Math 3150 – HW 2 Solutions

May 28, 2018

2.10, 2.19 fg, 2.20 fg, 2.22, 2.37, 2.42, 2.51, 2.56

1. 2.10

   Solution: The solution to $Lu = u'''$ is always of the form $u = ax^3 + bx^2 + cx + d$ (by integrating a bunch of times), so then we just apply the different boundary conditions to find a, b, c, d.

   Solution in back of the book.

2. 2.19fg

   Solution: For these, we basically just solve the ODE for the eigenvalue equation $Lu = \lambda u$, with λ provided. This question is a little stupid because we would *never* do this without boundary conditions.

   Solution in back of the book.

3. 2.20fg

   Solution: Much more reasonable than the last problem. Basically, we solve $Lu = \lambda u$ for the provided eigenvalue λ, find u, and then apply the boundary conditions. We're verifying that λ and its corresponding u really is an eigenvalue/eigenfunction of our operator (with boundary conditions!).

   Solution in back of the book.

4. 2.22

   Solution:

   In my opinion, the probably most important problem on this homework assignment. Note that we're provided a differential operator L *and* boundary conditions. Both are critical in determinining the eigenvalues. If I changed either, we would totally change the eigenvalues.

   There is a detailed solution in the back of the book. I would definitely check this out to prepare for the quiz!

5. 2.37

   Find a constant a so that $p_1(x) = 1$ and $p_2(x) = e^x + a$ are orthogonal on the interval $x \in [0, 1]$ with respect to the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

   Solution: The functions $p_1(x)$ and $p_2(x)$ are orthogonal if

   $$\int_0^1 (e^x + a)\, dx = 0,$$

   which is true if

   $$(e^x + ax)\Big|_0^1 = e + a - 1 = 0.$$

   Therefore, we require

   $$a = 1 - e.$$

6. 2.42

   Consider $f(x) = x^3 - x$ and $g(x) = x^2 - 1$ on $x \in [-1, 1]$.

   (a) Verify that $f(x)$ is an odd function and $g(x)$ is an even function, meaning $f(-x) = -f(x)$ and $g(-x) = g(x)$.

   Solution: First, check that $f(x)$ is odd:

   $$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

   Next, check that $g(x)$ is even:

   $$g(-x) = (-x)^2 - 1 = x^2 - 1 = g(x).$$

   (b) Directly compute that $\langle f(x), g(x) \rangle = 0$.

   Solution:

   $$\begin{aligned} \int_{-1}^{1} (x^3 - x)(x^2 - 1)dx &= \int_{-1}^{1} (x^5 - 2x^3 + x)dx \\ &= x^6/6 - x^4/2 + x^2/2 \Big|_{-1}^{1} = \left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right) - \left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right) = 0. \end{aligned}$$

7. 2.51

Solution: I guess I accidentally assign a problem I don't mean to on each homework assignment. This is exactly using the annoying trig identities I tell you never to worry about. Here, we want to prove that $\cos nx$is orthogonal to $\cos mx$ for $n \neq m$. Computing their inner product, we have

$$\langle \cos nx, \cos mx \rangle = \int_0^{\pi} \cos(nx)\cos(mx)\,\mathrm{d}x.$$

Here, the trig identity we'll use is

$$\cos A \cos B = \frac{1}{2}\left\{\cos(A + B) + \cos(A - B)\right\},$$

so our integral becomes

$$\begin{aligned} \int_0^{\pi} \cos(nx)\cos(mx)\,\mathrm{d}x &= \int_0^{\pi} \cos[(m+n)x] + \cos[(m-n)x]\,\mathrm{d}x \\ &= \frac{\sin(m+n)x}{2(m+n)}\Big|_{x=0}^{x=\pi} + \frac{\sin(m-n)x}{2(m-n)}\Big|_{x=0}^{x=\pi}. \end{aligned}$$

Here, we know that since m, n are integers, $m - n$ is also an integer. However, $\sin(0) = 0$ and $\sin(k\pi) = 0$, where k is an integer. Therefore, we can immediately conclude orthogonality

$$\langle \cos nx, \cos mx \rangle = 0.$$

If you wanted, you could also prove that

$$\langle \cos nx, \cos nx \rangle \neq 0,$$

which we would use the trig identity

$$\cos^2 nx = \frac{1 + \cos 2nx}{2},$$

but the details are really boring and not really worth doing since we know inner products have the property that

$$\langle u, u \rangle = 0$$

if and only if $u = 0$.

8. 2.56

   Compute the orthogonal projection $\hat{f}$ using the first four Haar wavelets that approximates $f(x) = |x - \frac{1}{2}|$.

Solution: Call the Haar wavelets $\{\psi_1, \psi_2, \psi_3, \psi_4\}$. Then the least squares approximation for f is

$$\hat{f} = a\psi_1 + b\psi_2 + c\psi_3 + d\psi_4.$$

Now it remains to calculate the coefficients a, b, c, d using orthogonal projection. This is pretty tedious. I hope you use (and encourage) some sort of computer assistance for annoying integrals like Mathematica or Maple. Happy to demonstrate how to use these if requested!

Calculate a:

$$a = \frac{\int_0^1 |x - \frac{1}{2}| dx}{\int_0^1 1^2 dx} = \int_0^{1/2} (1/2 - x) dx + \int_{1/2}^1 (x - 1/2) dx$$

$$= 1/2x - x^2/2\Big|_0^{1/2} + x^2/2 - 1/2x\Big|_{1/2}^1 = (1/4 - 1/8) - (1/8 - 1/4) = (1/8 + 1/8) = 1/4.$$

Calculate b:

$$b = \frac{\int_0^{1/2} |x - \frac{1}{2}| dx + \int_{1/2}^1 (-1)|x - \frac{1}{2}| dx}{\int_0^1 1 dx} = \int_0^{1/2} (1/2 - x) dx + \int_{1/2}^1 (1/2 - x) dx$$

$$= \int_0^1 (1/2 - x) dx = 1/2x - x^2/2\Big|_0^1 = 0.$$

Calculate c:

$$c = \frac{\int_0^{1/4} |x - \frac{1}{2}| dx + \int_{1/4}^{1/2} (-1)|x - \frac{1}{2}| dx}{\int_0^{1/2} 1 dx} = \frac{1}{2}\left(\int_0^{1/4} (1/2 - x) dx + \int_{1/4}^{1/2} (x - 1/2) dx\right)$$

$$= \frac{1}{2}\left(1/2x - x^2/2\Big|_0^{1/4} + x^2/2 - 1/2x\Big|_{1/4}^{1/2}\right) = \frac{1}{2}(1/8 - 1/32 + 1/8 - 1/4 - 1/32 + 1/8)$$

$$= \frac{1}{2}(3/32 - 4/32 + 3/32) = 1/32.$$

Calculate d:

$$d = \frac{\int_{1/2}^{3/4} |x - \frac{1}{2}| dx + \int_{3/4}^1 (-1)|x - \frac{1}{2}| dx}{\int_{1/2}^1 1 dx} = \frac{1}{2}\left(\int_{1/2}^{3/4} (x - 1/2) dx + \int_{3/4}^1 (1/2 - x) dx\right)$$

$$= \frac{1}{2}\left((x^2/2 - 1/2x)\Big|_{1/2}^{3/4} + (1/2x - x^2/2)\Big|_{3/4}^1\right) = \frac{1}{2}(9/32 - 3/8 - 1/8 + 1/4 - 3/8 + 9/32)$$

$$= -1/32$$

Therefore, the approximation is

$$\hat{f} = 1/4\psi_1 + 1/32\psi_3 - 1/32\psi_4.$$