier transform

Can be interpreted as Fourier series of samples (as in 1D)

Complexity $O(N^2 \log N)$ instead of $O(N^3)$ (FFT)

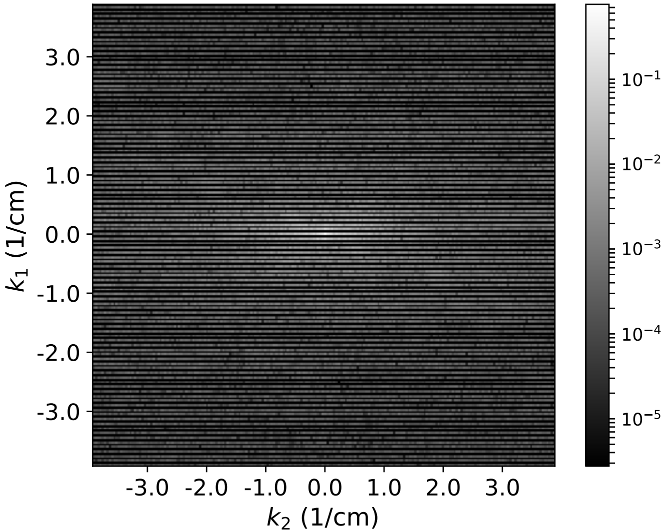
Undersampling in the frequency domain

Important goal in MRI: reduce scan time

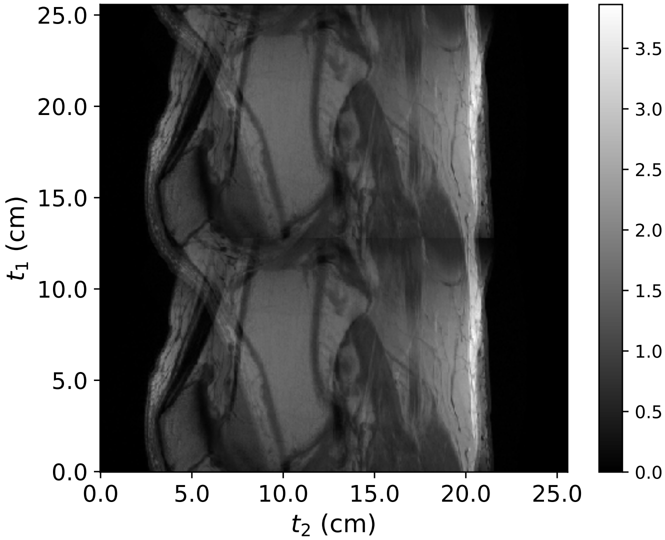
Can be achieved by measuring less frequency coefficients

What happens if we undersample in the Fourier domain?

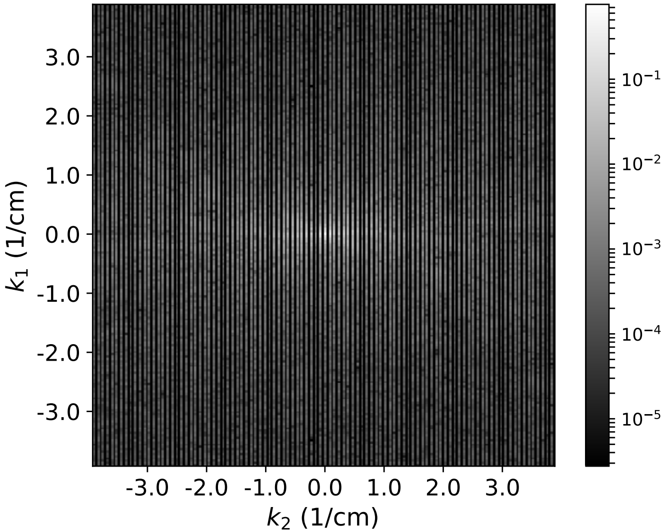
x2 undersampling (k_1)



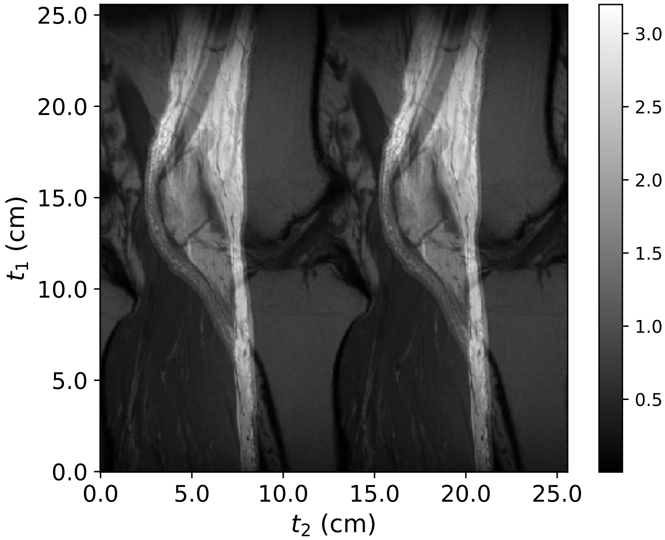
Aliasing in image domain



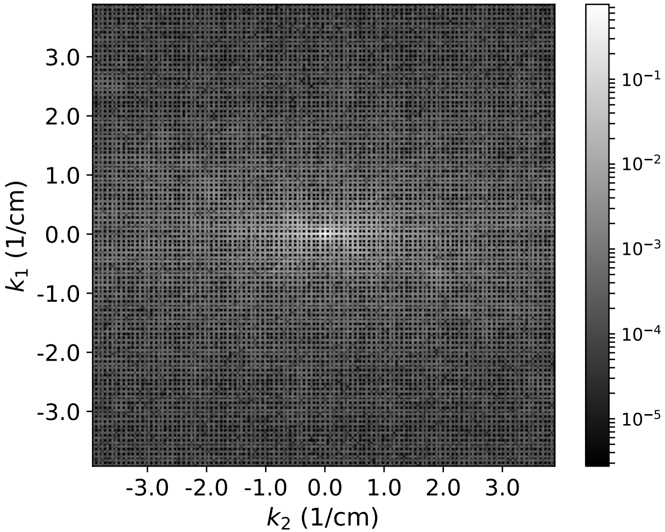
x2 undersampling (k_2)



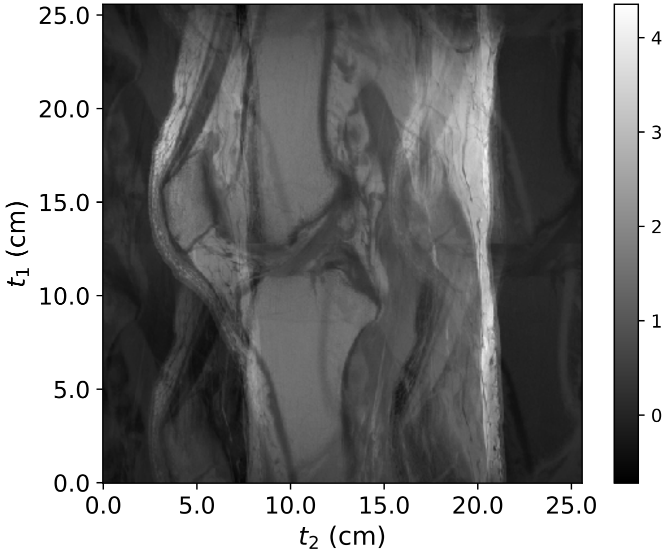
Aliasing in image domain



x2 undersampling (k_1 and k_2)



Aliasing in image domain



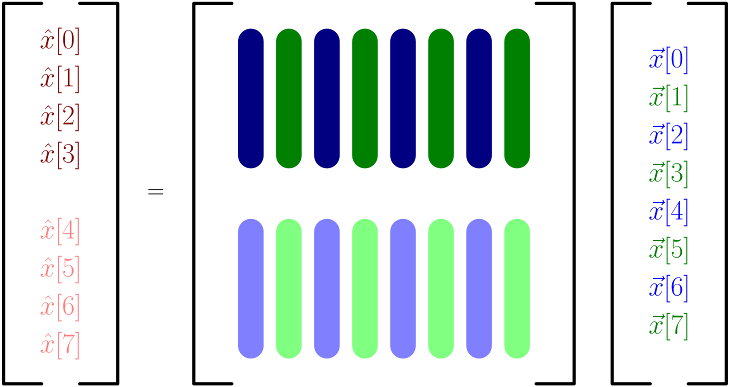
Aliasing in image domain

Separate the image into its left $N/2$ columns and its right $N/2$ columns

$$X := \begin{bmatrix} X_{\text{left}} & X_{\text{right}} \end{bmatrix}$$

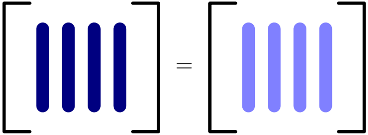
x2 undersampling in k_2

DFT matrix



$\hat{x}[0]$
$\hat{x}[1]$
$\hat{x}[2]$
$\hat{x}[3]$
$\hat{x}[4]$
$\hat{x}[5]$
$\hat{x}[6]$
$\hat{x}[7]$
$=$
$\vec{x}[0]$
$\vec{x}[1]$
$\vec{x}[2]$
$\vec{x}[3]$
$\vec{x}[4]$
$\vec{x}[5]$
$\vec{x}[6]$
$\vec{x}[7]$

DFT matrix



DFT matrix

$$(F_{[N]})_{\text{even}} := \begin{bmatrix} F_{[N/2]} \\ F_{[N/2]} \end{bmatrix}.$$

Undersampling x2 in k_2

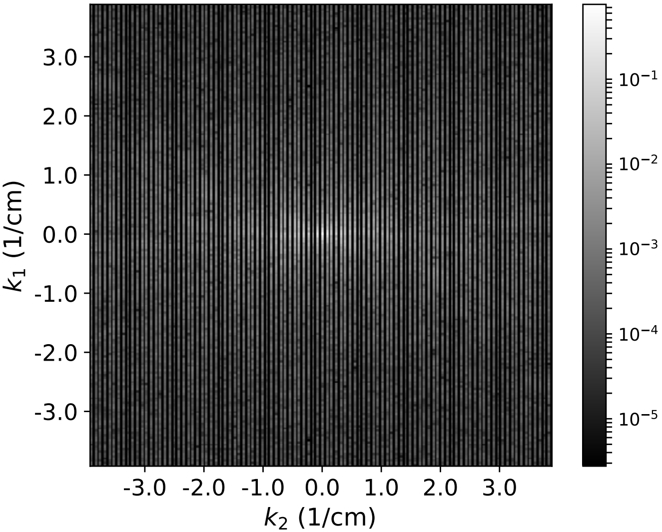
$$
\begin{aligned}
\widehat{X}_{even} &= F_{[N]} X (F_{[N]})_{\text{even}} \\
&= F_{[N]} \begin{bmatrix} X_{\text{left}} & X_{\text{right}} \end{bmatrix} \begin{bmatrix} F_{[N/2]} \\ F_{[N/2]} \end{bmatrix}
\end{aligned}
$$

Undersampling x2 in k_2

$$\begin{aligned}
\widehat{X}_{even} &= F_{[N]} X (F_{[N]})_{\text{even}} \\
&= F_{[N]} \begin{bmatrix} X_{\text{left}} & X_{\text{right}} \end{bmatrix} \begin{bmatrix} F_{[N/2]} \\ F_{[N/2]} \end{bmatrix} \\
&= F_{[N]} \left(X_{\text{left}} + X_{\text{right}} \right) F_{[N/2]}
\end{aligned}$$

Left and right halves are scrambled!

