Final Exam
Math 3150 - PDEs
June 20, 2018

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules or caught cheating will be asked to leave immediately. Point values are in the square to the left of the question. **If there are any other issues, please ask the proctor.**

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

Name: ______________________________

Signature: ______________________________

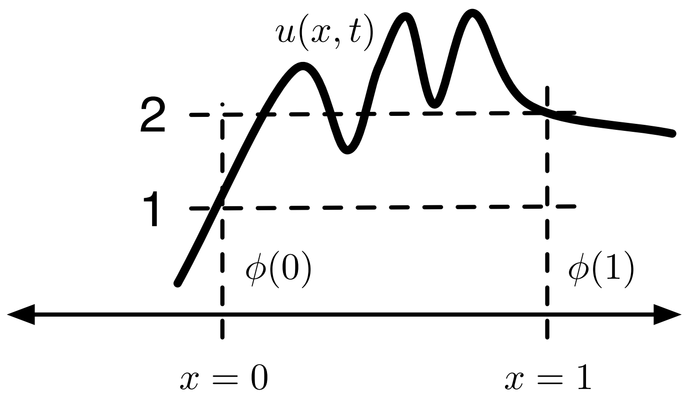
uID: ______________________________

Solutions

Question	Points	Score
1	20	
2	15	
3	20	
4	15	
5	5	
6	17½	
7	17½	
Total:	110	

Note: There are 7 questions on the exam with 110 points available but the exam will be graded out of 100. That is, if you get 90/110, your grade will be 90/100 = 90%.
For this reason, it is *strongly* recommended that you look through the exam to use your time to accumulate points wisely!

1. **(Flux, continuity)** Suppose at a fixed time t the concentration $u(x,t)$ of a chemical is given by the graph in the below figure.



Let $U(t) = \int_0^1 u(x,t)dx$ be the total mass in the interval $x \in [0,1]$. Suppose $u(x,t)$ satisfies the continuity equation

$$\partial_t u = \partial_{xx} u - \partial_x u.$$

Use the graph of $u(x,t)$, your knowledge of flux and the continuity equation to determine answers to the following questions.

To avoid ambiguity, $u_x(0,t) = 2$ and $u_x(1,t) = -1/2$.

Showing work is likely to help earn you partial credit if you select the wrong answer.

4 (a) **Flux rule:** $\phi(x) =$ ______________________ **(fill in the blank)**

4 (b) **What is the direction of $\phi(0)$? (circle one):**
Right Left No flux

4 (c) **What is the direction of $\phi(1)$? (circle one):**
Right Left No flux

4 (d) **Which is larger in magnitude? (circle one):**
$\phi(0)$ $\phi(1)$ They have the same magnitude

4 (e) **Is the total mass U growing or shrinking?** (circle one):
Growing Shrinking No change

Solution: **This is the exact same graph and PDE from Quiz 1, with the question just worded slightly differently.**

(a) Here, we start with the PDE and try to figure out the flux rule rather than the reverse (as on quiz 1). Either way, the key thing is the continuity equation for a general flux rule ϕ:

$$\partial_t u = -\partial_x \phi + R.$$

Clearly here, we have $R = 0$, so what ϕ gives us the PDE

$$\partial_t u = \partial_{xx} u - \partial_x u?$$

This gives us

$$-\partial_x\phi = \partial_{xx}u - \partial_x u,$$

and pulling out a minus sign and a derivative, we have

$$-\partial_x\phi = \partial_{xx}u - \partial_x u = -\partial_x \underbrace{\{-\partial_x u + u\}}_{=\phi(x)}.$$

(b) Once we know this ϕ, we can use the values on the graph for $\partial_x u$ and u to get to

$$\phi(0) = -2 + 1 = -1.$$

In one dimension, positive flux is to the *right*, so this is to the **left**.

