Math 3150 – HW 5 Solutions

June 11, 2018

Here are problems that would be on the last homework assignment.

1. 4.21 (good question)

   Find the particular solution to the heat equation on a rod $x \in [0, \pi]$ with $D = 1$ and insulated endpoints with a source $R/K_c = \cos(x)$ and initial condition $f(x) = -\cos(x)$.

Solution: The heat equation with the given specifications is written

$$u_t = u_{xx} + \cos(x),$$

with no-flux boundary conditions

$$u'(0, t) = u'(\pi, t) = 0.$$

The boundary conditions are homogeneous, but the source term means that the equation is not. So, since the problem is non-homogeneous, the solution must be of the form

$$u(x, t) = w(x, t) + u_{eq}(x),$$

where $w(x, t)$ is the solution to the homogeneous problem

$$w_t = w_{xx}, \quad w'(0, t) = w'(\pi, t) = 0$$

and $u_{eq}(x)$ is the solution to the equilibrium problem

$$0 = u''_{eq}(x) + \cos(x), \quad u'_{eq}(0) = u'_{eq}(\pi) = 0.$$

First, we find the general solution to the homogeneous problem, which we have solved before with the no-flux BCs in previous problems:

$$w(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-n^2 t}.$$

Next, we solve the ODE for the equilibrium problem by direct integration:

$$\begin{aligned} u''_{eq}(x) &= -\cos(x) \\ \Rightarrow u'_{eq}(x) &= -\sin(x) + c_1 \\ \Rightarrow u_{eq}(x) &= \cos(x) + c_1 x + c_2. \end{aligned}$$

Applying the boundary conditions, we get

$$\begin{aligned} u'_{eq}(0) &= -\sin(0) + c_1 = 0 \Rightarrow c_1 = 0, \\ u'_{eq}(\pi) &= -\sin(\pi) + c_1 = 0 \Rightarrow c_1 = 0. \end{aligned}$$

We do not yet have enough information with the boundary conditions to solve for c_2, so we leave it unknown for now.

Combining the two solutions gives us the general solution to our full nonhomogeneous problem

$$u_{gen}(x, t) = \cos(x) + c_2 + a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-n^2 t}.$$

Notice that since c_2 and a_0 are both constants, we could combine them together into one new constant A_0:

$$u_{gen}(x,t) = \cos(x) + A_0 + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-n^2 t}.$$

To solve for the unknown constants, we will now apply our initial condition.

$$u(x,0) = -\cos(x) = \cos(x) + A_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

We can rearrange the equation to look like a cosine series by subtracting $\cos(x)$ from both sides, which gives us

$$-2\cos(x) = A_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

We can now find the coefficients by inspection: in order to match terms on both sides, we need $a_1 = -2$ and $a_n = 0$ otherwise (including at A_0). Plugging in these coefficients gives us the particular solution:

$$u(x,t) = \cos(x) - 2\cos(x)e^{-t} = \cos x \left(1 - 2e^{-t}\right).$$

2. 5.9

Consider a string with fixed zero ends of length $L = 1$ with speed parameter $c = 1$ and initial position

$$u(x,0) = f(x) = \begin{cases} 2x, & x \in [0, 1/2] \\ 2(1-x), & x \in (1/2, 1] \end{cases}$$

and zero initial velocity $u_t(x,0) = 0$. Find the exact solution.

Solution: This string satisfies the equation

$$u_{tt} = u_{xx},$$

which we can solve with separation of variables. Plugging in the separation of variables ansatz $u(x,t) = p(x)q(t)$ and separating variables gives

$$\frac{q''(t)}{q(t)} = \frac{p''(x)}{p(x)} = \lambda.$$

First, we solve the eigenvalue problem associated with the spatial variable, which has solutions of the form

$$p(x) = A_n \cos(\sqrt{-\lambda}x) + B_n \sin(\sqrt{-\lambda}x),$$

with the associated boundary conditions $p(0) = p(1) = 0$. Applying these boundary conditions, we get

$$p(0) = A_n = 0,$$
$$p(1) = B_n \sin(\sqrt{-\lambda}) = 0 \Rightarrow \lambda = -(n\pi)^2.$$

Thus, the eigenfunctions are of the form $p_n(x) = \sin(n\pi x)$.

