Math 3150 – Review questions

I stole these from previous year's exams. Again, I don't claim these will be on the test, have correct solutions, or are even appropriate for our class, but hopefully they provide some useful studying directions. You should *definitely* look at the quizzes, worksheets, and homework first, especially the problems that seem to be good ones!

1. Suppose cars flow along an infinite highway with density $\rho(x, t)$. The flux of cars passing through each point x is determined by the flux rule $\phi(x, \rho) = v(\rho)\rho$, where $v(\rho) = 1/(1 + x^2)$. Let $M(t)$ be the mass of cars in a region $(0, 1)$ given by

$$M(t) = \int_0^1 \rho(x, t)dx$$

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   (a) Assume there cars are conserved on the infinite highway. Using the flux rule listed above, find an identity for dM/dt in terms of the flux rule.

   (b) Assuming the choice of interval $(0, 1)$ is arbitrary, determine the associated PDE for the density of cars $\rho(x, t)$.

Solution: (a)

$$dM/dt = \phi(0) - \phi(1) = \rho v(0) - \rho v(1) = \rho - \frac{\rho}{2} = \frac{\rho}{2}$$

(b)

$$\rho_t = -(\frac{\rho}{1 + x^2})_x = -\rho_x \frac{1}{1 + x^2} + \rho \frac{2x}{(1 + x^2)^2}$$

2. We define the following collection of basis functions on the interval $x \in [0, 1]$

$$p_0 = 1$$

$$p_1 = \begin{cases} 1, & x < 1/2 \\ -1, & x \geq 1/2 \end{cases}$$

$$p_2 = \begin{cases} 1, & x < 1/4 \\ -1, & 1/4 \leq x < 1/2 \\ 0, & else \end{cases}$$

$$p_3 = \begin{cases} 1, & 1/2 \leq x < 3/4 \\ -1, & 3/4 \leq x < 1 \\ 0, & else \end{cases}$$

   (a) Draw graphs of the four functions.

   (b) Verify that each function is orthogonal to the others. There are six combinations. Use the standard inner product:

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$

Solution: Check all pairs $\langle p_i, p_j \rangle$ and verify the integral is 0.

(c) Find orthogonal projections of

$$f(x) = \begin{cases} 1, & x < 1/4 \\ 0, & x \geq 1/4 \end{cases}$$

onto the set of basis functions given the standard inner product.

Solution:

$$c_0 = 1/4, \quad c_1 = 1/4, \quad c_2 = 1/4, \quad c_3 = 0$$

(d) Express the best approximation $\hat{f}$ using the basis functions given the standard inner product.

Solution:

$$\hat{f} = (1/4)p_0 + (1/4)p_1 + (1/2)p_2 + 0p_3$$

3. Find sine series of the function $f(x) = 1$, on $x \in [0, 1]$ by expanding over the odd extension.

Solution:

$$b_n = 2\int_0^1 \sin(\pi n x)dx = -2\frac{\cos(\pi n x)}{\pi n}|_0^1 = \frac{2(\cos(\pi n) - 1)}{n\pi}$$

$$\hat{f}(x) \sim \sum_{n=1}^{\infty} \frac{2(\cos(\pi n) - 1)}{n\pi}\sin(\pi n x)$$

4. Find cosine series of the function $f(x) = 1$, on $x \in [0, 1]$ by expanding over the even extension.

Solution:

$$a_0 = 1$$

$$a_n = 0, \; for \; n > 0$$

$$\hat{f}(x) = 1$$

5. Consider the heat equation on a rod of length L with an insulated boundary at $x = L$ and a fixed temperature boundary at $x = 0$: $u(0, t) = 0$.

(a) Use the hypothesis that $u(x, t) = p(x)q(t)$ to get the ODEs

$$\frac{dq}{dt} = -\lambda q$$

and

$$\frac{d^2p}{dx^2} = -\lambda p$$

with boundary conditions $\frac{dp}{dx}(L) = 0$ and $p(0) = 0$

Solution: plug $u(x,t) = p(x)q(t)$ into the PDE gives

$$\frac{dq}{dt}q = q\frac{d^2p}{dx^2}$$
$$\implies \frac{1}{q}\frac{dq}{dt} = \frac{1}{p}\frac{d^2p}{dx^2}$$

The left side depends only on t, the right side depends only on x so they both must be equal to a constant. Call that constant $-\lambda$. So we have

$$\frac{dq}{dt} = -\lambda q$$

and

$$\frac{d^2p}{dx^2} = -\lambda p$$

with BC $p'(L) = 0$ and $p(0) = 0$

(b) What is the general solution to the ODE for $q(t)$.