(c) Simialrly, we plug in the values at the end right point, and get

$$\phi(1) = -(-1/2) + 2 = 5/2$$

which means the flux is to the **right**.

(d) $|\phi(1)| = 5/2 = 2.5$ which is definitely bigger in magnitude than $|\phi(0)| = 1$, meaning more stuff is flowing through the right than the left.

(e) If we integrate the continuity equation for the density u from $x = 0$ to $x = 1$ and using the definition of $M(t)$ yields our other standard continuity equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \phi(0) - \phi(1).$$

plugging in the values we found above, we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -1 - 5/2 = -7/2.$$

Because this is negative, it says the growth of total amount of stuff is negative, so it is going **down (shrinking)**.

15 2. **(Eigenvalues)** Find all the eigenvalues and eigenfunctions of the differential operator L

$$Lu := u''(x)$$

with boundary values

$$u'(0) = 0, \qquad u(\pi) = 0.$$

Solution: This is a very slight modification from quiz 2.

This is a little messier than a lot of the easy problems we've done of this variety, but overall a straightforward problem if you've done enough of these.

We're considering the eigenvalue problem

$$Lu = \lambda u,$$

which for our particular operator yields

$$u'' = \lambda u$$

which has characteristic roots $\pm\sqrt{\lambda}$, meaning that the three cases we have are the typical $\lambda > 0, \lambda < 0$ and $\lambda = 0$. We must look into each of these.

case 1: $\lambda = 0$

In this case, our eigenvalue equation becomes

$$u'' = 0$$

so integrating twice yields

$$u = ax + b.$$

The left boundary condition ensures

$$u'(\pi) = a = 0 \qquad \Longrightarrow \qquad a = 0.$$

For the right to work, we also have

$$u(\pi) = 0 = b \qquad \Longrightarrow \qquad b = 0.$$

Thus, $u(x) = 0$, which is a scenario we toss out and therefore $\lambda = 0$ is not an eigenvalue. **case 2:** $\lambda > 0$

Here, our roots are real so solutions are of the form

$$u = c_1 e^{\omega x} + c_2 e^{-\omega x},$$

where I've abbreviated $\omega := \sqrt{\lambda}$. The derivative of this is

$$u'(x) = c_1 \omega e^{\omega x} - \omega c_2 e^{-\omega x}.$$

which, when we check the left boundary, we get

$$u'(0) = c_1 \omega - \omega c_2 = 0.$$

Clearly we don't want $\omega = 0$ (because this would make $\lambda = 0$), so $c_1 = c_2$. Now, the right boundary, we find

$$u(\pi) = c_1 e^{\omega \pi} + c_1 e^{-\omega \pi} = 0.$$

There are lots of ways of seeing this, but for instance, when I multiply by $e^{\omega\pi}$ we get

$$0 = c_1 \left[e^{2\omega\pi} - 1 \right],$$

we don't want $c_1 = 0$ because our whole solution is zero in that case, but the only way the stuff inside the brackets is zero is if $\omega = 0$ so we're backed into a corner and this isn't possible. Thus, we don't have any eigenvalues $\lambda > 0$.

case 3: $\lambda < 0$

At this point, you probably know the eigenvalues are nergative but they are a smidge messy even still. If $\lambda < 0$, we have complex roots so

$$u = c_1 \cos(\omega x) + c_2 \sin(\omega x),$$

where here $\omega := \sqrt{-\lambda}$, which is a real number. The derivative of this is

$$u'(x) = -c_1\omega\sin(\omega x) + \omega c_2\cos(\omega x)$$

which we can plug into the left boundary to find

$$0 = -c_1\omega\sin(0) + \omega c_2\cos(0),$$

but $\sin(0) = 0$ so this suggests $c_2 = 0$.