Next, we solve the ODE for the spatial variable $q_n''(t) = -(n\pi)^2 q(t)$, which has solutions of the form

$$q_n(t) = a_n \cos(n\pi t) + b_n \sin(n\pi t).$$

Combining these, we get the general solution

$$u_{gen}(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x) \left[a_n \cos(n\pi t) + b_n \sin(n\pi t)\right].$$

Now we find the particular solution by applying the initial conditions. The initial position satisfies

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x).$$

We can find the coefficients a_n via orthogonal projection/sine series:

$$a_n = \frac{\langle f(x), \sin(n\pi x)\rangle}{\langle \sin(n\pi x), \sin(n\pi x)\rangle} = 2\left[\int_0^{1/2} 2x\sin(n\pi x)dx + \int_{1/2}^{1} 2(1-x)\sin(n\pi x)dx\right]$$

$$= 4\left[-\frac{x}{n\pi}\cos(n\pi x)\Big|_0^{1/2} + \frac{1}{n\pi}\int_0^{1/2}\cos(n\pi x)dx - \frac{(1-x)}{n\pi}\cos(n\pi x)\Big|_{1/2}^{1} - \frac{1}{n\pi}\int_{1/2}^{1}\cos(n\pi x)dx\right]$$

$$= 4\left[-\frac{\cos(n\pi)}{n\pi} + \frac{1}{(n\pi)^2}\sin(n\pi x)\Big|_0^{1/2} - \frac{1}{(n\pi)^2}\sin(n\pi x)\Big|_{1/2}^{1}\right]$$

$$= \frac{4}{n\pi}\left[\frac{2}{n\pi}\sin(n\pi/2) - \cos(n\pi)\right].$$

Now, we find the coefficients b_n by applying the initial velocity:

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} b_n n\pi \sin(n\pi x),$$

which can only be satisfied if $b_n = 0$.

Plugging in our known coefficients gives us the particular solution:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\left[\frac{2}{n\pi}\sin(n\pi/2) - \cos(n\pi)\right]\sin(n\pi x)\cos(n\pi t)$$

or equivalent.

3. 4.24

Solution: See solution in back.

4. 5.10

Solve the wave equation with gravitation

$$u_{tt} = u_{xx} - g$$

with zero initial position and velocity $u(x,0) = u_t(x,0) = 0$, using the method of nonhomogeneous PDEs.

Solution: This nonhomogeneous PDE has a solution of the form

$$u(x,t) = w(x,t) + u_{eq}(x),$$

where $u_{eq}(x)$ is the equilibrium solution and $w(x,t)$ is the solution to the homogeneous problem. The solution to the homogeneous problem can be found via separation of variables as in the previous problem, and it is

$$w(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)\left[a_n\cos(\frac{n\pi}{L}t) + b_n\sin(\frac{n\pi}{L}t)\right].$$

The equilibrium solution satisfies the equation

$$u''_{eq}(x) = g,$$

and integrating twice yields the solution

$$u_{eq}(x) = \frac{g}{2}x^2 + c_1 x + c_2.$$

Applying boundary conditions gives

$$\begin{aligned} u_{eq}(0) = 0 &= c_2, \\ u_{eq}(L) = 0 &= \frac{g}{2}L^2 + c_1 L \Rightarrow c_1 = -\frac{gL}{2}. \end{aligned}$$

Combining these two solutions gives the general nonhomogeneous solution

$$u_{gen}(x,t) = \frac{g}{2}x(x-L) + \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)\left[a_n \cos(\frac{n\pi}{L}t) + b_n \sin(\frac{n\pi}{L}t)\right].$$

Now we apply the initial conditions to find the values for constants a_n and b_n. First, we apply the initial position, which satisfies

$$\begin{aligned} u(x,0) = 0 &= \frac{g}{2}x(x-L) + \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{L}x) \\ \Rightarrow \frac{g}{2}x(L-x) &= \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{L}x). \end{aligned}$$

We find a_n via orthogonal projection/sine series:

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L \frac{g}{2}x(L-x)\sin(\frac{n\pi}{L}x)dx = \frac{g}{L}\int_0^L x(L-x)\sin(\frac{n\pi}{L}x)dx \\ &= \frac{g}{L}\left[-x(L-x)\frac{L}{n\pi}\cos(\frac{n\pi}{L}x)\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos(\frac{n\pi}{L}x)(L-2x)dx\right] \\ &= \frac{g}{n\pi}\left[(L-2x)\frac{L}{n\pi}\sin(\frac{n\pi}{L}x)\Big|_0^L + 2\int_0^L \frac{L}{n\pi}\sin(\frac{n\pi}{L}x)dx\right] \\ &= \frac{2g}{n\pi}\left[\left(\frac{L}{n\pi}\right)^2 \cos(\frac{n\pi}{L}x)\Big|_0^L\right] \\ &= \frac{2gL^2}{(n\pi)^3}\left[\cos(n\pi) - 1\right]. \end{aligned}$$

Now, we apply the initial velocity, which satisfies

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} b_n \frac{n\pi}{L}\sin(\frac{n\pi}{L}x),$$

which can only be true if $b_n = 0$.

Now we can substitute in the known coefficients to get the particular solution:

$$u(x,t) = \frac{g}{2}x(x-L) + \sum_{n=1}^{\infty} \frac{2gL^2}{(n\pi)^3}\left[\cos(n\pi) - 1\right]\sin(\frac{n\pi}{L}x)\cos(\frac{n\pi}{L}t).$$

5. 5.11 (decent question)

Solution: See solution in back.