Set theory

1 Basic definitions

A set is a collection of objects. The set of all elements that we consider in a certain situation is called the universe and is usually denoted by $\Omega$. If an object $x$ in $\Omega$ belongs to set $S$, we say that $x$ is an element of $S$ and write $x \in S$. If $x$ is not an element of $S$ then we write $x \notin S$. The empty set, usually denoted by $\emptyset$, is a set such that $x \notin \emptyset$ for all $x \in \Omega$ (i.e. it has no elements). If all the elements in a set $B$ also belong to a set $A$ then $B$ is a subset of $A$, which we denote by $B \subseteq A$. If in addition there is at least one element of $A$ that does not belong to $B$ then $B$ is a proper subset of $A$, denoted by $B \subset A$.

The elements of a set can be arbitrary objects and in particular they can be sets themselves. This is the case for the power set of a set, defined in the next section.

A useful way of defining a set is through a statement concerning its elements. Let $S$ be the set of elements such that a certain statement $s(x)$ holds, to define $S$ we write

$$S := \{ x \mid s(x) \}.$$  

For example, $A := \{ x \mid 1 < x < 3 \}$ is the set of all elements greater than 1 and smaller than 3. Let us define some important sets and set operations using this notation.

2 Basic operations

Definition 2.1 (Set operations).

- The complement $S^c$ of a set $S$ contains all elements that are not in $S$.

$$S^c := \{ x \mid x \notin S \}.$$  

- The union of two sets $A$ and $B$ contains the objects that belong to $A$ or $B$.

$$A \cup B := \{ x \mid x \in A \text{ or } x \in B \}.$$  

This can be generalized to a sequence of sets $A_1, A_2, \ldots$

$$\bigcup_{n} A_n := \{ x \mid x \in A_n \text{ for some } n \},$$

where the sequence may be infinite.
• The **intersection** of two sets $A$ and $B$ contains the objects that belong to $A$ and $B$.

$$A \cap B := \{ x \mid x \in A \text{ and } x \in B \}.$$ 

Again, this can be generalized to a sequence,

$$\bigcap_n A_n := \{ x \mid x \in A_n \text{ for all } n \}.$$ 

• The **difference** of two sets $A$ and $B$ contains the elements in $A$ that are not in $B$.

$$A/B := \{ x \mid x \in A \text{ and } x / \in B \}.$$ 

• The **power set** $2^S$ of a set $S$ is the set of all possible subsets of $S$, including $\emptyset$ and $S$.

$$2^S := \{ S' \mid S' \subseteq S \}.$$ 

Two sets are equal if they have the same elements, i.e. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. It is easy to verify for instance that $(A^c)^c = A$, $S \cup \Omega = \Omega$, $S \cap \Omega = S$ or the following identities which are known as De Morgan’s laws.

**Theorem 2.2** (De Morgan’s laws). For any two sets $A$ and $B$

$$(A \cup B)^c = A^c \cap B^c,$$

$$(A \cap B)^c = A^c \cup B^c.$$ 

**Proof.** Let us prove the first identity; the proof of the second is almost identical.

First we prove that $(A \cup B)^c \subseteq A^c \cap B^c$. A standard way to prove the inclusion of a set in another set is to show that if an element belongs to the first set then it must also belong to the second. Any element $x$ in $(A \cup B)^c$ (if the set is empty then the inclusion holds trivially, since $\emptyset \subseteq S$ for any set $S$) is in $A^c$; otherwise it would belong to $A$ and consequently to $A \cup B$. Similarly, $x$ also belongs to $B^c$. We conclude that $x$ belongs to $A^c \cap B^c$, which proves the inclusion.

To complete the proof we establish $A^c \cap B^c \subseteq (A \cup B)^c$. If $x \in A^c \cap B^c$, then $x \notin A$ and $x \notin B$, so $x \notin A \cup B$ and consequently $x \in (A \cup B)^c$. 

\[ \square \]