$\hat{Y}$


k_1 (1/cm)
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
k_2 (1/cm)
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}

Recovered image

$$
\begin{aligned}
Y &= \frac{1}{N^2} F_{[N]}^* \widehat{Y} F_{[N]}^* \\
&= \frac{1}{N^2} F_{[N]}^* \widehat{X}_{\mathrm{even}} (F_{[N]})_{\mathrm{even}}^*
\end{aligned}
$$

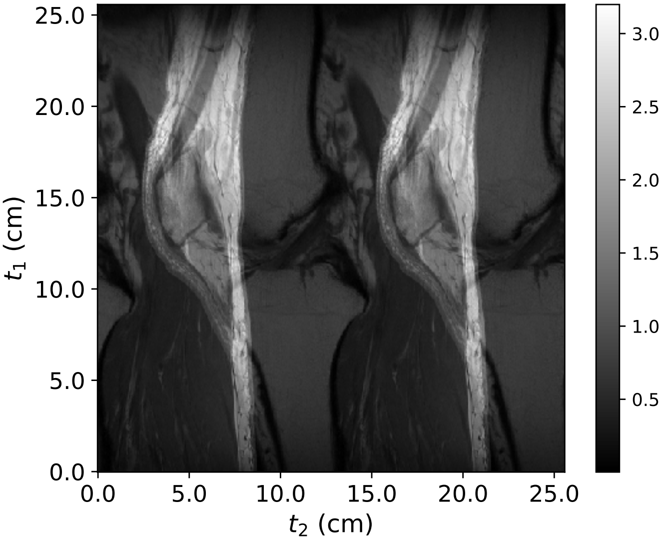
Recovered image

$$
\begin{aligned}
Y &= \frac{1}{N^2} F^*_{[N]} \widehat{Y} F^*_{[N]} \\
&= \frac{1}{N^2} F^*_{[N]} \widehat{X}_{\text{even}} (F_{[N]})^*_{\text{even}} \\
&= \frac{1}{N} F^*_{[N]} F_{[N]} \left(X_{\text{left}} + X_{\text{right}} \right) \frac{1}{N} F_{[N/2]} \begin{bmatrix} F^*_{[N/2]} & F^*_{[N/2]} \end{bmatrix}
\end{aligned}
$$

Recovered image

$$
\begin{aligned}
Y &= \frac{1}{N^2} F_{[N]}^{*} \widehat{Y} F_{[N]}^{*} \\
&= \frac{1}{N^2} F_{[N]}^{*} \widehat{X}_{\text{even}} (F_{[N]})_{\text{even}}^{*} \\
&= \frac{1}{N} F_{[N]}^{*} F_{[N]} \left(X_{\text{left}} + X_{\text{right}} \right) \frac{1}{N} F_{[N/2]} \begin{bmatrix} F_{[N/2]}^{*} & F_{[N/2]}^{*} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} X_{\text{left}} + X_{\text{right}} & X_{\text{left}} + X_{\text{right}} \end{bmatrix}
\end{aligned}
$$

Aliasing in image domain



Linear estimation

Goal: Estimate signal $\vec{x} \in \mathbb{R}^N$ from noisy data $\vec{y} \in \mathbb{R}^N$

Regression problem

Idea: Learn linear function from $\mathbb{R}^N$ to $\mathbb{R}^N$ using least squares

For images $N \approx 10^4$, for audio $N \approx 10^5$ (sampling rate ≥ 40 kHz)

Translation invariance

If a set is translation invariant, shifts of signals in the set also belong to it

Examples: Audio, images and videos

Circular translation

We denote by $\vec{x}^{\downarrow s}$ the sth circular translation of a vector $\vec{x} \in \mathbb{C}^N$

For all $1 \leq j \leq N$,

$$\vec{x}^{\downarrow s}[j] = \begin{cases} \vec{x}\,[j - s] & \text{if } 1 \leq j - s \leq N, \\ \vec{x}\,[N + j - s] & \text{if } j - s < 1 \end{cases}$$

Circular translation in 2D

For an $N \times N$ signal $X \in \mathbb{C}^{N \times N}$, circular translation by $(s_1, s_2) \in$ is denoted by $X^{\downarrow(s_1,s_2)}$

For all $1 \leq j_1, j_2 \leq N$,

$$X^{\downarrow(s_1,s_2)}[j_1, j_2] = \begin{cases} X[j_1 - s_1, j_2 - s_2] & \text{if } 1 \leq j_1 - s_1, j_2 - s_2 \leq N, \\ X[N + j_1 - s_1, j_2 - s_2] & \text{if } j_1 - s_1 < 1, \\ X[j_1 - s_1, N + j_2 - s_2] & \text{if } j_2 - s_2 < 1, \\ X[N + j_1 - s_1, N + j_2 - s_2] & \text{if } j_1 - s_1 < 1,\ j_2 - s_2 < 1. \end{cases}$$

Effect of shift on sinusoids

Shifting a sinusoid modifies its phase

$$\vec{\phi}_k^{\downarrow s}[l] = \exp\left(\frac{i2\pi k(l-s)}{N}\right)$$

Effect of shift on sinusoids

Shifting a sinusoid modifies its phase

$$\vec{\phi}_k^{\downarrow s}[l] = \exp\left(\frac{i2\pi k(l-s)}{N}\right)$$
$$= \exp\left(-\frac{i2\pi ks}{N}\right)\vec{\phi}_k[l]$$

Effect of shift on sinusoids

Shifting a sinusoid modifies its phase

$$\Phi_{k_1,k_2}^{\downarrow(s_1,s_2)} = \vec{\phi}_{k_1}^{\downarrow s_1} (\vec{\phi}_{k_2}^{\downarrow s_2})^T$$

Effect of shift on sinusoids

Shifting a sinusoid modifies its phase

$$\begin{aligned}\Phi_{k_1,k_2}^{\downarrow(s_1,s_2)} &= \vec{\phi}_{k_1}^{\downarrow s_1}(\vec{\phi}_{k_2}^{\downarrow s_2})^T \\ &= \exp\left(-\frac{i2\pi k_1 s_1}{N}\right)\exp\left(-\frac{i2\pi k_2 s_2}{N}\right)\Phi_{k_1,k_2}\end{aligned}$$

Effect of translation in Fourier domain

Let $\vec{x} \in \mathbb{C}^N$ with DFT $\hat{x}$ and $y := x^{\downarrow s}$

$$\hat{y}\,[k] := \langle \vec{x}^{\downarrow s}, \vec{\phi}_k \rangle$$

Effect of translation in Fourier domain

Let $\vec{x} \in \mathbb{C}^N$ with DFT $\hat{x}$ and $y := x^{\downarrow s}$

$$\begin{aligned}\hat{y}[k] &:= \langle \vec{x}^{\downarrow s}, \vec{\phi}_k \rangle \\ &= \langle \vec{x}, \vec{\phi}_k^{\downarrow -s} \rangle\end{aligned}$$

Effect of translation in Fourier domain

Let $\vec{x} \in \mathbb{C}^N$ with DFT $\hat{x}$ and $y := x^{\downarrow s}$

$$\begin{aligned}
\hat{y}[k] &:= \langle \vec{x}^{\downarrow s}, \vec{\phi}_k \rangle \\
&= \langle \vec{x}, \vec{\phi}_k^{\downarrow -s} \rangle \\
&= \left\langle \vec{x}, \exp\left(\frac{i2\pi ks}{N} \right) \vec{\phi}_k \right\rangle
\end{aligned}$$

Effect of translation in Fourier domain

Let $\vec{x} \in \mathbb{C}^N$ with DFT $\hat{x}$ and $y := x^{\downarrow s}$

$$\begin{aligned}
\hat{y}[k] &:= \langle \vec{x}^{\downarrow s}, \vec{\phi}_k \rangle \\
&= \langle \vec{x}, \vec{\phi}_k^{\downarrow -s} \rangle \\
&= \left\langle \vec{x}, \exp\left(\frac{i2\pi ks}{N}\right) \vec{\phi}_k \right\rangle \\
&= \exp\left(-\frac{i2\pi ks}{N}\right) \langle \vec{x}, \vec{\phi}_k \rangle
\end{aligned}$$

Effect of translation in Fourier domain

Let $\vec{x} \in \mathbb{C}^N$ with DFT $\hat{x}$ and $y := x^{\downarrow s}$

$$
\begin{aligned}
\hat{y}[k] &:= \langle \vec{x}^{\downarrow s}, \vec{\phi}_k \rangle \\
&= \langle \vec{x}, \vec{\phi}_k^{\downarrow -s} \rangle \\
&= \left\langle \vec{x}, \exp\left(\frac{i2\pi ks}{N}\right) \vec{\phi}_k \right\rangle \\
&= \exp\left(-\frac{i2\pi ks}{N}\right) \langle \vec{x}, \vec{\phi}_k \rangle \\
&= \exp\left(-\frac{i2\pi ks}{N}\right) \hat{x}[k]
\end{aligned}
$$

Effect of translation in Fourier domain

Let $X \in \mathbb{C}^{N \times N}$ with DFT $\widehat{X}$ and $Y := X^{\downarrow(s_1,s_2)}$, $0 \leq s_1, s_2 \leq N-1$

$$\widehat{Y}[k_1,k_2] := \exp\left(-\frac{i2\pi k s_1}{N}\right)\exp\left(-\frac{i2\pi k s_2}{N}\right)\widehat{X}[k_1,k_2], \quad 1 \leq k_1, k_2 \leq N$$

Linear translation-invariant maps

Linear map $\mathcal{D} : \mathbb{C}^N \to \mathbb{C}^N$ denoises class of translation-invariant signals

For input $\vec{y}$, denoised signal equals $\mathcal{D}(\vec{y})$

What if input is $\vec{y}^{\downarrow s}$?

Linear translation-invariant maps

Linear map $\mathcal{D} : \mathbb{C}^N \to \mathbb{C}^N$ denoises class of translation-invariant signals

For input $\vec{y}$, denoised signal equals $\mathcal{D}(\vec{y})$

What if input is $\vec{y}^{\,\downarrow s}$?

Output should be $\mathcal{D}(\vec{y})^{\downarrow s}$!