Solution:

$$q(t) = ce^{-\lambda t}$$

(c) Assume that $\lambda < 0$, show that the eigenvalues and corresponding eigenfunctions are of the form

$$\lambda_n = (\frac{\pi}{2L}(2n+1))^2, \qquad p_n(x) = \sin(\frac{\pi}{2L}(2n+1)$$

Solution: The solutions to the boundary value problem are of the form

$$p(x) = c_1\cos(\sqrt{-\lambda}x) + c_2\sin(\sqrt{-\lambda}x)$$

Apply the $x = 0$ BC:

$$p(0) = c_1 = 0$$

Apply the $x = L$ BC:

$$\frac{dp}{dx}(L) = \sqrt{-\lambda}(c_2\cos(\sqrt{-\lambda}L) = 0$$

However: $\cos(\sqrt{-\lambda}L) = 0 \therefore \sqrt{-\lambda}L = (2n+1)\pi/2, n = 0, 1, 2, ...$so the eigenvalues are

$$\lambda_n = -((2n+1)\pi/(2L))^2, \qquad n = 0, 1, 2, ...$$

and the corresponding eigenfunctions are

$$p(x) = \sin((2n+1)\pi x/(2L)) \qquad n = 0, 1, 2, ...$$

(d) Combine the eigenfunctions $p_n(x)$ and $q_n(t)$ to find the general solution to the heat equation.

Solution:

$$u(x,t) = \sum_{n=0}^{\infty} a_n \sin((2n+1)\pi x/(2L)) e^{-((2n+1)\pi/(2L))^2 t}$$

(e) Find the particular solution to the PDE when the initial condition is

$$u(x,0) = f(x) = 4\sin(\frac{\pi}{2L}) - \sin(\frac{\pi}{2L}3x)$$

Solution: THe IC is already in the form of a sum of eigenfunctions to the problem. Therefore the solution is

$$u(x,t) = 4\sin(\pi x/(2L))e^{-(\pi/2L)^2 t} - \sin(3\pi x/(2L))e^{-(3\pi/2L)^2 t}$$

6. Find the equilibrium solution for the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + R(x)$ with $R(x) = \sin(\pi x/L)$ and boundary conditions $u(0) = 1$ and $\frac{du}{dx}(L) = 0$.

Solution: At equilibrium, the solution becomes

$$0 = \frac{d^2 u}{dx^2} + \sin(\pi x/L)$$

Integrating twice yields

$$u(x) = \frac{L^2}{\pi^2}\sin(\pi x/L) + c_1 x + c_2$$

$u(0) = c_2 = 1$ and $u'(L) = 0 \therefore c_1 = L/\pi$

$$u(x) = \frac{L^2}{\pi^2}\sin(\pi x/L) + \frac{L}{\pi}x + 1$$

7. Consider the inner product on continuous functions with one continuous derivative on the interval $[0, L]$:

$$\langle u, v\rangle = \int_0^L u(x)v(x)dx$$

The family of functions $\{\sin(\pi x/(2L)), \sin(3\pi x/(2L)), \sin(5\pi x/(2L)), ...\}$ is orthogonal with respect to this inner product and

$$\langle \sin((2n+1)\pi x/(2)), \sin((2m+1)\pi x/(2L))\rangle = \begin{cases} L/2 \text{ if } n = m \\ 0 \text{ if } n \neq m \end{cases}$$

Assume that $f(x)$ can be written as:

$$f(x) = \sum_{n=0}^{\infty} a_n \sin((2n+1)\pi x/(2L))$$

give an expression for the a_n in terms of an integral involving $f(x)$.

Solution:

$$a_n = \frac{\langle f(x), \sin((2n+1)\pi x/(2L))\rangle}{\langle \sin((2n+1)\pi x/(2)), \sin((2n+1)\pi x/(2L))\rangle}$$
$$= \frac{2}{L}\int_0^L f(x)\sin((2n+1)\pi x/(2L))dx$$

8. A general form for the solution to the heat equation on a rod of length L with both ends insulated is:

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos(\frac{n\pi x}{L})e^{-(\frac{n\pi}{L})^2 t}$$

(a) What is the equilibrium solution?

Solution: Take limits:

$$\lim_{t\to\infty} u(x,t) = a_0$$

because

$$\lim_{t\to\infty} e^{-(\frac{n\pi}{L})^2 t} = 0$$

unless $n = 0$

(b) Consider the solutions:

$$u_1(x,t) = \cos(\pi x/L)e^{-(\pi/L)^2 t}$$
$$u_2(x,t) = \cos(3\pi x/L)e^{-(3\pi/L)^2 t}$$

Which one goes to the equilibrium solution faster? Why?