This leaves us with the boundary on the right, which says

$$u(\pi) = c_1\cos(\omega\pi) = 0.$$

We don't want $c_1 = 0$, so that means

$$\cos(\omega\pi) = 0.$$

When is $\cos(x) = 0$? At $x = \pi/2, 3\pi/2$ and generically $\pi/2 + n\pi$. This suggests

$$\omega\pi = \pi\left(\frac{1}{2} + n\right)$$

and therefore

$$\omega_n = \left(\frac{1}{2} + n\right)$$

so remembering $\omega = \sqrt{-\lambda}$, we can solve for the original eigenvalues

$$\lambda_n = -\left(\frac{1}{2} + n\right)^2$$

with corresponding eigenfunctions

$$u_n(x) = \cos\left[\left(\frac{1}{2} + n\right)x\right]$$

3. **(Inner products)**

5 (a) Explain two reasons that we defined the inner product $\langle u, v\rangle$ between two abstract objects u, v. That is, what two things do we gain?

5 (b) Consider the two functions

$$\phi_1(x) = 1, \qquad \phi_2(x) = x - \frac{1}{2}.$$

over the interval $x \in [0, 1]$ with the standard inner product.

Are ϕ_1, ϕ_2 orthogonal? Why or why not?

10 (c) Suppose we want to approximate

$$f(x) = \sqrt{x}$$

using a linear combination of our functions

$$\hat{f}(x) = c_1\phi_1(x) + c_2\phi_2(x).$$

What are the best choices for c_1, c_2?

Nice facts: $\int_0^1 \sqrt{x}(x - 1/2)\,dx = 1/15$, $\int_0^1 (x - 1/2)^2\,dx = 1/12$.

Solution: **These questions are taken direcly from quiz 2 and 3.**

(a) This wasn't meant to be a mind-reading question, but rather distill the inner product to its basic properties. The two big things are that the inner product gives us notions of **length/distance** and **angle**.

Specifically, we defined the norm (magnitude) as

$$\|u\| = \sqrt{\langle u, u\rangle},$$

which allows us to define the distance between two objects u, v as

$$d(u, v) := \|u - v\|.$$

The second big gain we get is that we can now talk about angles, specifically the angle θ between two objects u, v is defined to be

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|}.$$

From this, we can talk about (the very important!) property of orthogonality, which is simply $\langle u, v\rangle = 0$.

(b) Here, we use the definition of orthogonality: the inner product between the two objects equaling zero. Computing this inner proudct:

$$\langle \phi_1, \phi_2\rangle = \int_0^1 1 \cdot (x - \frac{1}{2}) = \left[\frac{x^2}{2} - \frac{1}{2}x\right]_{x=0}^{x=1} = 0.$$

(c) We know that the orthogonal project formula gives us the best values of c_j, which takes the form

$$c_j = \frac{\langle f, \phi_j\rangle}{\langle \phi_j, \phi_j\rangle}.$$

In this particular problem, we find

$$\begin{aligned} c_1 &= \frac{\langle f, \phi_1\rangle}{\langle \phi_1, \phi_1\rangle} \\ &= \frac{\int_0^1 \sqrt{x} \cdot 1 \, \mathrm{d}x}{\int_0^1 1 \cdot 1 \, \mathrm{d}x} \\ &= \frac{2/3}{1} = \frac{2}{3}. \end{aligned}$$

For the second coefficient,

$$\begin{aligned} c_2 &= \frac{\langle f, \phi_2\rangle}{\langle \phi_2, \phi_2\rangle} \\ &= \frac{\int_0^1 \sqrt{x} \cdot (x - 1/2) \, \mathrm{d}x}{\int_0^1 (x - 1/2)^2 \, \mathrm{d}x} \\ &= \frac{1/15}{1/12} = \frac{12}{15}. \end{aligned}$$

Putting this all together, we have

$$\hat{f} = \frac{2}{3}1 + \frac{12}{15}(x - 1/2) = \frac{4}{15} + \frac{4x}{5}.$$

