Math 3150 – PDEs WS 4

Name: ______________________________

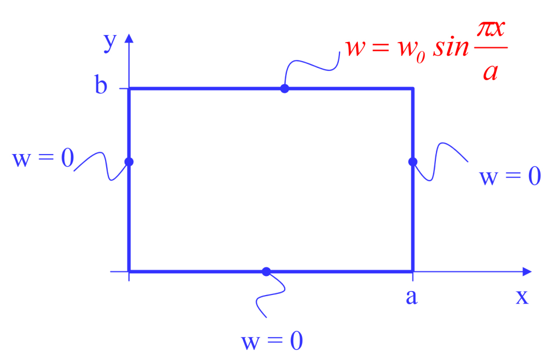
1. Consider the partial differential equation for $w(x, y)$

$$\Delta w = \nabla^2 w = \partial_{xx} w + \partial_{yy} w = 0. \tag{1}$$

This PDE is called **Laplace's equation** and shows up in tons of different fields. A small sampling:

1. **electrostatics:** The potential V (voltage) from an electric field satisfies the relation $\nabla^2 V = -\rho/\varepsilon_0$, where ρ is the charge density and ε_0 is the permitivity of space. Therefore, if we want to know the voltage in a charge-free region, (1) is the PDE we study.
2. **heat transfer:** we know the heat equation describing temperature is $\partial_t T = D\nabla^2 T$, so (1) can be thought of as the equilibrium heat distribution for a 2 dimensional object.
3. **structures:** In (1) $w(x, y)$ could be the deformation (out of the paper) of some membrane, where the boundary conditions will determine how the membrane is attached.
4. **fluids:** If the velocity field of a fluid v can be described by some potential $v = \nabla\phi$, then if the fluid is incompressible (i.e. water, but not air) ϕ satisfies (1).
5. **image processing**: one intuitive behavior of solutions of (1) is that it smoothes things out. One application is then to use solutions of this PDE to smooth out noisy edges in images.

In today's worksheet, we'll solve (1) with the boundary conditions shown below using *separation of variables*.



2. Suppose the solution is of the form $w(x, y) = p(x)q(y)$. (It's a little easier to keep the negative sign on the y ODE.) Plug this in and get some ODEs involving a constant λ. Don't solve them yet.

Solution: Plugging in the guess $w = p(x)q(y)$, we get

$$p''q + pq'' = 0,$$

so rearranging, we find

$$\frac{p''(x)}{p(x)} = -\frac{q''(y)}{q(y)} = \lambda,$$

where we've concluded that these must equal a constant, because they're distinctly a function of x and y separately. This leads to the two ODEs

$$p'' = \lambda p, \qquad -q'' = \lambda q.$$

We'll discuss the boundary conditions in a bit, when we try to characterize what λ could be.

3. First consider the case where $\lambda = 0$. Are these eigenvalues? *Hint:* start with $x = 0$, then $y = 0$, and then $x = a$ boundaries.

Solution: In the case that $\lambda = 0$, our solutions both become $p(x) = Ax + B$ and $q(y) = Cy + D$. Now, we must start enforcing boundaries:

- at the left $x = 0$, we have $w(0, y) = 0 = (A0 + B)(Cy + D)$ so either $B = 0$ or $C, D = 0$, the latter we do not want (since it would make q entirely zero, so we'll assume $b = 0$.
- However, we also have at $y = 0$, $w(x, 0) = 0 = (Ax)(C0 + D)$, so either $A = 0$ or $D = 0$. Again, if $A = 0$ then our whole thing is zero, so we'll take $D = 0$.
- Finally, at $x = a$, we have $w(a, y) = 0 = (Aa)(Cy)$, so either $a = 0$ or $c = 0$, but either produces the fully zero solution, so we can rule out the $\lambda = 0$ case entirely.

4. I'll just tell you that $\lambda > 0$ does not work here (it might be nice to convince yourself of this) but it's similar to the previous problem.

   Consider $\lambda < 0$ and find the eigenfunctions.