Linear translation-invariant (LTI) map

A map $\mathcal{L}$ from $\mathbb{C}^N$ to $\mathbb{C}^N$ is linear if for any $\vec{x}, \vec{y} \in \mathbb{C}^N$ and any $\alpha \in \mathbb{C}$

$$\mathcal{L}(\vec{x} + \vec{y}) = \mathcal{L}(\vec{x}) + \mathcal{L}(\vec{y}),$$
$$\mathcal{L}(\alpha\vec{x}) = \alpha\mathcal{L}(\vec{x}),$$

and translation invariant if for any shift $0 \leq s \leq N - 1$

$$\mathcal{L}(\vec{x}^{\downarrow s}) = \mathcal{L}(\vec{x})^{\downarrow s}$$

Linear translation-invariant (LTI) map

A map $\mathcal{L}$ from $\mathbb{C}^{N\times N}$ to $\mathbb{C}^{N\times N}$ is linear if for any $X, Y \in \mathbb{C}^{N\times N}$ and any $\alpha \in \mathbb{C}$

$$\mathcal{L}(X+Y) = \mathcal{L}(X) + \mathcal{L}(Y),$$
$$\mathcal{L}(\alpha X) = \alpha \mathcal{L}(X),$$

and translation invariant if for any $0 \leq s_1, s_2 \leq N-1$

$$\mathcal{L}(X^{\downarrow(s_1,s_2)}) = \mathcal{L}(X)^{\downarrow(s_1,s_2)}$$

Impulse response

Vectors with only one nonzero entry $\vec{e}_0$, $\vec{e}_1$, $\vec{e}_2$, ... are called impulses

LTI are characterized by their impulse response

$$\vec{h}_{\mathcal{L}} := \mathcal{L}(\vec{e}_0)$$

In 2D

$$H_{\mathcal{L}} := \mathcal{L}(E_0)$$

where $E[0,0] = 1$ and $E[j_1, j_2] = 0$ otherwise

Circular convolution

The circular convolution between two vectors $\vec{x}, \vec{y} \in \mathbb{C}^N$ is defined as

$$\vec{x} * \vec{y}\,[j] := \sum_{s=0}^{N-1} \vec{x}\,[s]\,\vec{y}^{\downarrow s}\,[j]\,, \quad 0 \leq j \leq N-1$$

Circular convolution

The 2D circular convolution between $X \in \mathbb{C}^{N\times N}$ and $Y \in \mathbb{C}^{N\times N}$ is

$$X * Y\left[j_1, j_2\right] := \sum_{s_1=0}^{N-1} \sum_{s_2=0}^{N-1} X\left[s_1, s_2\right] Y^{\downarrow(s_1,s_2)}\left[j_1, j_2\right], \quad 0 \leq j_1, j_2 \leq N-1$$

LTI maps as convolution with impulse response

For any LTI map $\mathcal{L} : \mathbb{C}^N \to \mathbb{C}^N$ and any $\vec{x} \in \mathbb{C}^N$

$$\mathcal{L}\left(\vec{x}\right) = \vec{x} * \vec{h}_{\mathcal{L}}$$

For any 2D LTI map $\mathcal{L} : \mathbb{C}^{N \times N} \to \mathbb{C}^{N \times N}$ and any $X \in \mathbb{C}^{N \times N}$

$$\mathcal{L}\left(X\right) = X * H_{\mathcal{L}}$$

Proof

$$\mathcal{L}\left(\vec{x}\right) = \mathcal{L}\left(\sum_{j=0}^{N-1} \vec{x}\,[j]\vec{e}_j\right)$$

Proof

$$\begin{aligned}
\mathcal{L}\left(\vec{x}\right) &= \mathcal{L}\left(\sum_{j=0}^{N-1} \vec{x}\left[j\right]\vec{e}_j\right) \\
&= \mathcal{L}\left(\sum_{j=0}^{N-1} \vec{x}\left[j\right]\vec{e}_0^{\downarrow j}\right)
\end{aligned}$$

Proof

$$
\begin{aligned}
\mathcal{L}\left(\vec{x}\right) &= \mathcal{L}\left(\sum_{j=0}^{N-1} \vec{x}\,[j]\vec{e}_j\right) \\
&= \mathcal{L}\left(\sum_{j=0}^{N-1} \vec{x}\,[j]\vec{e}_0^{\downarrow j}\right) \\
&= \sum_{j=0}^{N-1} \vec{x}\,[j]\mathcal{L}\left(\vec{e}_0^{\downarrow j}\right)
\end{aligned}
$$

Proof

$$\begin{aligned}
\mathcal{L}(\vec{x}) &= \mathcal{L}\left(\sum_{j=0}^{N-1} \vec{x}[j]\vec{e}_j\right) \\
&= \mathcal{L}\left(\sum_{j=0}^{N-1} \vec{x}[j]\vec{e}_0^{\downarrow j}\right) \\
&= \sum_{j=0}^{N-1} \vec{x}[j]\mathcal{L}\left(\vec{e}_0^{\downarrow j}\right) \\
&= \sum_{j=0}^{N-1} \vec{x}[j]\mathcal{L}(\vec{e}_0)^{\downarrow j}
\end{aligned}$$

Proof

$$
\begin{aligned}
\mathcal{L}\left(\vec{x}\right) &= \mathcal{L}\left(\sum_{j=0}^{N-1} \vec{x}[j]\vec{e}_j\right) \\
&= \mathcal{L}\left(\sum_{j=0}^{N-1} \vec{x}[j]\vec{e}_0^{\downarrow j}\right) \\
&= \sum_{j=0}^{N-1} \vec{x}[j]\mathcal{L}\left(\vec{e}_0^{\downarrow j}\right) \\
&= \sum_{j=0}^{N-1} \vec{x}[j]\mathcal{L}\left(\vec{e}_0\right)^{\downarrow j} \\
&= \sum_{j=0}^{N-1} \vec{x}[j]\vec{h}_{\mathcal{L}}^{\downarrow j}
\end{aligned}
$$

Convolution in time is multiplication in frequency

Let $\vec{y} := \vec{x}_1 * \vec{x}_2$, $\vec{x}_1, \vec{x}_2 \in \mathbb{C}^N$. Then

$$\hat{y}[k] = \hat{x}_1[k]\,\hat{x}_2[k], \quad 0 \le k \le N-1$$

Convolution in time is multiplication in frequency

Let $Y := X_1 * X_2$ for $X_1, X_2 \in \mathbb{C}^{N \times N}$. Then

$$\widehat{Y}[k_1, k_2] = \widehat{X}_1[k_1, k_2]\,\widehat{X}_2[k_1, k_2]$$

Proof

$$\hat{y}[k] := \left\langle \vec{x}_1 * \vec{x}_2, \vec{\phi}_k \right\rangle$$

Proof

$$\hat{y}[k] := \left\langle \vec{x}_1 * \vec{x}_2, \vec{\phi}_k \right\rangle$$
$$= \left\langle \sum_{s=0}^{N-1} \vec{x}_1[s]\, \vec{x}_2^{\downarrow s}, \vec{\phi}_k \right\rangle$$

Proof

$$
\begin{aligned}
\hat{y}[k] &:= \left\langle \vec{x}_1 * \vec{x}_2, \vec{\phi}_k \right\rangle \\
&= \left\langle \sum_{s=0}^{N-1} \vec{x}_1[s] \, \vec{x}_2^{\downarrow s}, \vec{\phi}_k \right\rangle \\
&= \left\langle \sum_{s=0}^{N-1} \vec{x}_1[s] \, \frac{1}{N} \sum_{j=0}^{N-1} \exp\left(-\frac{i 2\pi j s}{N} \right) \hat{x}_2[j] \vec{\phi}_j, \vec{\phi}_k \right\rangle
\end{aligned}
$$

Proof

$$\begin{aligned}
\hat{y}[k] &:= \left\langle \vec{x}_1 * \vec{x}_2, \vec{\phi}_k \right\rangle \\
&= \left\langle \sum_{s=0}^{N-1} \vec{x}_1[s]\, \vec{x}_2^{\downarrow s}, \vec{\phi}_k \right\rangle \\
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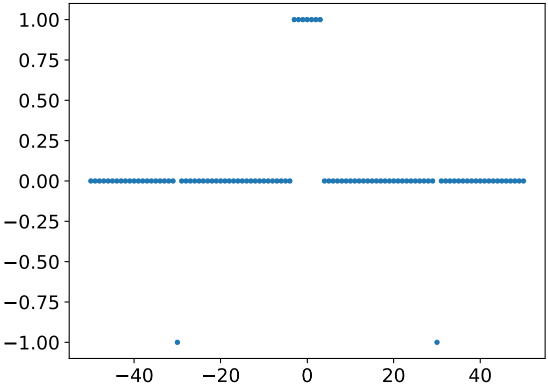
&= \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \vec{x}_1[s] \exp\left(-\frac{i2\pi js}{N}\right) \frac{1}{N} \left\langle \hat{x}_2[j]\vec{\phi}_j, \vec{\phi}_k \right\rangle
\end{aligned}$$

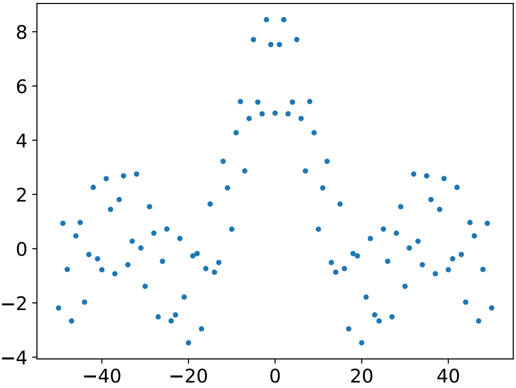
Proof

$$
\begin{aligned}
\hat{y}\left[k\right] &:= \left\langle \vec{x}_1 * \vec{x}_2, \vec{\phi}_k \right\rangle \\
&= \left\langle \sum_{s=0}^{N-1} \vec{x}_1\left[s\right] \vec{x}_2^{\downarrow s}, \vec{\phi}_k \right\rangle \\
&= \left\langle \sum_{s=0}^{N-1} \vec{x}_1\left[s\right] \frac{1}{N} \sum_{j=0}^{N-1} \exp\left(-\frac{i2\pi js}{N}\right) \hat{x}_2[j]\vec{\phi}_j, \vec{\phi}_k \right\rangle \\
&= \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \vec{x}_1\left[s\right] \exp\left(-\frac{i2\pi js}{N}\right) \frac{1}{N} \left\langle \hat{x}_2[j]\vec{\phi}_j, \vec{\phi}_k \right\rangle \\
&= \sum_{j=0}^{N-1} \hat{x}_1[j]\hat{x}_2[j] \frac{1}{N} \left\langle \vec{\phi}_j, \vec{\phi}_k \right\rangle \\
&= \hat{x}_1[k]\hat{x}_2[k]
\end{aligned}
$$

