Math 3150 – HW 1 Solutions

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1.2, 1.3, 1.4 1.12, 1.14, 1.18, 1.22, 2.2, 2.4

1. 1.2

Solution:

(a) The starting point, and probably most important thing to kow here is the constituitive (continuity) equation for the density $u(x, t)$

$$\frac{\partial}{\partial t} u(x, t) = -\frac{\partial}{\partial x} \phi + R. \tag{1}$$

While this will serve more useful later, there is actually a more concrete version of the equation that deals with the *net* change in total stuff. Call

$$M(t) = \int_a^b u(x, t)\, \mathrm{d}x = \text{total stuff},$$

then, by integrating (1) (try this yourself!) with respect to x, we get the more useful equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \phi(a) - \phi(b) + \int_a^b R(x)\, \mathrm{d}x.. \tag{2}$$

(2) will be the one that we use for problems of this type.

For this part, we know it is in equilibrium, so $\partial_t u = 0$ or $\partial_t M = 0$. Therefore, plugging this into (2), we find

$$0 = \phi(a) - \phi(b) + \int_a^b R\, \mathrm{d}x.$$

In other words, the net flux balances the net sink/source. For this particular problem, we have

$$\phi(a) - \phi(b) = .0005\mathrm{m}^2 \cdot (-0.3\mathrm{m}/s) + .001\mathrm{m}^2 \cdot (0.1\mathrm{m/s}) = -.00005,$$

where the negative sign comes from the direction of the flow. Specifically, it flows out at the left and in at the right. Therefore, this must be balanced by the net source

$$\text{net source} = \int R\, \mathrm{d}x = .00005 > 0,$$

so indeed we have a source (not a sink).

(b) For part 2, we are not in equilibrium, but any difference is made up by the net rate of change. That is

$$\text{net rate of change} = \text{net source} + \text{net flux}$$

so we have the flux from the previous part (doesn't change)

$$.0001 = \text{net source} - .00005$$

and therefore

$$\text{net source} = .00015\mathrm{m}^3/\mathrm{s}.$$

Again, the net source is positive.

2. 1.3

Solution: This is just an application of the conservation principle. Again, we're in equilibrium, so $\frac{dM}{dt} = 0$ in (2), meaning

$$\text{net source} = \int_a^b R(x)\,dx = \int_0^2 \sin(\pi x) = 0.$$

Therefore, the net flux is also zero.

3. 1.4

Solution: Identitcal to the previous.

$$\text{net source} = \int_0^2 x(2-x)\,dx = 4/3,$$

and therefore in equilibrium

$$\text{net flux} = -\text{net flux} = -4/3,$$

and therefore the net flux is out.

4. 1.12

Solution:

(a) Here, because the question first asks about the how the *density* $u(x,t)$ is changing, we use equation (1), which says

$$\partial_t u = -\partial_x \phi + \underbrace{R}_{}0.$$

In this particular case,

$$\partial_t u = -2\pi \cos(2\pi x),$$

If we plot this, the behavior is easy to see. The density is staying the same when this $= 0$, which is at $x = 1/4, 2/4, 3/4$. The density is decreasing when $\partial_t u < 0$, which is between $(0, 1/4), (3/4, 5/4)$, and so on. Finally, the density is increasing in $(1/4, 3/4)$, etc.

(b) If the system is in equilibrium, then $R(x) = \partial_x \phi$, so $R(x) = 2\pi \cos(2\pi x)$ and therefore is a mixture of being a source and sink in different regions.

This problem is somewhat deep in the sense that being in equilibrium is a statement that the flux and sources balance *everywhere* (at each point), not just in their added up versions (which is a much weaker statement).

5. 1.14

Solution:

(a) This first part is kinda stupid. Very similar to the previous question, we use the *density* version of the continuity law

$$\partial_t u = -\partial_x \phi + R(x).$$

However, this requires you to eyeball the derivative (slope) of ϕ, but it isn't really constant in any interval.

(b) This question is much better. To determine the *net* change, we can instead use the endpoints of ϕ.

For instance, in the interval $[0, 1]$, we use the formula (2)

$$\frac{dM}{dt} = \phi(a) - \phi(b) + \int_a^b R\,dx.$$

Eyeballing this a bit with $a = 0, b = 1$, we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} \approx -1/2 - 0 + 1 = 1/2 > 0,$$

so the amount of stuff in that interval is going up. Note that I ballparked the area under R as 1, as each square grid is exactly area 1. In the second interval, we instead have

$$\frac{\mathrm{d}M}{\mathrm{d}t} \approx 0 - (-1/2) + 1/3 = 5/6 > 0,$$

so again it is going up.

6. 1.18

Solution: I can't believe I assigned this. I think it was a mistake. Ignore it. Happy to talk about it if you'd like, just let me know.

7. 1.22

Solution: Here, because it's asking whether the total mass is increasing or decreasing, we get to use the (2) version where we just care about the endpoints. In particular

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \phi(a) - \phi(b).$$

(a) For this part, $\phi = u$, so we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \phi(a) - \phi(b) = u(a) - u(b) < 0 \text{(for panel 1)}.$$

(b) Skipping a few of these that aren't interesting, take part 3: $\phi = -\partial_x u$, so in the first panel we can estimate that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \phi(a) - \phi(b) = -\partial_x u(a) - -\partial_x u(b) \approx -1 - 2 < 0,$$

so the amount of stuff is going down.

8. 2.2

Solution: This was meant as a little linear algebra review. First, we should compute what $\vec{w}$ actually is based on the relation

$$\vec{w} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}.$$

From this, it's a little easy to eyeball that we need -1 copies of $\vec{u}^\star$ and -5 copies of $\vec{v}^\star$, but if you wanted to do this more precisely you could set up the linear system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

and use the usual techniques to solve.

9. 2.4

Solution:

Not certain I meant to assign this one either, but it's a nice exercise in remembering what these terms mean. Basically, if $\vec{u}, \vec{v}$ form a basis, that means that I can pull any other element out of the vector space and give it a name, say $\vec{z}$ and we're guaranteed that some c_1, c_2 exist such that

$$\vec{z} = c_1\vec{u} + c_2\vec{v}.$$

In other words, $\vec{u}, \vec{v}$ form the fundamental building blocks (which we "build" by linear combinations) for all the elements of my vector space V.

The question is, if I pull another random thing out of the hat, and call it $\vec{s}$, can I build it with $\vec{w} = \vec{u} + \vec{v}$ and $\vec{x} = \vec{u} - \vec{v}$?

Well, let's try with a linear combination of $\vec{w}, \vec{x}$

$$\begin{aligned}\vec{s} &= d_1\vec{w} + d_2\vec{x} \\ &= d_1(\vec{u} + \vec{v}) + d_2(\vec{u} - \vec{v}) \\ &= (d_1 + d_2)\vec{u} + (d_1 - d_2)\vec{v} \\ &= g_1\vec{u} + g_2\vec{v}.\end{aligned}$$

Hence, writing $\vec{s}$ as a linear combination of $\vec{x}$ and $\vec{w}$ actually boils down to writing it as a linear combination of $\vec{u}, \vec{v}$, but we're *guaranteed* we can do this, because those vectors form a basis! We're done.