Math 3150 – PDEs WS 2

Name: ____________________________________

1. **Euler-Bernoulli beam equation**

   Consider a horizontal beam of length 1 (for simplicity) that is is fixed at each end. Due to force from gravity, the beam will sag and not be perfectly horizontal. Let $y(x, t)$ be the vertical displacement of the beam at time t.

   The equation (PDE) describing $y(x, t)$ satisfies

   $$\partial_{tt} y = E \partial_{xxxx} y + \rho g,$$

   where E is the Young's modulus (stiffness), ρ is the density of the beam, and g the gravitational constant. *(Does this look like other PDEs we've discussed? What is the flux rule?)*

   In equilibrium, this becomes the ODE for $y(x)$:

   $$E \frac{\mathrm{d}^4}{\mathrm{d}x^4} y = -\rho g,$$

   Suppose we know the boundaries

   $$y(0) = y(1) = 0, \qquad y'(0) = y'(1) = 0.$$

   (a) Solve for the equilibrium displacement $y(x)$ with the given boundary conditions.

   (b) Suppose $E = 1, \rho g = 1$ and use some sort of software (or not) to plot the solution and draw a sketch here. Does this agree with your intuition?

   (c) What kind of behavior do you think we could get out of this equation if we considered the *time-dependent* PDE $y(x, t)$?

   Solution: Overall, don't worry about this problem at all. I screwed up here, and basically all I want you to know is that the beam equation exists (and is kinda a kooky looking PDE with 4 x derivatives. **This won't show up on a quiz, exam, etc.**

2. Find the corresponding eigenfunction for the linear operator L and given eigenvalue λ

   $$\lambda = -3, \qquad Ly := y' + y, \qquad y(0) = 1.$$

Solution: Using the definition of an eigenvalue, we have

$$Ly = \lambda y$$

which, plugging these in yields

$$y'(x) + y(x) = -3y(x),$$

and rearranging gives us

$$y'(x) = -4y(x).$$

There are a few ways to solve this at this point, but the most obvious one to me at the time was to divide both sides and integrate, so

$$\int \frac{y'(x)}{y(x)}\,\mathrm{d}x = \ln y \qquad = \qquad \int -4\,\mathrm{d}x = -4x + c, ,$$

logs are kinda ugly, so rearranging this a bit, we have

$$y(x) = ae^{-4x},$$

where a is an unknown constant, but plugging in $y(0) = 1$ we find $a = 1$.

Therefore, the **eigenvector/eigenfunction** corresponding to the eigenvalue $\lambda = -3$ is $y(x) = e^{-4x}$.

3. Find all eigenvalues of

$$Ly := \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}, \qquad y'(0) = 0, y'(\pi) = 0.$$

Solution: Apparently I also screwed this one up in class (rough few days, sorry!) but it's overall somewhat minor. The idea was correct.

Taking the ODE, we have our eigenvalue equation

$$Ly = \lambda y$$

which yields

$$y'' = \lambda y,$$

and gives us the same characteristic equation $r^2 = \lambda$ so $r = \pm\sqrt{\lambda}$. Again, we have three scenarios here: $\lambda > 0$ so r is real, $\lambda < 0$ so r is complex, and $\lambda = 0$. We investigate each.

Case 1: $\lambda = 0$.

If we plug in $\lambda = 0$, we find our equation becomes

$$y'' = 0$$

so integrating twice gives us

$$y(x) = c_1 x + c_2.$$

Plugging in the boundary at $x = 0$ gives us

$$y'(0) = c_1 \qquad = \qquad 0,$$

so we can conclude that $c_1 = 0$. Note that if we try the other boundary, we get

$$y'(\pi) = c_1 = 0,$$

so we satisfy this automatically. But what is c_2? An unknown constant. Therefore, our eigenfunction corresponding to $\lambda = 0$ is simply the *constant* solution

$$y(x) = c_2 = \text{constant}.$$

Case 2:$\lambda > 0$:

Here, when $\lambda > 0$, our roots are real so our solution looks like

$$y(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x}.$$

Plugging in $x = 0$, we get

$$y'(0) = -c_1\sqrt{\lambda}e^{-\sqrt{\lambda}0} + \sqrt{\lambda}c_2 e^{\sqrt{\lambda}0} = 0,$$

so $c_1 = c_2$. Plugging in $y'(\pi)$, we get

$$y'(\pi) = -c_2\sqrt{\lambda}e^{-\sqrt{\lambda}\pi} + \sqrt{\lambda}c_2 e^{\sqrt{\lambda}\pi} = 0,$$

but note, the only way this is possible is if $c_2 = 0$ (since the exponentials can never cancel each other out), so our solution becomes identically zero $y(x) = 0$, which is *not* a valid eigenfunction, so $\lambda > 0$ is not possible.

Case 3:$\lambda < 0$: In this scenario, we have complex roots (since $\sqrt{\lambda}$ is complex) so our solutions are of the form

$$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

Plugging in $y'(0) = 0$, we find

$$y'(0) = -c_1 \sin(0) + \sqrt{\lambda}c_2 \cos(0) = c_2\sqrt{\lambda} = 0,$$

so necessarily $c_2 = 0$. In the second condition, we find

$$y'(\pi) = -c_1\sqrt{\lambda}\sin(\pi\sqrt{\lambda}) = 0.$$

Clearly we don't want $c_1 = 0$ otherwise our whole solution would be zero. So that means we must have

$$\sin(\pi\sqrt{\lambda}) = 0.$$

Where is $\sin(x) = 0$? At exactly $0 + n\pi$ (infinitely many times), so we have

$$\sqrt{\lambda}\pi = n\pi,$$

and squaring, we find

$$\lambda = -n^2.$$

Note we have the negative square root because we assumed λ was negative! These are valid eigenvalues for this operator and boundary pair.