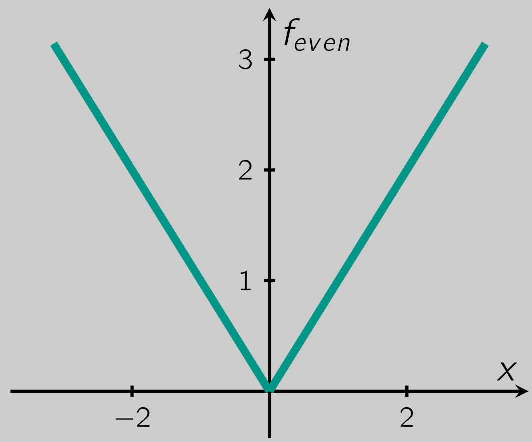
4. **(Fourier series, Laplace's equation)** Consider $f(x) = x$, on $x \in [0, \pi]$.

5 (a) Draw the even extension of the function over $x \in [-\pi, \pi]$.

10 (b) Find the cosine series of the function by specifying every coefficient exactly and computing all integrals fully.
Potentially useful information: $\int_0^L \cos(n\pi x/L)^2\,\mathrm{d}x = L/2$ and $\int_0^L x\cos(n\pi x/L)\,\mathrm{d}x = L^2(\cos(n\pi)-1)/(n^2\pi^2)$. Be careful about bounds of integration!

Solution: **This was taken from the homework.**

(a) This is effectively just a definitional question, you to recall that the even extension just satisfies $f(-x) = f(x)$, or that we reflect *horizontally* across $x = 0$:



(b) This problem was a little more involved, but with the hint still effectively definitional. Note that *Quiz 3* was describing the steps of doing exactly this problem!

To compute the Fourier cossine series on $[0, \pi]$ of some function $f(x)$, we can simply compute the full Fourier series of f_{even} on $[-\pi, \pi]$, in which case, the Fourier series becomes

$$\hat{f}_{cos} = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where we can obtain the coefficients by using the orthogonal projection formula

$$a_0 = \frac{\langle f_{even}, 1\rangle}{\langle 1, 1\rangle}, \qquad a_n = \frac{\langle f_{even}, \cos nx\rangle}{\langle \cos nx, \cos nx\rangle},$$

where it's implied that the inner product is over the whole region $[-\pi, \pi]$ (otherwise why would we bother defining the even extension)? *Note:* we know there are no sine terms by construction (sine is an odd function, so we don't need any of these building blocks to represent our even function f_{even}).

Computing these integrals, we find

$$a_0 = \frac{\int_{-\pi}^{\pi} f_{even}\,\mathrm{d}x}{\int_{-\pi}^{\pi} 1\,\mathrm{d}x} = \frac{2\int_0^{\pi} x\,\mathrm{d}x}{2\pi} = \frac{\pi}{2}.$$

Note that in the above, I've used the fact that the integral from $-\pi, \pi$ of an even function is just twice that of the integral from $0, \pi$, a trick we've used several times.

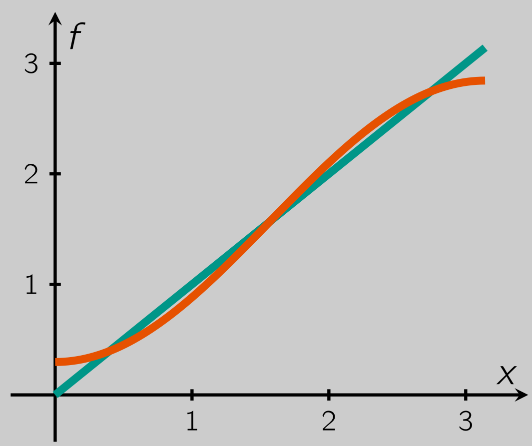
Computing the other coefficients, we can use the hint to do a lot of the heavy lifting

$$\begin{aligned} a_n &= \frac{\int_{-\pi}^{\pi} f_{even}(x)\cos nx\,dx}{\int_{-\pi}^{\pi}\cos nx\,dx} \\ &= \frac{2\int_0^{\pi} x\cos nx\,dx}{2\int_0^{\pi}\cos nx\,dx} \\ &= \frac{2\pi^2\left[\cos(n\pi)-1\right]/(n^2\pi^2)}{\pi} \\ &= \frac{2(\cos(\pi n)-1)}{\pi n^2}. \end{aligned}$$

