Name:____ Quiz Score:____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the box to the left of the question. If there are any other issues, please ask the instructor.

10 1. Let $u(x,t)$ represent the population of fishes in a river described by $x \in [0,1]$. The population of fish can do two things: diffuse (move) around and die at a rate proportional to how many fishes there are. Consequently, the PDE that describes their spatial population is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u.$$

Fish (obviously) can't swim in or out of the river, so take the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0.$$

Using separation of variables, find the solution to the PDE subject to the initial conditions

$$u(x,0) = 2 + \cos \pi x.$$

Hint: the spatial ODE eigenvalue problem should be the same as the classic heat equation, for which the eigenfunctions/values are $\cos(n\pi x/L)$ and $-(\pi n/L)^2$ respectively and you can just take this to be a fact.

Solution: As with all separation of variables problems, we start with the guess

$$u(x,t) = p(x)q(t).$$

Plugging this into our PDE, we get

$$p\dot{q} = p''q + pq, \qquad \dot{q} := \frac{\mathrm{d}q}{\mathrm{d}t}, \qquad p' := \frac{\mathrm{d}p}{\mathrm{d}x}.$$

Thus, we can separate this into

$$\frac{\dot{q}}{q} = \frac{p'' + p}{p} = \frac{p''}{p} - 1,$$

but the hint suggests we want the spatial (p) ODE to be the same as the heat equation, were we had p''/p. Thus, move the 1 over to yield

$$\frac{\dot{q}}{q} + 1 = \frac{p''}{p} = \lambda,$$

where we know that if $f(t) = g(x)$ then they must both be equal to some constant, which we can call λ.

Thus, we're left with the spatial eigenvalue problem

$$p''(x) = \lambda p, \qquad p'(0) = p'(1) = 0.$$

We know, from class, this is exactly the same eigenvalue problem as the no-flux heat equation, which has eigenvalues and eigenfunctions

$$\lambda_n = -\left(\frac{\pi n}{L}\right)^2, \qquad p_n(x) = \cos\frac{\pi n x}{L},$$

but here we have $L = 1$, so we've got the eigenvalues and eigenfunctions

$$\lambda_n = -(\pi n)^2, \qquad p_n(x) = \cos(\pi n x).$$

We should note that we also have $\lambda = 0$, with $p_0(x) =$ constant as the corresponding eigenfunction. Now we turn to the q ODE, which says

$$\dot{q} + q = \lambda q,$$

which, rearranged, yields

$$\frac{\dot{q}}{q} = (\lambda - 1),$$

which has solutions

$$q_n(t) = e^{(\lambda_n - 1)t} = e^{-\left[(\pi n)^2 + 1\right]t}.$$

Notice that this form makes sense. The only difference from our heat equation solution is that this one decays a little bit faster. This makes sense since we added a decay term to the PDE.

Now we know our eigenfunctions, so we combine them in a linear combination (because we currently don't know which n's to include), so we see our general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n q_n(t) p_n(x) = A_0 e^{-t} + \sum_{n=1}^{\infty} A_n \cos(n\pi x) e^{-\left[(\pi n)^2 + 1\right]t}.$$

Note that I've pulled out the $n = 0$ case to make it more explicit, but this wasn't completely necessary. Now, to determine the A_n's we need to use the initial condition. The initial condition says

$$u(x,0) = 2 + \cos \pi x,$$

but this has to be equal to our solution at $t = 0$, which yields

$$2 + \cos \pi x = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = A_0 + A_1 \cos(\pi x) + A_2 \cos(2\pi x) + \cdots .$$

It's clear from this that $A_0 = 2$, $A_1 = 1$ and $A_{\geq 2} = 0$. Putting this back into our general (time dependent) solution, yields the final solution to the PDE

$$u(x,t) = 2e^{-t} + \cos(\pi x) e^{-\left[\pi^2 + 1\right]t}.$$

Note that as $t \to \infty$, this decays to 0, which makes sense since no fish are fluxing into the river but they are dying at some rate.