Math 3150 Food for Thought #4

Friday, September 16

Group #: Solutions

Names :

All group members listed here must be present and actively contributing to receive credit for this assignment. Any violation of this policy will result in a zero for all group members.

In today's assignment, we'll be using the orthogonal projection method to approximate the pulse function

$$f(x) = \begin{cases} 0 & \text{if } x \in [-\pi, 0) \\ 1 & \text{if } x \in [0, \pi/2] \\ 0 & \text{if } x \in [\pi/2, \pi) \end{cases}.$$

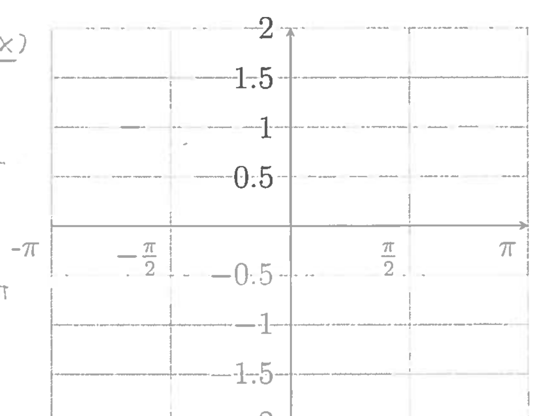
We will check if accuracy of our approximation improves if we use more orthogonal functions, and finally use these notions to develop Fourier Series.

1. Find an approximation $\hat{f}_0(x)$ to $f(x)$ using the orthogonal functions $\{1, \cos(x), \sin(x)\}$. Carefully plot $f(x)$ and $\hat{f}_0(x)$ together on the provided axes.

Note (*)

$\sin^2(x) = \frac{1-\cos(2x)}{2}$

$\cos^2(x) = \frac{1+\cos(2x)}{2}$



See plots on last page.

$\hat{f}_0(x) = a_0 + a_1\cos(x) + b_1\sin(x)$

$$a_0 = \frac{\langle 1, f\rangle}{\langle 1, 1\rangle} = \frac{\int_0^{\pi/2} dx}{\int_{-\pi}^{\pi} dx} = \frac{\pi/2}{2\pi} = \frac{1}{4}$$

$$a_1 = \frac{\langle \cos(x), f\rangle}{\langle \cos(x), \cos(x)\rangle} = \frac{\int_0^{\pi/2}\cos(x)\,dx}{\int_{-\pi}^{\pi}\cos^2(x)\,dx} = \frac{1}{\pi}$$

(*)

$$b_1 = \frac{\langle \sin(x), f\rangle}{\langle \sin(x), \sin(x)\rangle} = \frac{\int_0^{\pi/2}\sin(x)\,dx}{\int_{-\pi}^{\pi}\sin^2(x)\,dx} = \frac{1}{\pi}$$

2. Find an approximation $\hat{f}_1(x)$ to $f(x)$ using the orthogonal functions

$$\{1, \cos(x), \cos(2x), \sin(x), \sin(2x)\}.$$

Carefully plot $f(x)$ and $\hat{f}_1(x)$ together on the provided axes. Do you think your approximation $\hat{f}_1(x)$ is better than the approximation $\hat{f}_0(x)$ from the previous question?

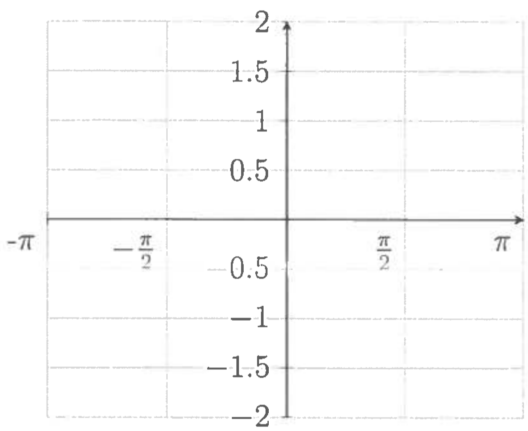
$$\hat{f}_1 = a_0 + a_1 \cos(x) + a_2 \cos(2x) + b_1 \sin(x) + b_2 \sin(2x)$$

$$\left.\begin{array}{l} a_0 = \frac{1}{4} \\ a_1 = \frac{1}{\pi} \\ b_1 = \frac{1}{\pi} \end{array}\right\} \text{from Q1.}$$

$$a_2 = \frac{\langle \cos(2x), f\rangle}{\langle \cos(2x), \cos(2x)\rangle} = \frac{\int_0^{\pi/2} \cos(2x)\,dx}{\int_{-\pi}^{\pi} \cos^2(2x)\,dx} = \frac{0}{\pi} = 0$$

$$b_2 = \frac{\langle \sin(2x), f\rangle}{\langle \sin(2x), \sin(2x)\rangle} = \frac{\int_0^{\pi/2} \sin(2x)\,dx}{\int_{-\pi}^{\pi} \sin^2(2x)\,dy} = \frac{1}{\pi}$$

$$\therefore \hat{f}_1(x) = \frac{1}{4} + \frac{1}{\pi}\cos(x) + \frac{1}{\pi}\sin(x) + \frac{1}{\pi}\sin(2x)$$