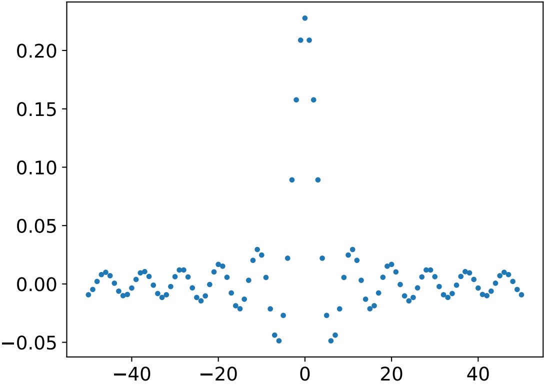
$\vec{x}$



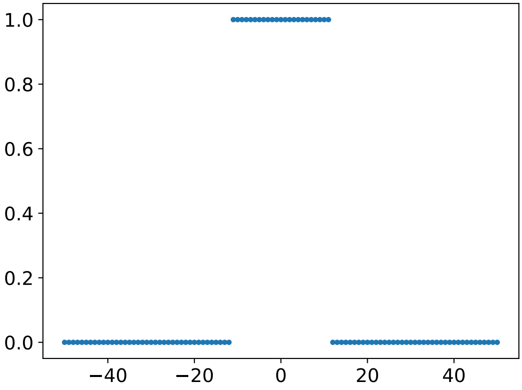
$\hat{x}$



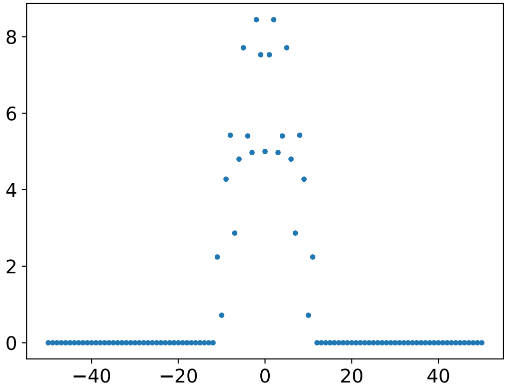
$\vec{y}$



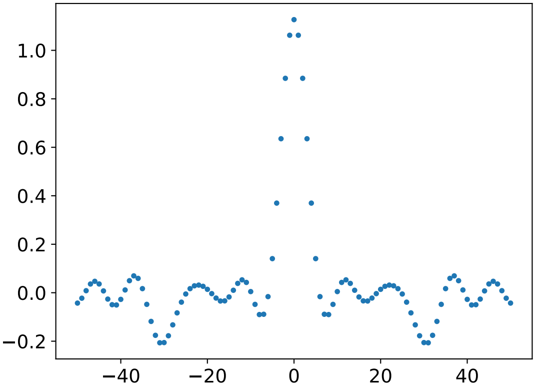
$\hat{y}$



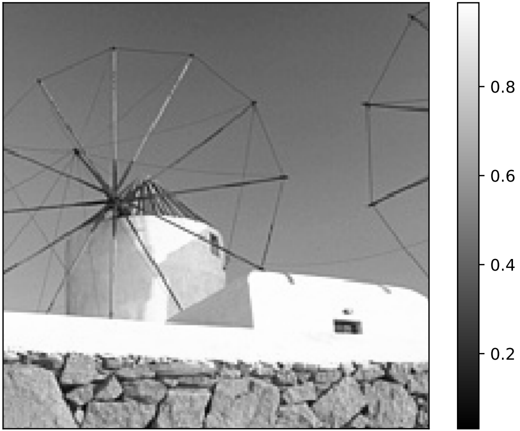
$\hat{x} \circ \hat{y}$



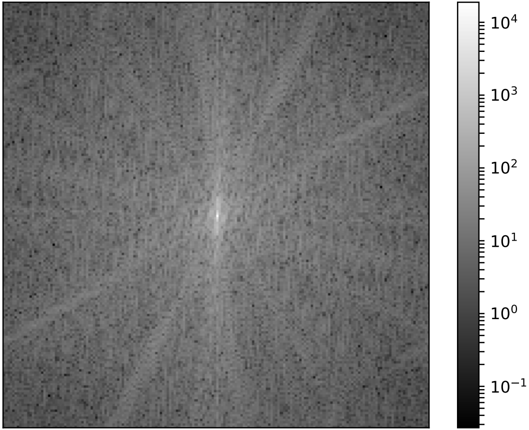
$\vec{x} * \vec{y}$



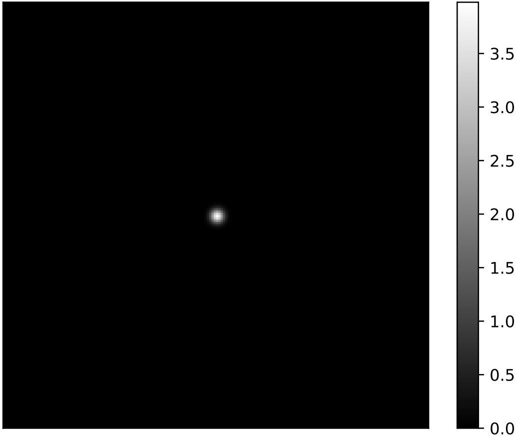
X



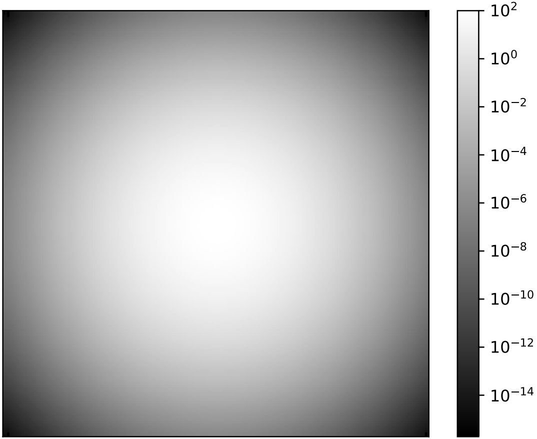
$\widehat{X}$



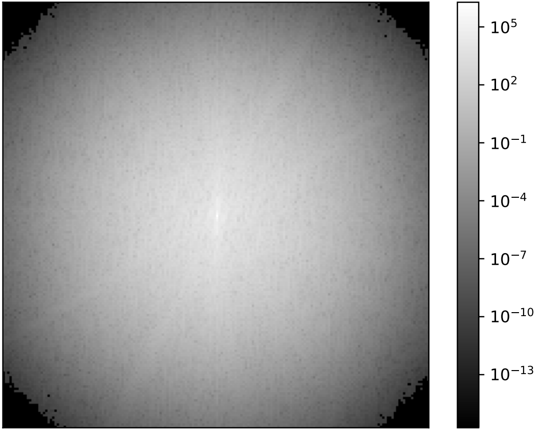
Y



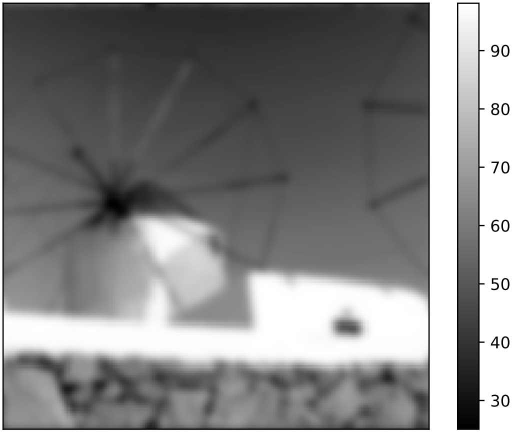
$\widehat{Y}$



$\widehat{X} \circ \widehat{Y}$



$X * Y$



Convolution in time is multiplication in frequency

LTI maps just scale Fourier coefficients!

DFT of impulse response is the transfer function of the map

For any LTI map $\mathcal{L}$ and any $\vec{x} \in C^N$

$$\mathcal{L}(\vec{x}) = \sum_{k=0}^{N-1} \hat{h}_{\mathcal{L}}[k]\hat{x}[k]\vec{\phi}_k.$$

For any 2D LTI map $\mathcal{L}$ and any $X \in C^{N \times N}$

$$\mathcal{L}(X) = \sum_{k_1=0}^{N-1} \sum_{k_2=1}^{N} \hat{H}_{\mathcal{L}}[k_1, k_2]\widehat{X}[k_1, k_2]\Phi_{k_1,k_2}$$

Signal estimation

Training set $(\vec{x}^{[1]}, \vec{y}^{[1]})$, ..., $(\vec{x}^{[n]}, \vec{y}^{[n]})$

Optimal translation-invariant estimator is solution to

$$\min_{\vec{w} \in \mathbb{C}^N} \sum_{j=1}^{n} \left|\left| \vec{x}^{[j]} - \vec{w} * \vec{y}^{[j]} \right|\right|_2^2$$

Signal estimation

Training set $(\vec{x}^{[1]}, \vec{y}^{[1]})$, ..., $(\vec{x}^{[n]}, \vec{y}^{[n]})$

Optimal translation-invariant estimator is solution to

$$\min_{\vec{w} \in \mathbb{C}^N} \sum_{j=1}^{n} \left|\left|\vec{x}^{[j]} - \vec{w} * \vec{y}^{\,[j]}\right|\right|_2^2$$

Solution is Wiener filter with transfer function

$$\hat{w}_{\text{opt}}[k] = \frac{\sum_{j=1}^{n} \hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]}}{\sum_{j=1}^{n} \left|\hat{y}^{[j]}[k]\right|^2}, \quad 1 \leq k \leq N$$

2D Wiener filter

Training set $(X^{[1]}, Y^{[1]}), \ldots, (X^{[n]}, Y^{[n]})$

Optimal translation-invariant estimator is

$$\widehat{W}_{\text{opt}}[k_1, k_2] = \frac{\sum_{j=1}^{n} \widehat{X}^{[j]}[k_1, k_2]\overline{\widehat{Y}^{[j]}[k_1, k_2]}}{\sum_{j=1}^{n} \left|\widehat{Y}^{[j]}[k_1, k_2]\right|^2}, \quad 1 \leq k_1, k_2 \leq N$$

Proof

$$C\left(w\right) := \frac{1}{2}\sum_{j=1}^{n}\left|\left|\vec{x}^{[j]} - \vec{w} * \vec{y}^{[j]}\right|\right|_2^2$$

Proof

$$C(w) := \frac{1}{2}\sum_{j=1}^{n} \left|\left|\vec{x}^{[j]} - \vec{w} * \vec{y}^{[j]}\right|\right|_2^2$$

$$= \frac{1}{2}\sum_{j=1}^{n} \left|\left|\frac{1}{N} F_{[N]}^{*}(\hat{x}^{[j]} - \hat{w} \circ \hat{y}^{[j]})\right|\right|_2^2$$

Proof

$$
\begin{aligned}
C(w) &:= \frac{1}{2}\sum_{j=1}^{n}\left|\left|\vec{x}^{[j]} - \vec{w} * \vec{y}^{[j]}\right|\right|_2^2 \\
&= \frac{1}{2}\sum_{j=1}^{n}\left|\left|\frac{1}{N}F_{[N]}^{*}(\hat{x}^{[j]} - \hat{w} \circ \hat{y}^{[j]})\right|\right|_2^2 \\
&= \frac{1}{2N}\sum_{j=1}^{n}\left|\left|\hat{x}^{[j]} - \hat{w} \circ \hat{y}^{[j]}\right|\right|_2^2
\end{aligned}
$$