Writing the first few terms of this series by plugging in values of n, we get

$$\hat{f}_{cos}(x) \approx \frac{\pi}{2} - \frac{4\cos(x)}{\pi} - \frac{4\cos(3x)}{9\pi} - \frac{4\cos(5x)}{25\pi} + \cdots$$

the first two terms are plotting in orange below with f in green



5 5. Explain one physics/chemistry/engineering/other example where **Laplace's equation** for $w(x, y)$

$$\partial_{xx}w + \partial_{yy}w = 0. \tag{1}$$

describes something. What does $w(x, y)$ represent in this application?

Solution: **This is taken directly from worksheet 4.** Any of the following would suffice:

1. **electrostatics:** The potential V (voltage) from an electric field satisfies the relation $\nabla^2 V = -\rho/\varepsilon_0$, where ρ is the charge density and ε_0 is the permitivity of space. Therefore, if we want to know the voltage in a charge-free region, (1) is the PDE we study.
2. **heat transfer:** we know the heat equation describing temperature is $\partial_t T = D\nabla^2 T$, so (1) can be thought of as the equilibrium heat distribution for a 2 dimensional object.
3. **structures:** In (1) $w(x, y)$ could be the deformation (out of the paper) of some membrane, where the boundary conditions will determine how the membrane is attached.
4. **fluids:** If the velocity field of a fluid v can be described by some potential $v = \nabla\phi$, then if the fluid is incompressible (i.e. water, but not air) ϕ satisfies (1).
5. **image processing**: one intuitive behavior of solutions of (1) is that it smoothes things out. One application is then to use solutions of this PDE to smooth out noisy edges in images.

$17\frac{1}{2}$ 6. **(Separation of variables)** Let $u(x, t)$ represent the population of fishes in a river described by $x \in [0, 1]$. The population of fish can do two things: diffuse (move) around and die at a rate proportional to how many fishes there are. Consequently, the PDE that describes their spatial population is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u.$$

Fish (obviously) can't swim in or out of the river, so take the boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0.$$

Using separation of variables, find the solution to the PDE subject to the initial conditions

$$u(x, 0) = 2 + \cos \pi x.$$

Hint: You can take

$$p'' = \lambda p, \quad p'(0) = 0, \quad p'(L) = 0 \qquad \Longrightarrow \qquad \lambda_n = -(n\pi/L)^2, \quad p_n = \cos(n\pi x/L)$$

for granted in this problem.

Solution: This is taken directly from quiz 4.

As with all separation of variables problems, we start with the guess

$$u(x, t) = p(x)q(t).$$

Plugging this into our PDE, we get

$$p\dot{q} = p''q + pq, \qquad \dot{q} := \frac{\mathrm{d}q}{\mathrm{d}t}, \qquad p' := \frac{\mathrm{d}p}{\mathrm{d}x}.$$

Thus, we can separate this into

$$\frac{\dot{q}}{q} = \frac{p'' + p}{p} = \frac{p''}{p} - 1,$$

but the hint suggests we want the spatial (p) ODE to be the same as the heat equation, were we had p''/p. Thus, move the 1 over to yield

$$\frac{\dot{q}}{q} + 1 = \frac{p''}{p} = \lambda,$$

where we know that if $f(t) = g(x)$ then they must both be equal to some constant, which we can call λ.

