Math 3150 – HW 4 Solutions

June 13, 2018

4.2, 4.3, 4.14(e), 4.15*(b)*,(d), 4.18, 4.19

1. (4.2)

   Consider the equilibrium diffusion equation for the mass density $u(x)$ with no sources and no-flux boundary conditions on the domain $x \in [0, 100]$. Suppose there is 10 kg total of material in the domain. Determine the solution $u(x)$.

   Solution: The heat equation in equilibrium with no sources/sinks satisfies

   $$u''(x) = 0,$$
   $$\Rightarrow u'(x) = c_1,$$
   $$\Rightarrow u(x) = c_1 x + c_2.$$

   Now, we apply the no-flux boundary conditions to find the unknown coefficients:

   $$u'(0) = 0 = c_1.$$
   $$u'(100) = 0 = c_1.$$

   From the boundary conditions, we can only see that the temperature must be constant throughout the domain, i.e.

   $$u(x) = c_2.$$

   To find c_2, we apply the additional piece of information that there is a total of 10 kg of material in the domain. This means that

   $$U = 10 = \int_0^{100} u(x)dx = \int_0^{100} c_2 dx,$$

   so $c_2 = 1/10$. Therefore, the equilibrium solution is

   $$u(x) = \frac{1}{10}.$$

2. (4.3)

   Consider the standard equilibrium heat equation $u_t = 0 = u_{xx}$ on $x \in [0, 1]$. Given the following parameter values and boundary conditions, determine (1) the equilibrium solution and draw a graph, and (2) compute the flux ϕ and indicate on the solution graph the magnitude and direction of the flux. Alternatively, specify if the solution does not exist or is not fully determined.

   (a) u(0)=10, u(1) = 21.

   Solution: The general form for the solution to the given heat equation at equilibrium is

   $$u''(x) = 0$$
   $$\Rightarrow u'(x) = c_1$$
   $$\Rightarrow u(x) = c_1 x + c_2,$$

   and the flux is $\phi(x) = -\frac{du}{dx} = -c_1$. This general solution and flux will be used for all subsequent parts.

For this part, apply the boundary conditions to see that $u(0) = 10 = c_2$, and $u(1) = 21 = c_1 + 10 \Rightarrow c_1 = 11$. So, the equilibrium solution is

$$u(x) = 11x + 10$$

and the flux is

$$\phi(x) = -11.$$

That is, flux is to the left with magnitude 11.

(b) u(0)=2, u'(0)=0.

Solution: Applying the boundary conditions give

$$u(0) = 2 = c_2,$$
$$u'(0) = 0 = c_1.$$

So, the equilibrium solution is

$$u(x) = 2$$

and the flux is $\phi(x) = 0$, no flux.

(c) u'(0) = 4, u(1)=0

Solution: Applying the boundary conditions give

$$u'(0) = 4 = c_1,$$
$$u(1) = 0 = 4 + c_2 \Rightarrow c_2 = -4.$$

So, the equilibrium solution is

$$u(x) = 4x - 4$$

and the flux is $\phi(x) = -4$, that is, flux is to the left with magnitude 4.

(d) u'(0)=3, u'(1)=3

Solution: Applying the boundary conditions give

$$u'(0) = 3 = c_1,$$
$$u'(1) = 3 = c_1$$

So, the equilibrium solution is not fully determined, but has the form

$$u(x) = 3x + c_2,$$

and the flux is $\phi(x) = -3$, that is, flux is to the left with magnitude 3.

(e) u'(0)=10, u'(1)=0

Solution: Applying the boundary conditions give

$$u'(0) = 10 = c_1,$$
$$u'(1) = 0 = c_1.$$

The equilibrium solution does not exist, because both boundary conditions cannot be satisfied at the same time.

(f) u(0)=u(1), u'(0)=u'(1)

Solution: Applying the boundary conditions give

$$u(0) = u(1) \Rightarrow c_2 = c_1 + c_2,$$
$$u'(0) = u'(1) \Rightarrow c_1 = c_1.$$

Equation (2) is always satisfied, but Equation (1) can only be satisfied if $c_1 = 0$. So, the equilibrium solution cannot be fully determined, although it has the form

$$u(x) = c_2$$

and the flux is $\phi(x) = 0$, that is, no flux.