3. Find an approximation $\hat{f}_2(x)$ to $f(x)$ using the orthogonal functions

$$\{1, \cos(x), \cos(2x), \cos(3x), \sin(x), \sin(2x), \sin(3x)\}.$$

Carefully plot $f(x)$ and $\hat{f}_2(x)$ together on the provided axes. Compare the approximation to your previous two approximations.

$a_0 = \frac{1}{4}$

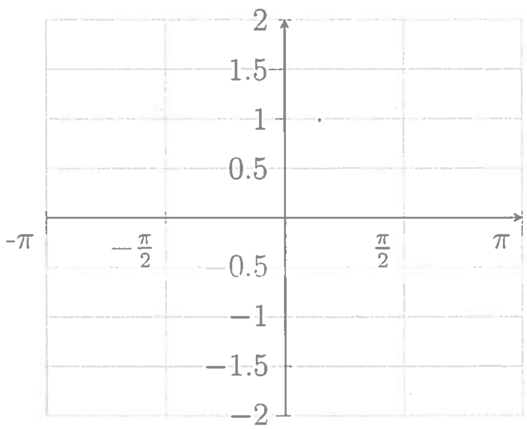
$a_1 = \frac{1}{\pi}$, $a_2 =$

$b_1 = \frac{1}{\pi}$, $b_2 =$

$$a_3 = \frac{\langle f, \cos(3x)\rangle}{\langle \cos(3x), \cos(3x)\rangle} = \frac{\int_0^{\pi/2} \cos(3x)\,dx}{\int_{-\pi}^{\pi} \cos^2(3x)\,dx} = \frac{-1/3}{\pi} = -\frac{1}{3\pi}$$

$$b_3 = \frac{\langle f, \sin(3x)\rangle}{\langle \sin(3x), \sin(3x)\rangle} = \frac{\int_0^{\pi/2} \sin(3x)\,dx}{\int_{-\pi}^{\pi} \sin^2(3x)\,dx} = \frac{1/3}{\pi} = \frac{1}{3\pi}$$

$$\hat{f}_2(x) = \frac{1}{4} + \frac{1}{\pi}\cos(x) - \frac{1}{3\pi}\cos(3x) + \frac{1}{\pi}\sin(x) + \frac{1}{\pi}\sin(2x) + \frac{1}{3\pi}\sin(3x)$$



4. Find an approximation $\hat{f}(x)$ to $f(x)$ using the orthogonal basis functions $\{1, \cos(nx), \sin(nx)\}_{n=}^{\infty}$ Your coefficients for your approximation will be defined in terms of n. *Note: This approximation is called the Fourier series of $f(x)$, and the coefficients a_0, a_n, b_n are called Fourier coefficients.*

$$\hat{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_0 = \frac{1}{4}$$

$$a_n = \frac{\langle f, \cos(nx)\rangle}{\langle \cos(nx), \cos(nx)\rangle} = \frac{\int_0^{\pi/2} \cos(nx)\,dx}{\int_{-\pi}^{\pi} \cos^2(nx)\,dx} = \frac{\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)}{\pi} = \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{1}{n\pi} & n = 1, 5, 9, \dots \\ -\frac{1}{n\pi} & n = 3, 7, 11, \dots \end{cases}$$

$$b_n = \frac{\langle f, \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle} = \frac{\int_0^{\pi/2} \sin(nx)\,dx}{\int_{-\pi}^{\pi} \sin^2(nx)\,dx} = \frac{-\frac{1}{n}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]}{\pi}$$

$$= \begin{cases} \frac{1}{n\pi} & n \text{ odd} \\ 0 & n = 4, 8, 12, \dots \\ \frac{2}{n\pi} & n = 2, 6, 10, \dots \end{cases}$$

5. Write down the Fourier series $\hat{f}(x)$ for some general function $f(x)$ with $x \in [-\pi, \pi]$, then find the general formulas for the coefficients.