Thus, we're left with the spatial eigenvalue problem

$$p''(x) = -\lambda p, \qquad p'(0) = p'(1) = 0.$$

We know, from class, this is exactly the same eigenvalue problem as the no-flux heat equation, which has eigenvalues and eigenfunctions

$$\lambda_n = -\left(\frac{\pi n}{L}\right)^2, \qquad p_n(x) = \cos\frac{\pi n x}{L},$$

but here we have $L = 1$, so we've got the eigenvalues and eigenfunctions

$$\lambda_n = -(\pi n)^2, \qquad p_n(x) = \cos(\pi n x).$$

We should note that we also have $\lambda = 0$, with $p_0(x) = \text{constant}$ as the corresponding eigenfunction. Now we turn to the q ODE, which says

$$\dot{q} + q = \lambda q,$$

which, rearranged, yields

$$\frac{\dot{q}}{q} = (\lambda - 1),$$

which has solutions

$$q_n(t) = e^{(\lambda_n - 1)t} = e^{-\left[(\pi n)^2 + 1\right]t}.$$

Notice that this form makes sense. The only difference from our heat equation solution is that this one decays a little bit faster. This makes sense since we added a decay term to the PDE.

Now we know our eigenfunctions, so we combine them in a linear combination (because we currently don't know which n's to include), so we see our general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n q_n(t) p_n(x) = A_0 e^{-t} + \sum_{n=1}^{\infty} A_n \cos(n\pi x) e^{-\left[(\pi n)^2 + 1\right]t}.$$

Note that I've pulled out the $n = 0$ case to make it more explicit, but this wasn't completely necessary. Now, to determine the A_n's we need to use the initial condition. The initial condition says

$$u(x, 0) = 2 + \cos \pi x,$$

but this has to be equal to our solution at $t = 0$, which yields

$$2 + \cos \pi x = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = A_0 + A_1 \cos(\pi x) + A_2 \cos(2\pi x) + \cdots .$$

It's clear from this that $p2, A_1 = 1$ and $A_{\geq 2} = 0$. Putting this back into our general (time dependent) solution, yields the final solution to the PDE

$$u(x,t) = 2e^{-t} + \cos(\pi x) e^{-\left[\pi^2 + 1\right]t}.$$

Note that as $t \to \infty$, this decays to 0, which makes sense since no fish are fluxing into the river but they are dying at some rate.

$17\frac{1}{2}$ 7. **(Non-homogeneous PDEs)**

Find the particular solution to the heat equation

$$\partial_t u = \partial_{xx} u + \cos(x)$$

on a rod $x \in [0, \pi]$ with insulated endpoints and initial condition $f(x) = -\cos(x)$.

Hint: Same hint as the previous problem. You can take

$$p'' = \lambda p, \quad p'(0) = 0, \quad p'(L) = 0 \qquad \Longrightarrow \qquad \lambda_n = -(n\pi/L)^2, \quad p_n = \cos(n\pi x/L)$$

for granted in this problem.

Solution:

This is taken directly from homework 5.

The big hint for this one was the label that this is a *non-homogeneous* problem. We talked about two ways in which this could be the case. Here, the boundary conditions

$$\partial_x u(0,t) = \partial_x(\pi,t) = 0.$$

are homogeneous, so that's not the problem. However, the explicit source in the PDE definitely makes it of this variety. We know we can solve stuff like this by assuming the form

$$u(x,t) = w(x,t) + u_{eq}(x),$$

where $w(x,t)$ is the solution to the *homogeneous* problem, and $u_eq(x)$ is the solution to the *equilibrium problem*.