Proof

$$
\begin{aligned}
C(w) &:= \frac{1}{2}\sum_{j=1}^{n}\left|\left|\vec{x}^{[j]} - \vec{w} * \vec{y}^{[j]}\right|\right|_2^2 \\
&= \frac{1}{2}\sum_{j=1}^{n}\left|\left|\frac{1}{N}F_{[N]}^{*}(\hat{x}^{[j]} - \hat{w} \circ \hat{y}^{[j]})\right|\right|_2^2 \\
&= \frac{1}{2N}\sum_{j=1}^{n}\left|\left|\hat{x}^{[j]} - \hat{w} \circ \hat{y}^{[j]}\right|\right|_2^2 \\
&= \frac{1}{2N}\sum_{j=1}^{n}\sum_{k=1}^{N}\left|\hat{x}^{[j]}[k] - \hat{w}[k]\hat{y}^{[j]}[k]\right|^2 := \frac{1}{N}\sum_{k=1}^{N}\widetilde{C}_k\left(\hat{w}[k]\right)
\end{aligned}
$$

Proof

$$
\begin{aligned}
\widetilde{C}_k(\alpha) &:= \frac{1}{2}\sum_{j=1}^{n}\left|\hat{x}^{[j]}[k] - \alpha\,\hat{y}^{[j]}[k]\right|^2 \\
&= \frac{1}{2}\sum_{j=1}^{n}\left|\hat{x}^{[j]}[k]\right|^2 - 2\,\mathrm{Re}\left\{\hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]\alpha}\right\} + \left|\hat{y}^{[j]}[k]\right|^2|\alpha|^2 \\
&= \frac{1}{2}\sum_{j=1}^{n}\left|\hat{x}^{[j]}[k]\right|^2 - 2\,\mathrm{Re}\left\{\hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]}\right\}\alpha_R - 2\,\mathrm{Im}\left\{\hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]}\right\}\alpha_I \\
&\quad + \left|\hat{y}^{[j]}[k]\right|^2\left(\alpha_R^2 + \alpha_I^2\right),
\end{aligned}
$$

Proof

$$\widetilde{C}_k(\alpha) := \frac{1}{2}\sum_{j=1}^{n}\left|\hat{x}^{[j]}[k] - \alpha\,\hat{y}^{[j]}[k]\right|^2$$

$$= \frac{1}{2}\sum_{j=1}^{n}\left|\hat{x}^{[j]}[k]\right|^2 - 2\,\mathrm{Re}\left\{\hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]\alpha}\right\} + \left|\hat{y}^{[j]}[k]\right|^2|\alpha|^2$$

$$= \frac{1}{2}\sum_{j=1}^{n}\left|\hat{x}^{[j]}[k]\right|^2 - 2\,\mathrm{Re}\left\{\hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]}\right\}\alpha_R - 2\,\mathrm{Im}\left\{\hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]}\right\}\alpha_I$$

$$+ \left|\hat{y}^{[j]}[k]\right|^2\left(\alpha_R^2 + \alpha_I^2\right),$$

$$\arg\min \widetilde{C}_k(\alpha) = \frac{\sum_{j=1}^{n}\mathrm{Re}\left\{\hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]}\right\}}{\sum_{j=1}^{n}\left|\hat{y}^{[j]}[k]\right|^2} + i\,\frac{\sum_{j=1}^{n}\mathrm{Im}\left\{\hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]}\right\}}{\sum_{j=1}^{n}\left|\hat{y}^{[j]}[k]\right|^2}$$

Denoising

Data modeled as n iid samples from random vector

$$\vec{\mathbf{y}} = \vec{\mathbf{x}} + \vec{\mathbf{z}},$$

where $\vec{\mathbf{z}}$ is zero-mean Gaussian noise with variance σ^2, independent of $\vec{\mathbf{x}}$

Fact

Linear transformation $A\vec{\mathbf{z}}$ of a Gaussian vector with mean $\vec{\mu}$ and covariance matrix Σ is Gaussian with mean $A\vec{\mu}$ and cov. matrix $A\Sigma A^*$

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Linear transformation $A\vec{z}$ of a Gaussian vector with mean $\vec{\mu}$ and covariance matrix Σ is Gaussian with mean $A\vec{\mu}$ and cov. matrix $A\Sigma A^*$

Fourier coefficients of noise are Gaussian with zero mean and covariance matrix $F_{[N]}\sigma^2 I F_{[N]}^* = N\sigma^2 I$

Fact

Linear transformation $A\vec{z}$ of a Gaussian vector with mean $\vec{\mu}$ and covariance matrix Σ is Gaussian with mean $A\vec{\mu}$ and cov. matrix $A\Sigma A^*$

Fourier coefficients of noise are Gaussian with zero mean and covariance matrix $F_{[N]}\sigma^2 I F_{[N]}^* = N\sigma^2 I$ (iid Gaussian with variance $N\sigma^2$)

Wiener filter

Numerator is sample crosscorrelation of Fourier coefficients

$$\frac{1}{n}\sum_{j=1}^{n} \hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]} \approx \mathrm{E}\left(\hat{\boldsymbol{x}}[k]\overline{\hat{\boldsymbol{y}}[k]}\right)$$

Wiener filter

Numerator is sample crosscorrelation of Fourier coefficients

$$
\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n} \hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]} &\approx \mathrm{E}\left(\hat{\boldsymbol{x}}[k]\overline{\hat{\boldsymbol{y}}[k]}\right) \\
&= \mathrm{E}\left(\left|\hat{\boldsymbol{x}}[k]\right|^2\right) + \mathrm{E}\left(\hat{\boldsymbol{x}}[k]\overline{\hat{\boldsymbol{z}}[k]}\right)
\end{aligned}
$$

Wiener filter

Numerator is sample crosscorrelation of Fourier coefficients

$$\frac{1}{n}\sum_{j=1}^{n} \hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]} \approx \mathrm{E}\left(\hat{\boldsymbol{x}}[k]\overline{\hat{\boldsymbol{y}}[k]}\right)$$

$$= \mathrm{E}\left(\left|\hat{\boldsymbol{x}}[k]\right|^2\right) + \mathrm{E}\left(\hat{\boldsymbol{x}}[k]\overline{\hat{\boldsymbol{z}}[k]}\right)$$

$$= \mathrm{E}\left(\left|\hat{\boldsymbol{x}}[k]\right|^2\right) + \mathrm{E}\left(\hat{\boldsymbol{x}}[k]\right)\mathrm{E}\left(\overline{\hat{\boldsymbol{z}}[k]}\right)$$

Wiener filter

Numerator is sample crosscorrelation of Fourier coefficients

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n} \hat{x}^{[j]}[k]\overline{\hat{y}^{[j]}[k]} &\approx \mathrm{E}\left(\hat{\boldsymbol{x}}[k]\overline{\hat{\boldsymbol{y}}[k]}\right) \\
&= \mathrm{E}\left(\left|\hat{\boldsymbol{x}}[k]\right|^2\right) + \mathrm{E}\left(\hat{\boldsymbol{x}}[k]\overline{\hat{\boldsymbol{z}}[k]}\right) \\
&= \mathrm{E}\left(\left|\hat{\boldsymbol{x}}[k]\right|^2\right) + \mathrm{E}\left(\hat{\boldsymbol{x}}[k]\right)\mathrm{E}\left(\overline{\hat{\boldsymbol{z}}[k]}\right) \\
&= \mathrm{E}\left(\left|\hat{\boldsymbol{x}}[k]\right|^2\right)
\end{aligned}$$

Wiener filter

Denominator is sample mean square of the data

$$\frac{1}{n}\sum_{j=1}^{n}\left|\hat{y}^{[j]}[k]\right|^2 \approx \mathrm{E}\left(|\hat{\boldsymbol{y}}[k]|^2\right)$$

Wiener filter

Denominator is sample mean square of the data

$$\frac{1}{n}\sum_{j=1}^{n}\left|\hat{y}^{[j]}[k]\right|^2 \approx \mathrm{E}\left(|\hat{\boldsymbol{y}}[k]|^2\right)$$
$$= \mathrm{E}\left(|\hat{\boldsymbol{x}}[k]|^2\right) + \mathrm{E}\left(|\hat{\boldsymbol{z}}[k]|^2\right)$$

Wiener filter

Denominator is sample mean square of the data

$$\frac{1}{n}\sum_{j=1}^{n}\left|\hat{y}^{[j]}[k]\right|^2 \approx \mathrm{E}\left(|\hat{\boldsymbol{y}}[k]|^2\right)$$

$$= \mathrm{E}\left(|\hat{\boldsymbol{x}}[k]|^2\right) + \mathrm{E}\left(|\hat{\boldsymbol{z}}[k]|^2\right)$$

$$= \mathrm{E}\left(|\hat{\boldsymbol{x}}[k]|^2\right) + N\sigma^2_{\text{noise}}$$

Wiener filter

Transfer function of optimal denoising Wiener filter is given by

$$\hat{w}_{\mathrm{opt}}[k] = \frac{\mathrm{E}\left(|\hat{\boldsymbol{x}}[k]|^2\right)}{\mathrm{E}\left(|\hat{\boldsymbol{x}}[k]|^2\right) + N\sigma^2}, \quad 1 \leq k \leq N$$

Wiener filter

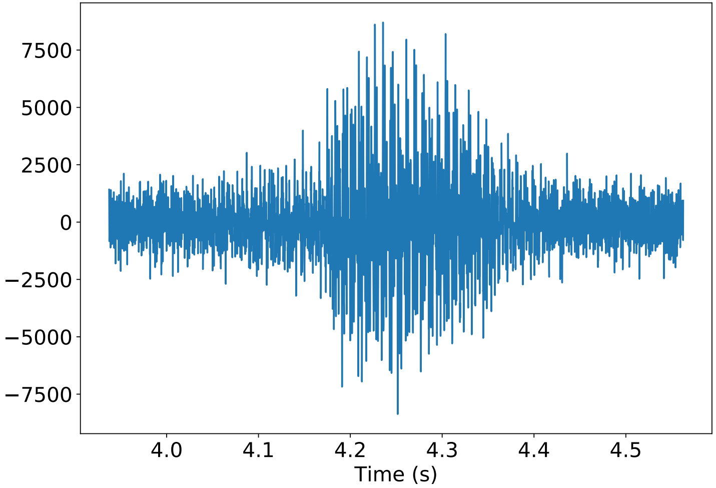
Transfer function of optimal denoising Wiener filter is given by

$$\hat{w}_{\text{opt}}[k] = \frac{\mathrm{E}\left(|\hat{\boldsymbol{x}}[k]|^2\right)}{\mathrm{E}\left(|\hat{\boldsymbol{x}}[k]|^2\right) + N\sigma^2}, \quad 1 \leq k \leq N$$

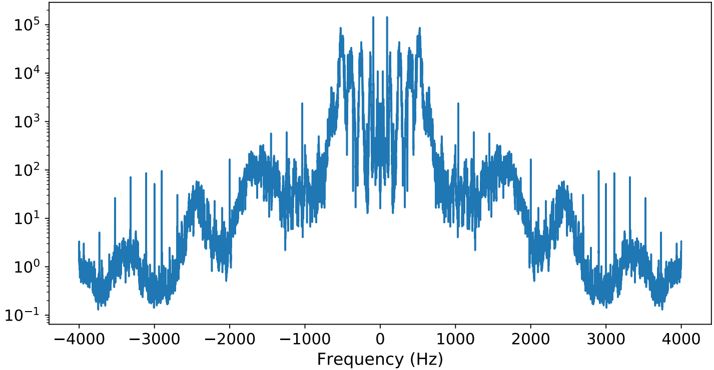
In 2D

$$\widehat{W}_{\text{opt}}[k_1, k_2] = \frac{\mathrm{E}\left(\left|\widehat{\boldsymbol{X}}[k_1, k_2]\right|^2\right)}{\mathrm{E}\left(\left|\widehat{\boldsymbol{X}}[k_1, k_2]\right|^2\right) + N^2\sigma^2}, \quad 1 \leq k_1, k_2 \leq N$$