$$\hat{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

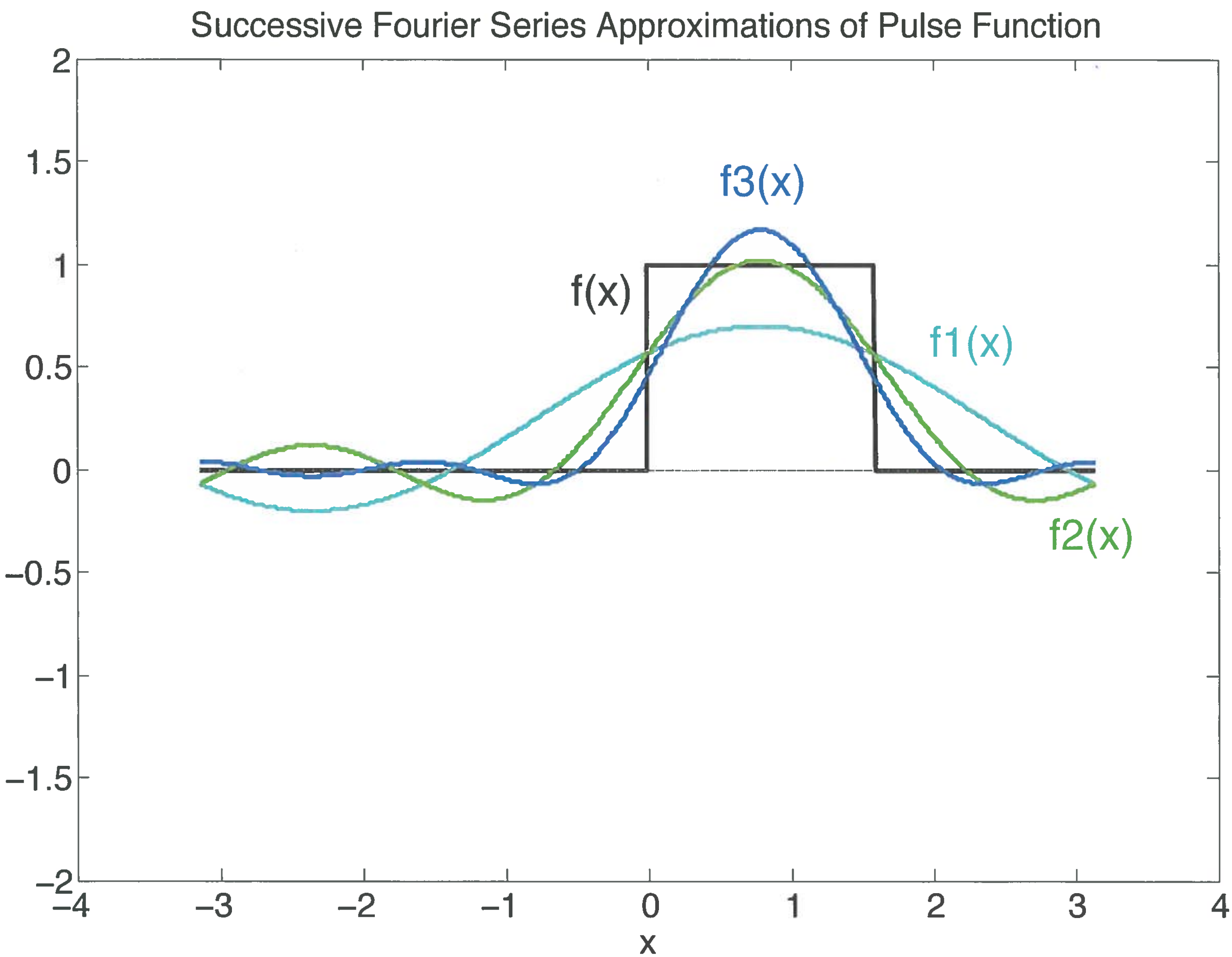
$$a_0 = \frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} = \frac{\int_{-\pi}^{\pi} f(x)\,dx}{\int_{-\pi}^{\pi} dx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{\langle \cos(nx), f \rangle}{\langle \cos(nx), \cos(nx) \rangle} = \frac{\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}{\int_{-\pi}^{\pi} \cos^2(nx)\,dx} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{\langle \sin(nx), f \rangle}{\langle \sin(nx), \sin(nx) \rangle} = \frac{\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx}{\int_{-\pi}^{\pi} \sin^2(nx)\,dx} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

These coefficient formulas hold for <u>any</u> Fourier series on <u>the domain</u> $x \in [-\pi, \pi]$.

FFT 4 Score: **/10**



Successive Fourier Series Approximations of Pulse Function
f3(x)
f(x)
f1(x)
f2(x)
x
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−4
−3
−2
−1
0
1
2
3
4