Solution: The magnitude of the constant multiplying t in $u_2(x,t)$ is much larger than that in $u_1(x,t)$ so it will go to zero faster

9. Find the general solution to the wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ with free ends: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$

Solution: Step 1: $L_t = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ and $L_x = \frac{\partial^2}{\partial x^2}$

Step 2: The eigenvalue problems are

$$p'' = \lambda p, \qquad p'(0) = p'(L) = 0$$
$$q'' = c^2\lambda q$$

Step 3: The solution to the spatial eigenvalue problem is

$$p_n(x) = \cos(n\pi x/L) \qquad \lambda = -(n\pi/L)^2$$

Step 4: The time eigenvalue problem is

$$q_n'' = -(c\frac{n\pi}{L})^2 q_n$$

So the solution is

$$q_n(t) = a_n \cos(cn\pi t/L) + b_n \sin(cn\pi t/L)$$

Step 5: The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n p_n(x) q_n(t) = \sum_{n=1}^{\infty} \cos(n\pi x/L)[a_n \cos(cn\pi t/L) + b_n \sin(cn\pi t/L)]$$

Because no initial conditions were provide, we cannot find a particular solution.

10. Find the general solution to the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + R(x)$ on a rod $x \in [0, \pi]$ with insulated endpoints and $R(x) = \cos(x)$.

Solution: The PDE is non-homogenous s so the solution is of the form

$$u(x,t) = u_{eq}(x) + w(x,t)$$

Where $u_{eq}(x)$ is the equilibrium solution to the PDE and $w(x,t)$ is the solution to the homogeneous problem.

The solution to the equilibrium problem $0 = \frac{d^2 u}{dx^2} + \cos(x)$ is

$$u(x) = \cos(x) + c_1 x + c_2$$

$$u'(0) = -\sin(0) + c_1 = 0 \implies c_1 = 0$$

$$u(x) = \cos(x) + c_2$$

Because we only have information about the derivative, we cannot determine the value of c_2.

It is shown in the book that

$$w(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-n^2 t} \cos(nx)$$

So the solution is

$$u(x,t) = \cos(x) + a_0 + \sum_{n=1}^{\infty} e^{-n^2 t} \cos(nx)$$

where the c_2 constant has been absorbed into a_0.

11. Consider a polluted lake modeled as a one dimensional medium on a domain $[0, 100]$ in meters with pollution density $u(x,t)$. The lake contains no currents and has no inlets or outlets so there are zero-flux boundaries. The pollution both diffuses through the lake and degrades:

$$u_t = u_{xx} - u$$

at time $t = 0$ the pollution density is $u(x, 0) = 1 + \cos(\frac{\pi}{50}x)$. Find the exact solution.

Solution: Step 1: $L_t = \frac{\partial}{\partial t}$ and $L_x = \frac{\partial^2}{\partial x^2} - \alpha$

Step 2: The eigenvalue problems are

$$p'' - p = \lambda p, \qquad p'(0) = p'(100) = 0$$

$$q' = \lambda q$$

Step 3: The solution to the spatial eigenvalue problem is

$$p_n(x) = \cos(n\pi x/L) \qquad n = -(n\pi/L)^2$$

Step 4: The time eigenvalue problem is

$$q_n'' = -(c\frac{n\pi}{L})^2 q_n$$

So the solution is

$$q_n(t) = a_n \cos(cn\pi t/L) + b_n \sin(cn\pi t/L)$$

Step 5: The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n p_n(x) q_n(t) = \sum_{n=1}^{\infty} \cos(n\pi x/L)[a_n \cos(cn\pi t/L) + b_n \sin(cn\pi t/L)]$$

Because no initial conditions were provide, we cannot find a particular solution.

12. Use d'Alembert's formula to determine the solution to the equation

$$u_{tt} - u_{xx} = 0,$$

with initial position $u(x,0) = \sin(x)$ and initial velocity $u_t(x,0) = x^2$. Plot the solution over the interval $x \in [0,\pi]$ at various times $t > 0$.

Solution: The solution must have the form

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(v)dv.$$

Substituting in the known functions $f(x)$ (initial position) and $g(x)$ (initial velocity) as well as the wave speed $c = 1$ we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\sin(x+t) + \sin(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} v^2 dv \\ \Rightarrow u(x,t) &= \frac{1}{2}\left[\sin(x+t) + \sin(x-t)\right] + \frac{1}{6}v^3\Big|_{x-t}^{x+t} \\ \Rightarrow u(x,t) &= \frac{1}{2}\left[\sin(x+t) + \sin(x-t)\right] + \frac{1}{6}\left((x+t)^3 - (x-t)^2\right) \\ \Rightarrow u(x,t) &= \sin(x)\cos(t) + x^2 t + \frac{1}{3}t^3. \end{aligned}$$

The last step is just a simplification where we have applied the sum to product trig identity, expanded the polynomials, and collected like terms.

It is good practice to plot the solution at multiple time points as well.