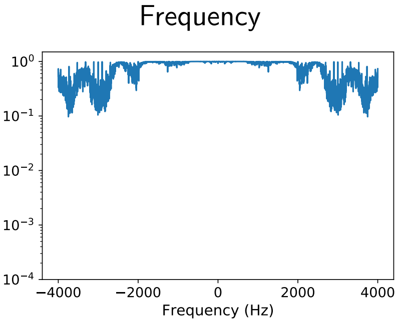
Audio data

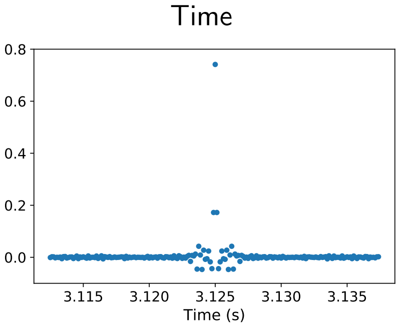


Audio data: Mean square of Fourier coefficients

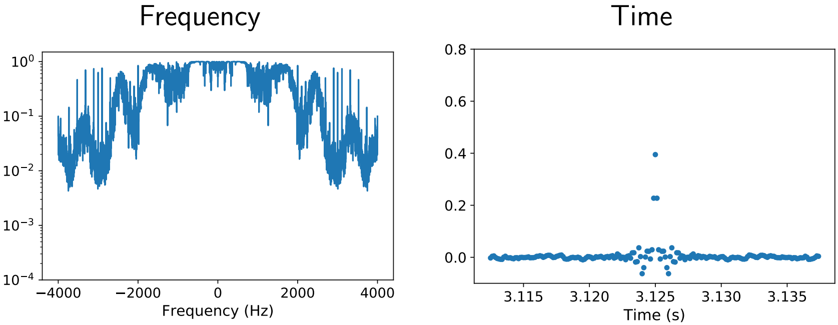


Wiener filter: $\sigma = 0.02$

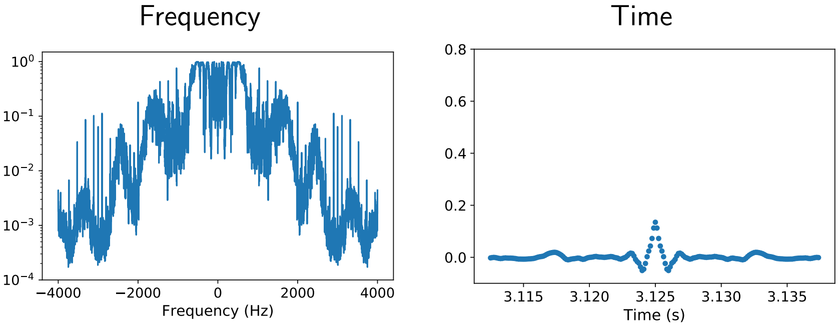




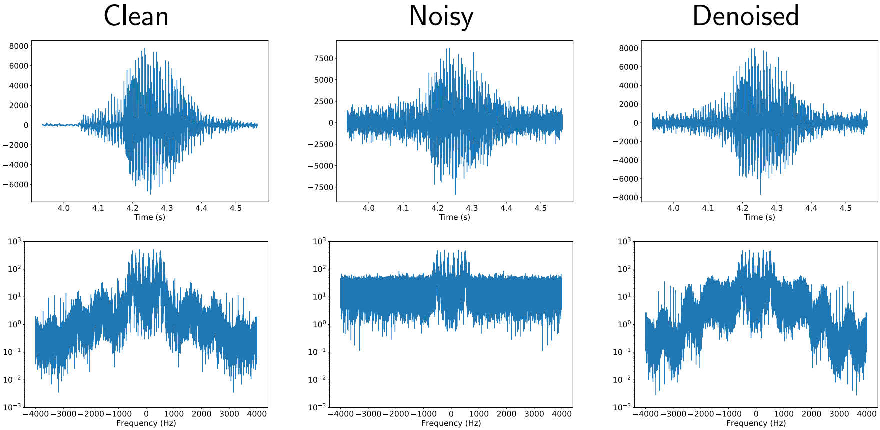
Wiener filter: $\sigma = 0.1$



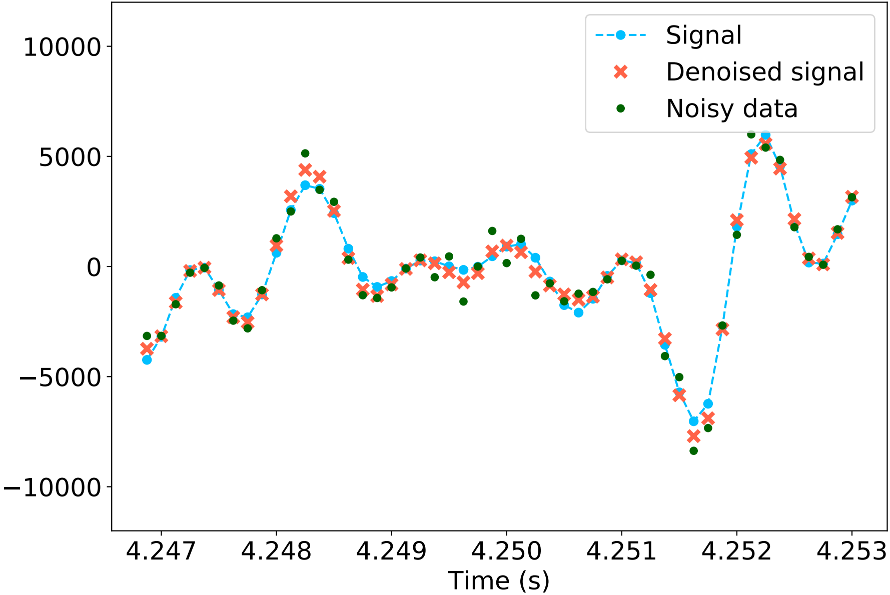
Wiener filter: $\sigma = 0.5$



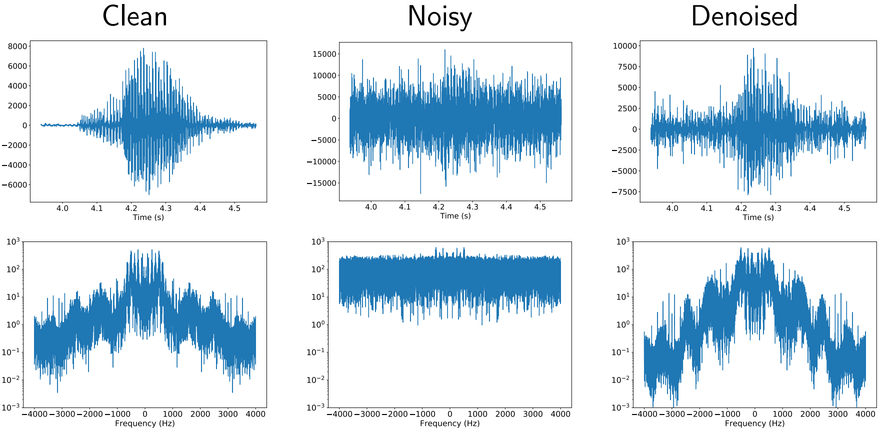
Example: $\sigma = 0.1$



Example: $\sigma = 0.1$



Example: $\sigma = 0.5$



Example: $\sigma = 0.5$

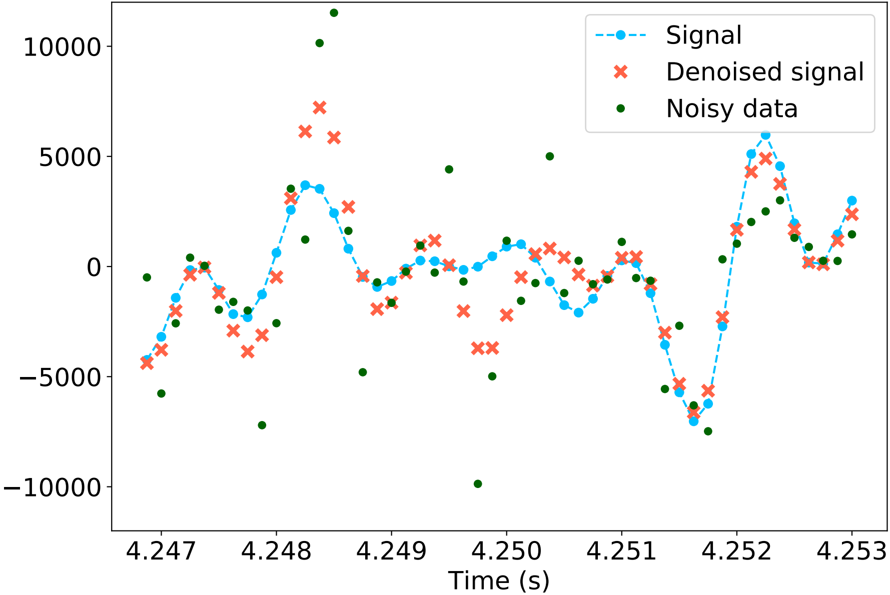


Image data

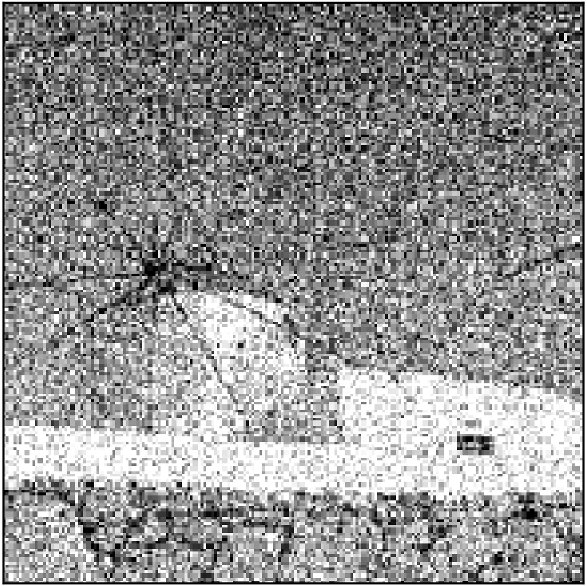
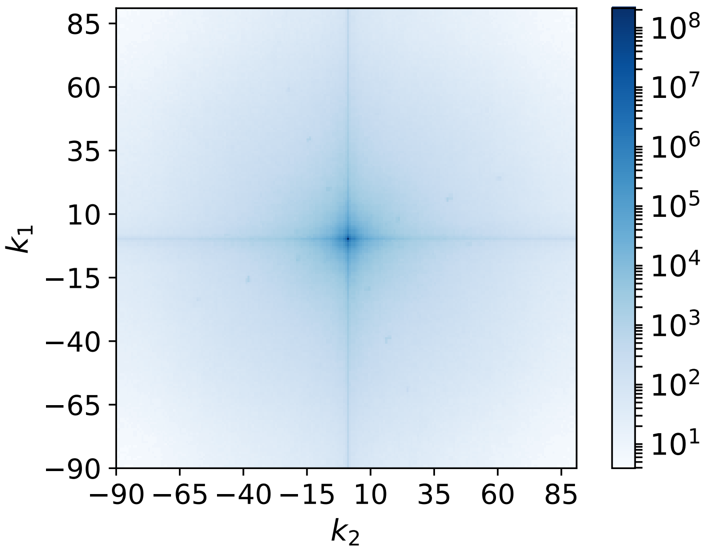
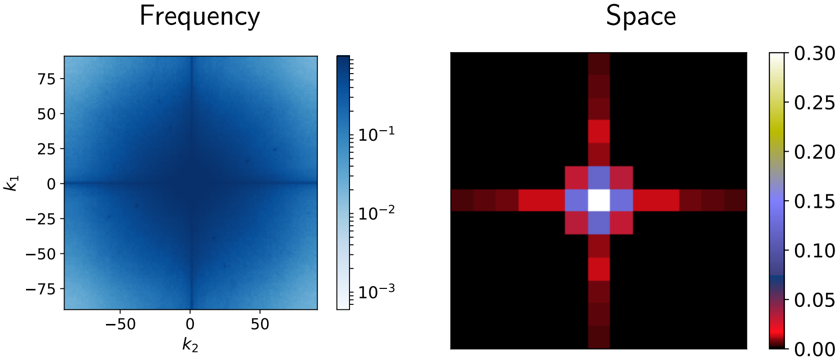


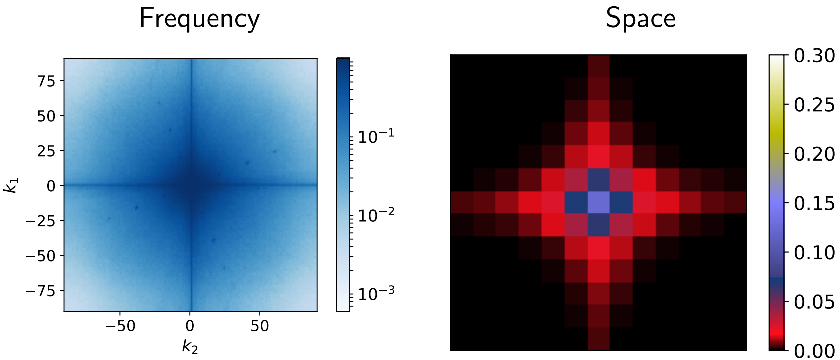
Image data: Mean square of Fourier coefficients