Solution: Call $\omega\sqrt{-\lambda}$ for brevity. Then, our ODE solutions become (complex, real roots respectively)

$$p(x) = A\cos\omega x + B\sin\omega x, \qquad p(y) = Ce^{\omega x} + De^{-\omega x}.$$

We again go case-by-case to look at the boundaries to find the solutions

- At $x = 0$ we have $w(0, y) = 0 = A(Ce^{\omega y} + De^{-\omega y})$, which means either $A = 0$ or $C, D = 0$, but we don't want the latter so we'll take $A = 0$.

- At $y = 0$: we have $w(x, 0) = B\sin(\omega x)(C + D) = 0$. Clearly$B \neq 0$, so we'll take $D = -C$.

- At $x = a$, we have $w(a, y) = B\sin(\omega a)(Ce^{\omega y} - Ce^{-\omega y})$. This is a little tough to see. It's honestly a bit easier to see if you take the $\sinh(\omega y), \cosh(\omega y)$ forms instead of exponential, but the class seemed to hate this, so we'll roll with exponential. Basically we need to rule out the second term being zero, so

$$\begin{aligned} 0 &= (Ce^{\omega y} - Ce^{-\omega y}) \\ &= C(e^{\omega y} - e^{-\omega y}) \\ &= C(e^{2\omega y} - 1). \end{aligned}$$

  But clearly, unless $\omega = 0$ (not possible), this is never solvable. Therefore, $\sin\omega a = 0$, so we can use the zeros of $\sin(x)$ to deduce the resulting eigenvalues

$$\sin\omega a = 0 \qquad \Longrightarrow \qquad \omega_n = \frac{n\pi}{a}.$$

  Putting this together, the resulting eigenfunctions are

$$w_n(x, y) = p_n(x)q_n(y) = c_n\sin(\frac{n\pi x}{a})\left(e^{n\pi y/a} - e^{-n\pi y/a}\right).$$

  Again, a nicer (but definitely different) way of writing this is

$$\tilde{w}_n(x, y) = p_n(x)q_n(y) = d_n\sin(\frac{n\pi x}{a})\sinh(\frac{n\pi y}{a}).$$

5. Whether you like exponentials or hyperbolic trig functions, I think the easiest way to write the resulting solution is

$$w(x, y) = \sum_{n=1}^{\infty} c_n\sin(\frac{n\pi x}{a})\sinh(\frac{n\pi y}{a}).$$

   Find the c_n values using the last boundary conditions. The orthogonality of Fourier modes will help tremendously.

Solution: To do this, we apply the last boundary condition at $y = b$, so we get

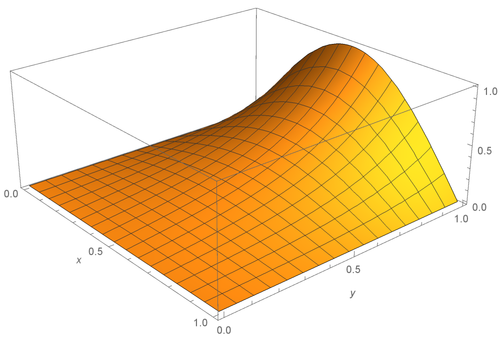
$$w(x, b) = w_0\sin\frac{\pi x}{a} = \sum_{n=1}^{\infty} c_n\sin\frac{nx\pi}{a}\sinh\frac{n\pi b}{a}.$$

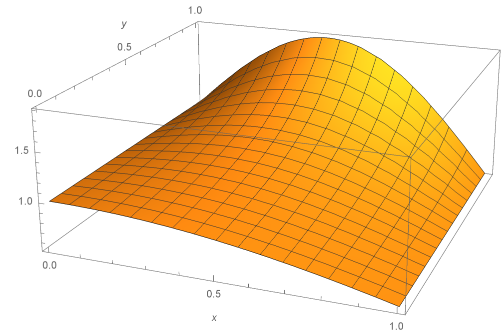
Here, by the orthogonality of sines, the only thing that can contribute is $n = 1$, so we have

$$w_0\sin\frac{\pi x}{a} = c_1\sin\frac{\pi x}{a}\sinh\frac{\pi b}{a},$$

and therefore $c_1 = w_0/\sin(\pi b/a)$ and c_n=0 otherwise.

6. Which of the two graphs is the real plot of the answer? Why?





Solution: The left is the one that satisfies all the boundaries.