3. (4.14(e))

Find particular solutions to the heat equation on a rod $x \in [0, \pi]$ with $D = 1$ and zero temperature endpoints given the initial conditions $f(x) = x$.

Solution: The heat equation with the given specifications is

$$u_t = u_{xx}$$

with boundary conditions

$$u(0, t) = u(\pi, t) = 0.$$

We solve this problem using separation of variables, and we begin by assuming that the solution has the form $u(x, t) = p(x)q(t)$. Plugging this assumption into the equation and separating variables gives

$$\frac{q'(t)}{q(t)} = \frac{p''(x)}{p(x)} = \lambda.$$

First, we solve the eigenvalue problem that is associated with the spatial variable

$$p''(x) = \lambda p(x).$$

In order to satisfy the boundary conditions, we know that we need $\lambda < 0$, so solutions to this eigenvalue problem must have the form

$$p(x) = a_n \cos(\sqrt{-\lambda} x) + b_n \sin(\sqrt{-\lambda} x).$$

Applying the boundary conditions $p(0) = p(\pi) = 0$, we get

$$p(0) = 0 = a_n,$$
$$p(\pi) = 0 = b_n \sin(\sqrt{-\lambda}\pi).$$

We can satisfy these boundary conditions if we take $\lambda = -n^2$, where $n = 1, 2, 3, \ldots$ so our eigenfunctions have the form

$$p_n(x) = b_n \sin(nx), \quad n = 1, 2, 3, \ldots$$

Next, we solve the ODE that is associated with the time variable:

$$q'(t) = \lambda q(t) \Rightarrow q(t) = ce^{\lambda t} \Rightarrow q_n(t) = ce^{-n^2 t}.$$

We combine our work to get the general solution

$$u_{gen}(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}.$$

Finally, we can apply the initial condition to get the particular solution, so we must satisfy the equation

$$u(x,0) = x = \sum_{n=1}^{\infty} b_n \sin(nx).$$

We can find the unknown coefficients with orthogonal projection:

$$\begin{aligned}
b_n = \frac{\langle x, \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle} &= \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)dx\right] \\
&= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right] \\
&= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos(n\pi)\right] \\
&= -\frac{2}{n}(-1)^n \\
&= \frac{2}{n}(-1)^{n+1}
\end{aligned}$$

So, the particular solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)e^{-n^2 t}$$

or equivalent.

4. (4.15(d))

Find the particular solution to the heat equation on a rod $x \in [0, 2\pi]$ with $D = 4$ and insulated endpoints, given the initial condition $f(x) = \sum_{n=1}^{\infty} \frac{1}{(2n)^2}\cos(nx)$.

Solution: The heat equation with the given specifications is

$$u_t = 4u_{xx}$$

with boundary conditions

$$u'(0,t) = u'(2\pi,t) = 0.$$

We solve this problem using separation of variables, and we begin by assuming that the solution has the form $u(x,t) = p(x)q(t)$. Plugging this assumption into the equation and separating variables gives

$$\frac{q'(t)}{4q(t)} = \frac{p''(x)}{p(x)} = \lambda.$$

First, we solve the eigenvalue problem that is associated with the spatial variable

$$p''(x) = \lambda p(x).$$

In order to satisfy the boundary conditions, we know that we need $\lambda < 0$, so solutions to this eigenvalue problem must have the form

$$p(x) = a_n\cos(\sqrt{-\lambda}x) + b_n\sin(\sqrt{-\lambda}x).$$

Applying the boundary conditions $p(0) = p(2\pi) = 0$, we get

$$\begin{aligned}
p'(0) &= 0 = b_n, \\
p'(2\pi) &= 0 = -a_n\sqrt{-\lambda}\sin(\sqrt{-\lambda}2\pi).
\end{aligned}$$

We can satisfy these boundary conditions if we take $\lambda = -\frac{n}{2}^2$, where $n = 0, 1, 2, 3, \ldots$ so our eigenfunctions have the form

$$p_n(x) = a_n \cos(\frac{n}{2}x), \quad n = 0, 1, 2, 3, \ldots$$

Next, we solve the ODE that is associated with the time variable:

$$q'(t) = 4\lambda q(t) \Rightarrow q(t) = ce^{4\lambda t} \Rightarrow q_n(t) = ce^{-4\left(\frac{n}{2}\right)^2 t}.$$