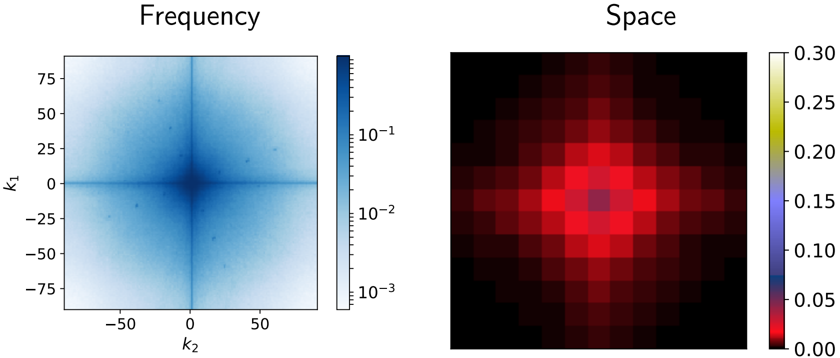
Wiener filter: $\sigma = 0.04$



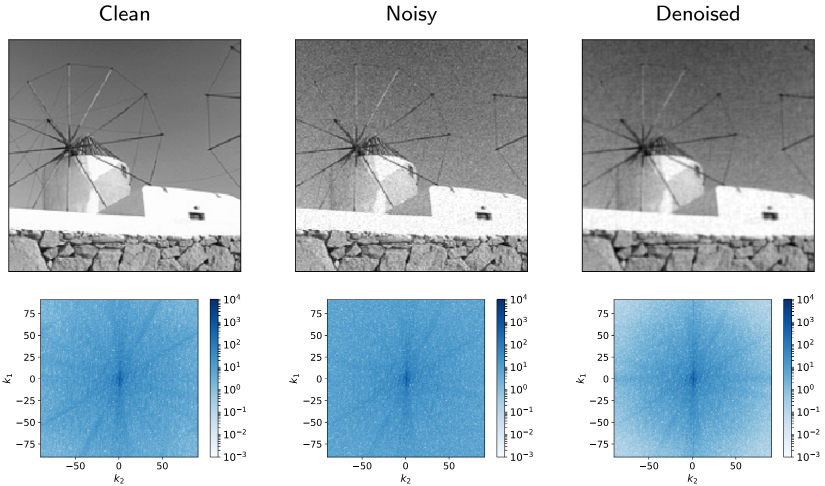
Wiener filter: $\sigma = 0.1$



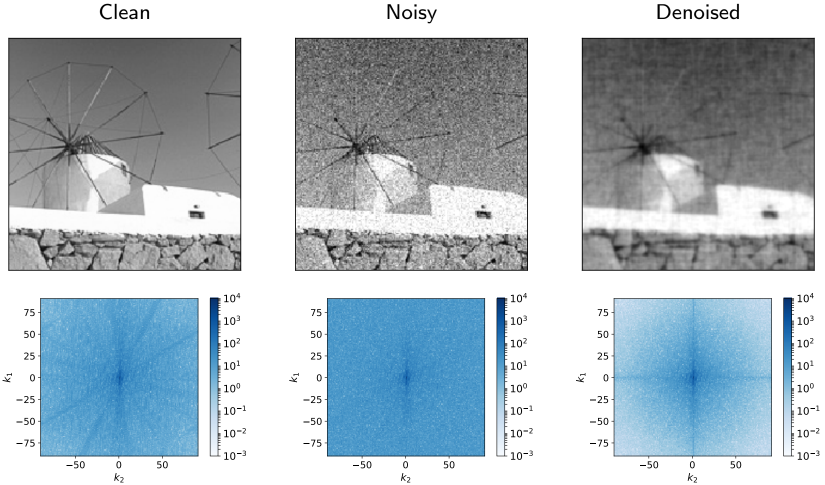
Wiener filter: $\sigma = 0.2$



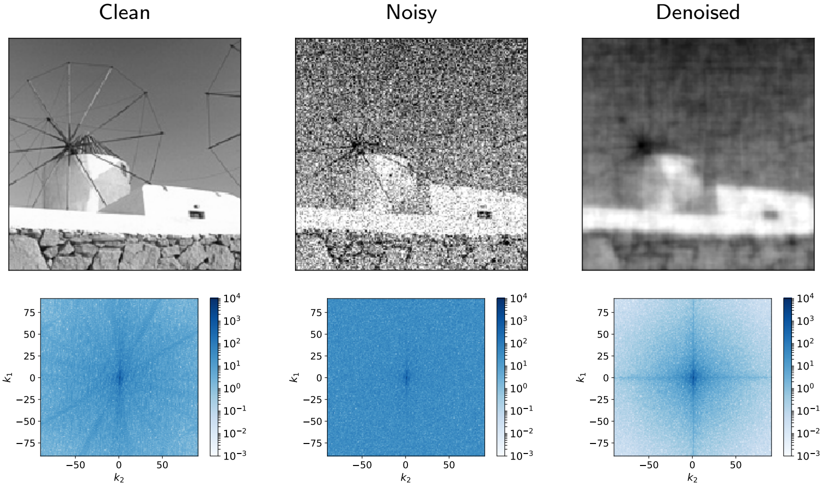
Example: $\sigma = 0.04$



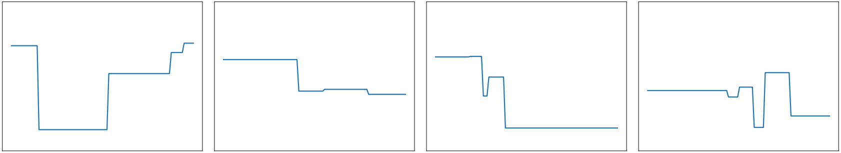
Example: $\sigma = 0.1$



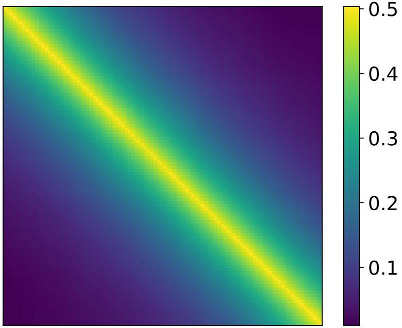
Example: $\sigma = 0.2$



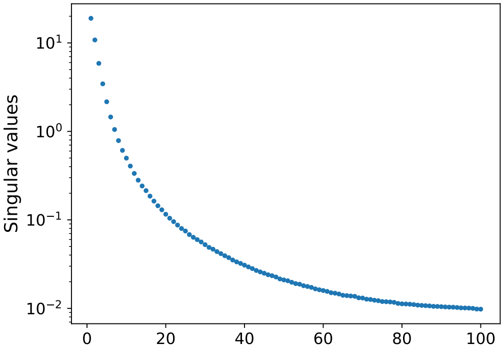
PCA of translation-invariant signals



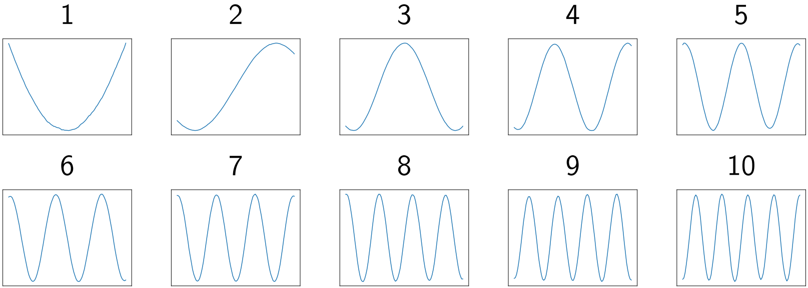
Sample covariance matrix



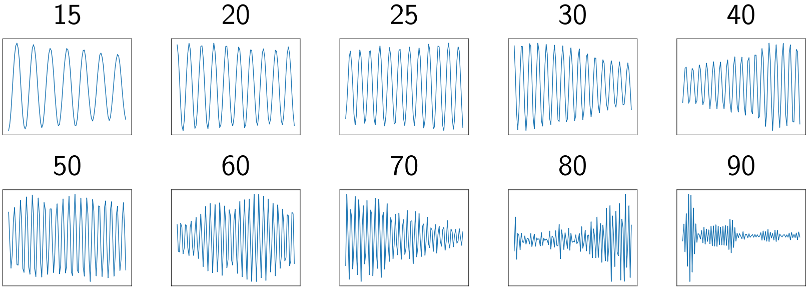
Singular values



Principal directions



Principal directions



Stationary signals

Translation-invariant signals can be characterized as samples from stationary random vector