The homogeneous problem is then

$$w_t = w_{xx}, \quad w'(0,t) = w'(\pi,t) = 0$$

and $u_{eq}(x)$ is the solution to the equilibrium problem

$$0 = u''_{eq}(x) + \cos(x), \quad u'_{eq}(0) = u'_{eq}(\pi) = 0.$$

First, we find the general solution to the homogeneous problem, which we have solved before with the no-flux BCs in previous problems:

$$w(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-n^2 t}.$$

Next, we solve the ODE for the equilibrium problem by direct integration:

$$\begin{aligned} u''_{eq}(x) &= -\cos(x) \\ \Rightarrow u'_{eq}(x) &= -\sin(x) + c_1 \\ \Rightarrow u_{eq}(x) &= \cos(x) + c_1 x + c_2. \end{aligned}$$

Applying the boundary conditions, we get

$$\begin{aligned} u'_{eq}(0) &= -\sin(0) + c_1 = 0 \Rightarrow c_1 = 0, \\ u'_{eq}(\pi) &= -\sin(\pi) + c_1 = 0 \Rightarrow c_1 = 0. \end{aligned}$$

We do not yet have enough information with the boundary conditions to solve for c_2, so we leave it unknown for now.

Combining the two solutions gives us the general solution to our full nonhomogeneous problem

$$u_{gen}(x,t) = \cos(x) + c_2 + a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-n^2 t}.$$

Notice that since c_2 and a_0 are both constants, we could combine them together into one new constant A_0:

$$u_{gen}(x,t) = \cos(x) + A_0 + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-n^2 t}.$$

To solve for the unknown constants, we will now apply our initial condition.

$$u(x,0) = -\cos(x) = \cos(x) + A_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

We can rearrange the equation to look like a cosine series by subtracting $\cos(x)$ from both sides, which gives us

$$-2\cos(x) = A_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

We can now find the coefficients by inspection: in order to match terms on both sides, we need $a_1 = -2$ and $a_n = 0$ otherwise (including at A_0). Plugging in these coefficients gives us the particular solution:

$$u(x,t) = \cos(x) - 2\cos(x)e^{-t} = \cos x\left(1 - 2e^{-t}\right).$$

Bonus Questions

8. **(Fourier)** Joseph Fourier accompanied ________________________ (a famous conqueror) as a military scientific advisor to conquer ________________________ (a place in northern Africa), where Fourier was later named governor.

Solution: Napolean was the conqueror. Egypt was the place.

9. **(Math)** Suppose we have some function $f(x)$ defined on $[0,1]$ and, as usual we want to construct an approximation

$$\hat{f} = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x)$$

where $\phi_j(x)$ are some orthogonal functions. We want our approximation to be accurate in the sense that $f(x)$ and $\hat{f}(x)$ do not deviate significantly at any point, so we want to minimize

$$\text{approximation error} = \max_{x\in[0,1]} \left| f(x) - \hat{f}(x) \right|.$$

Does the orthogonal projection formula for c_j work here? Why or why not? If not, what could we change so that it does work?

Solution: Basically, all I was hoping for was some explanation that the orthogonal projection formula is only appropriate for minimizing the squared error

$$\text{squared error} = \|f - \hat{f}\|^2 = \int_0^1 \left[f(x) - \hat{f}(x)\right]^2 \,\mathrm{d}x.$$

Because this error is an integral, discrepancies in individual points (for instance, I could take $\hat{f}(1/2) = 543$) do *not* appear as contributions when this is used as the way of measuring error. I wasn't expecting anyone to come up with a way to solve this problem, but you did get an extra point or two if you noticed that the way I defined the error actually suggests a different norm/inner product than the one we used throughout the class.

10. **(Rhode Island)** While you are taking this exam, Chris is in the strange state of Rhode Island. In Providence, there are more ________________________ shops per capita than anywhere else in the world. *Hint:* it accompanies the state's official drink of coffee milk, an abominable form of coffee.

Solution: Donuts. How did so many people get this wrong? New England has a Dunkin Donuts every 3 feet.

11. **(Other)** Feel free to express any other thoughts or feelings here.

Solution: Obviously not a question that has a solution, but if you're reading this I just wanted to express that teaching the class was a pleasure for me (Chris) and I appreciate you taking the time to care about these solutions!