We combine our work to get the general solution

$$u_{gen}(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n}{2}x)e^{-n^2 t}.$$

Finally, we can apply the initial condition to get the particular solution, so we must satisfy the equation

$$u(x, 0) = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} \cos(kx) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n}{2}x),$$

where we have used k as the indexing variable on the lefthand side so that we don't confuse it with the indexing variable n on the righthand side. We can find the unknown coefficients by inspection (matching terms). First, in order to match the lefthand and the righthand side, we must set $a_0 = 0$. Now, we notice that for n odd, there are no corresponding terms on the lefthand side, since all the cosine terms are integer multiples of x. Therefore, we can conclude that $a_n = 0$ for n odd. Then, for the even terms, we cover all the integer multiples of x and we can explicitly match the given coefficients to get $a_n = \frac{1}{(2n)^2}$ for n even. Plugging these values in to our general solution gives the particular solution

$$u(x, t) = \sum_{n \text{ even}} \frac{1}{(2n)^2} \cos(\frac{n}{2}x)e^{-n^2 t}$$

or equivalent.

5. (4.18)

   Consider the advection on a ring system in Example 4.12, but include a degradation term $-\alpha u$:

$$u_t = -vu_x - \alpha u,$$

   where $\alpha > 0$ specifies the rate of degradation. Let $v = 2\pi$ and $\alpha = 0.1$, and let the initial condition be $f(x) = 1 + \cos(x)$. Find the exact solution.

Solution: Begin with the separation of variables ansatz $u(x, t) = p(x)q(t)$. Plugging this into the PDE and separating variables yields

$$p(x)q'(t) = -vq(t)p'(x) - \alpha p(x)q(t)$$
$$\Rightarrow \frac{q'(t)}{q(t)} = \frac{-vp'(x) - \alpha p(x)}{p(x)}.$$

The spatial problem is

$$p'(x) = -\frac{\lambda + \alpha}{v} p(x),$$

with boundary condition $p(-\pi) = p(\pi)$. The function $p(x)$ has the form

$$p(x) = Ce^{-\frac{\lambda+\alpha}{v}x}.$$

Applying the boundary conditions shows us that

$$p(-\pi) = Ce^{\frac{\lambda+\alpha}{v}\pi}, \text{ and } p(\pi) = Ce^{-\frac{\lambda+\alpha}{v}\pi},$$

So we must have

$$Ce^{\frac{\lambda+\alpha}{v}\pi} = Ce^{-\frac{\lambda+\alpha}{v}\pi},$$

which is true either for $C = 0$ (trivial solution) or when we have imaginary eigenvalues $\frac{\lambda+\alpha}{v} = in$, for $n = 0, 1, 2, \ldots$ which implies that we have eigenvalues $\lambda_n = inv - \alpha$ with associated eigenfunctions

$$p_n(x) = e^{-inx}, \quad n = 0, 1, 2, \ldots$$

Now we solve the time-dependent ODE:

$$q_n'(t) = \lambda_n q_n(t) = (inv - \alpha)\, q_n(t)$$
$$\Rightarrow q_n(t) = e^{(inv-\alpha)t}.$$

Combining these gives us our general solution

$$u_{gen}(x,t) = \sum_{n=0}^{\infty} c_n e^{-inx} e^{(inv-\alpha)t} = \sum_{n=0}^{\infty} c_n e^{-\alpha t} e^{-in(x-vt)}$$
$$= \sum_{n=0}^{\infty} e^{-\alpha t} \left(a_n \cos(n(x-vt)) + b_n \sin(n(x-vt))\right).$$

Plugging in $v = 2\pi$ and $\alpha = 0.1$, this is

$$u_{gen}(x,t) = \sum_{n=0}^{\infty} e^{-0.1t} \left(a_n \cos(n(x-2\pi t)) + b_n \sin(n(x-2\pi t))\right).$$

Now, we must satisfy the initial condition:

$$u(x,0) = 1 + \cos(x) = \sum_{n=0}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

It is clear by inspection that this equation is exactly satisfied for $a_0 = 1$, $a_1 = 1$, and all other constants set equal to 0. So, our particular solution is

$$u(x,t) = e^{-0.1t} + e^{-0.1t}\cos(x-2\pi t) = e^{-0.1t}\left(1 + \cos(x-2\pi t)\right).$$

So, the mass density is moving with velocity $v = 2\pi$ around the ring, but as it moves it is also slowly spreading out due to the e^{-01t} factor.