$\vec{x}$ is wide-sense or weak-sense stationary if

1. it has a constant mean

$$\mathrm{E}\left(\vec{x}[j]\right) = \mu, \quad 1 \leq j \leq N$$

2. it has a translation-invariant covariance, there is a function κ such that

$$\mathrm{E}\left(\vec{x}[j_1]\vec{x}[j_2]\right) = \kappa(|j_2 - j_1|), \quad 1 \leq j_1, j_2 \leq N$$

κ is the autocovariance of $\vec{x}$

Covariance matrix

Covariance matrix of a wide-sense stationary random vector is Toeplitz symmetric

$$\Sigma = \begin{bmatrix} \kappa(0) & \kappa(1) & \kappa(2) & \kappa(3) \\ \kappa(1) & \kappa(0) & \kappa(1) & \kappa(2) \\ \kappa(2) & \kappa(1) & \kappa(0) & \kappa(1) \\ \kappa(3) & \kappa(2) & \kappa(1) & \kappa(0) \end{bmatrix}.$$

Circulant matrix

Each row vector is a unit circular shift of previous row

$$\begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix},$$

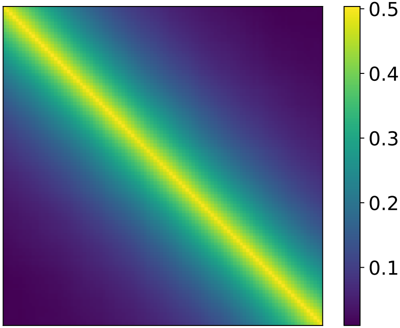
Covariance matrix

In real signals, autocovariance often decays

In that case, covariance is well approximated as circulant

$$\Sigma = \begin{bmatrix} \kappa(0) & \kappa(1) & 0 & 0 & 0 \\ \kappa(1) & \kappa(0) & \kappa(1) & 0 & 0 \\ 0 & \kappa(1) & \kappa(0) & \kappa(1) & 0 \\ 0 & 0 & \kappa(1) & \kappa(0) & \kappa(1) \\ 0 & 0 & 0 & \kappa(1) & \kappa(0) \end{bmatrix} \approx \begin{bmatrix} \kappa(0) & \kappa(1) & 0 & 0 & \kappa(1) \\ \kappa(1) & \kappa(0) & \kappa(1) & 0 & 0 \\ 0 & \kappa(1) & \kappa(0) & \kappa(1) & 0 \\ 0 & 0 & \kappa(1) & \kappa(0) & \kappa(1) \\ \kappa(1) & 0 & 0 & \kappa(1) & \kappa(0) \end{bmatrix}$$

Sample covariance matrix



Eigendecomposition of circulant matrix

Any circulant matrix $C \in \mathbb{C}^{n \times n}$ can be written as

$$C := \frac{1}{N} F_{[N]}^{*} \Lambda F_{[N]},$$

where $F_{[N]}$ is the DFT matrix and Λ is a diagonal matrix

Proof

For any vector $\vec{x} \in \mathbb{C}^n$

$$C\vec{x} = \vec{c} * \vec{x}$$

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For any vector $\vec{x} \in \mathbb{C}^n$

$$\begin{aligned} C\vec{x} &= \vec{c} * \vec{x} \\ &= \frac{1}{N} F_{[N]}^* \operatorname{diag}(\hat{c}) F_{[N]} \vec{x} \end{aligned}$$

SVD of circulant covariance matrix

Eigenvalues are nonnegative, so

$$\frac{1}{\sqrt{N}} F_{[N]}^{*} \operatorname{diag}(\hat{c}) \frac{1}{\sqrt{N}} F_{[N]}$$

is a valid SVD

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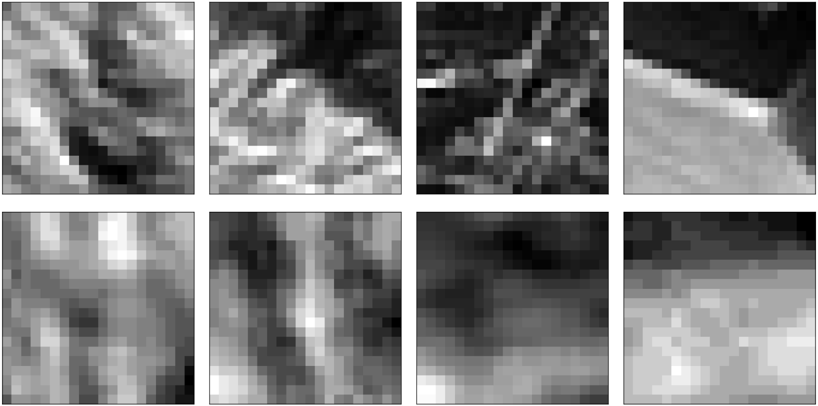
is a valid SVD

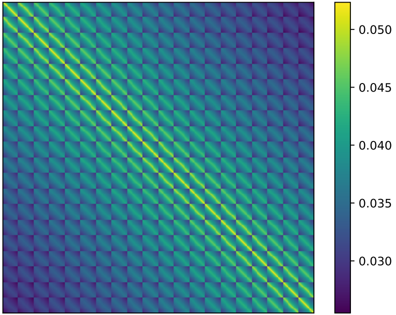
If $\hat{c}$ have different values, singular vectors are sinusoids!

Image patches

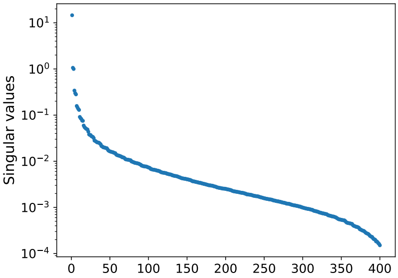


Covariance matrix

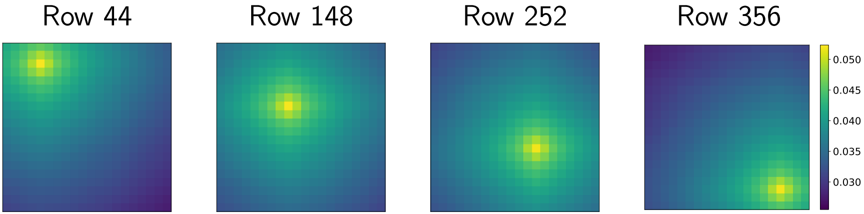
Sample covariance matrix



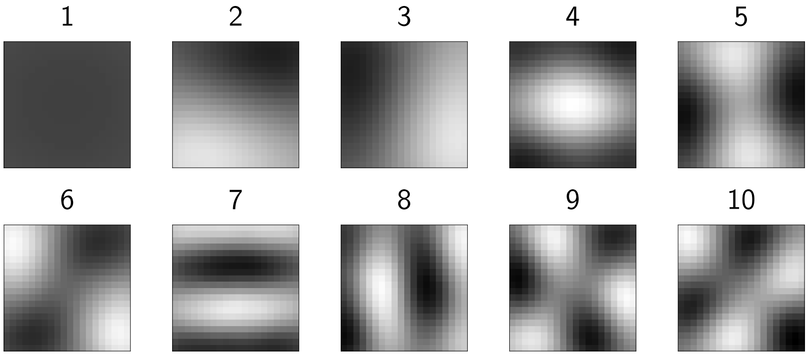
Singular values



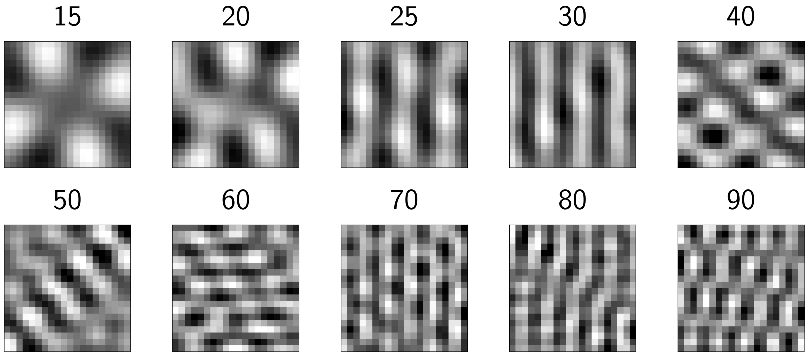
Rows of covariance matrix



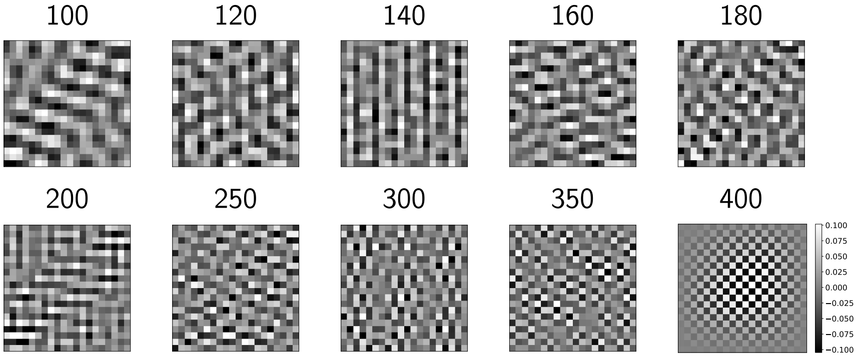
Principal directions



Principal directions



Principal directions



PCA of natural images

Principal directions tend to be sinusoidal

This suggests using 2D sinusoids for dimensionality reduction

PCA of natural images

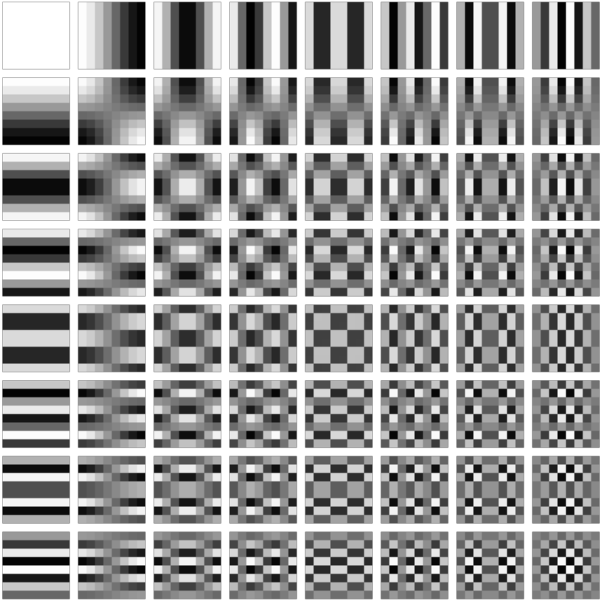
Principal directions tend to be sinusoidal

This suggests using 2D sinusoids for dimensionality reduction

JPEG compresses images using discrete cosine transform (DCT):

1. Image is divided into 8×8 patches
2. Each DCT band is quantized differently (more bits for lower frequencies)

DCT basis vectors



Projection of each 8x8 block onto first DCT coefficients