6. (4.19)

Consider a polluted lake modeled as a one-dimensional medium on a domain $x \in [0, 100]$ in meters, with pollution density $u(x,t)$. The lake contains no currents and has no inlets or outlets, so there are zero-flux boundaries. The pollution both diffuses through the lake and degrades:

$$u_t = Du_{xx} - \alpha u,$$

where $\alpha = 0.01 > 0$ specifies the rate of degradation and $D = 0.1$. At time $t = 0$, the pollution density is $u(x,0) = 1 + \cos\left(\frac{\pi}{50}x\right)$. Find the exact solution.

Solution: First, we begin with the ansatz $u(x,t) = p(x)q(t)$. Separating variables gives us

$$p(x)q'(t) = Dq(t)p''(x) - \alpha p(x)q(t)$$
$$\Rightarrow \frac{p''(x)}{p(x)} = \frac{q'(t) + \alpha q(t)}{Dq(t)} = \lambda.$$

Therefore the spatial eigenvalue problem is

$$p''(x) = \lambda p(x)$$

with no-flux boundary conditions $p'(0) = 0$ and $p'(100) = 0$. Solutions to this problem are of the form

$$p(x) = a_n \cos(\sqrt{-\lambda}x) + b_n \sin(\sqrt{-\lambda}x),$$
$$\Rightarrow p'(x) = -a_n\sqrt{-\lambda}\sin(\sqrt{-\lambda}x) + b_n\sqrt{-\lambda}\cos(\sqrt{-\lambda}x)$$

Applying the boundary conditions, we get

$$p'(0) = b_n\sqrt{-\lambda} = 0 \Rightarrow b_n = 0, \text{ and}$$
$$p'(100) = -a_n\sqrt{-\lambda}\sin(\sqrt{-\lambda}100) = 0, \quad n = 0, 1, 2, \ldots$$

The second equation can be satisfied nontrivially if

$$\sqrt{-\lambda}100 = n\pi, \quad \Rightarrow \lambda_n = -\left(\frac{n\pi}{100}\right)^2.$$

These eigenvalues have associated eigenfunctions

$$p_n(x) = \cos(\frac{n\pi}{100}x).$$

Now we solve the equation for $q(t)$:

$$q_n'(t) = (-D\lambda_n - \alpha)\, q(t) \Rightarrow q_n(t) = c_n e^{(-D\lambda_n - \alpha)t} = c_n e^{\left(D\left(\frac{n\pi}{100}\right)^2 - \alpha\right)t}.$$

Therefore our general solution is

$$u_{gen}(x,t) = \sum_{n=0}^{\infty} a_n e^{\left(D\left(\frac{n\pi}{100}\right)^2 - \alpha\right)t} \cos\left(\frac{n\pi}{100}x\right),$$

and plugging in $\alpha = 0.1$ and $D = 0.1$ we have

$$u_{gen}(x,t) = \sum_{n=0}^{\infty} a_n e^{\left(0.1\left(\frac{n\pi}{100}\right)^2 - 0.1\right)t} \cos\left(\frac{n\pi}{100}x\right),$$

To find the exact solution, we apply the initial condition:

$$u(x,0) = 1 + \cos\left(\frac{\pi}{50}x\right) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{100}x\right),$$

which once again we can see by inspection that this equation is satisfied if $a_0 = 1$, $a_2 = 1$, and $a_n = 0$ otherwise. Plugging in these coefficients yields our exact solution:

$$u(x,t) = e^{-0.1t} + e^{\left(0.1\left(\frac{\pi}{50}\right)^2 - 0.1\right)t} \cos\left(\frac{\pi}{50}x\right).$$