Math 3140 – Wave Equation Worksheet

Due: July 18, 2016

Since I'll be gone for the week of July 11, you will instead work through this guided tour of the wave equation. If you have any questions, I'll be available via email the whole week and even by Skype (or similar) if necessary.

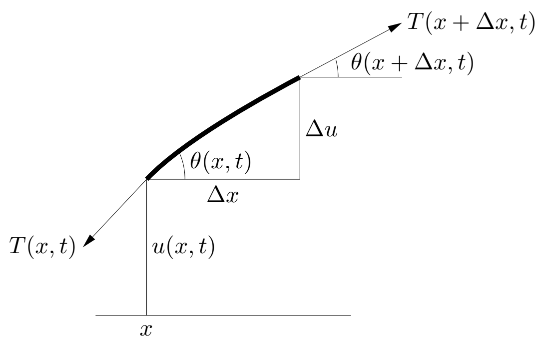
Derivation

In this section, we'll derive the wave equation from basic physical principles.

1. Consider a string of length of length L and denote x as the position along it. We are interested in how the string vibrates. Let's first define some things

$$\begin{aligned} u(x,t) &= \textit{vertical} \text{ displacement of the string at position } x \text{ and time } t \\ \theta(x,t) &= \text{angle between the string and a horizontal line at position } x \\ T(x,t) &= \text{tension (force) in the spring at position } x \\ \rho &= \text{mass density of the string.} \end{aligned}$$

Now, we consider a tiny slice of the string of length Δx.



2. What the approximate mass of our slice? Call this M.

Solution: Although the density is non-constant in the slice, if we take our slice small enough, it is approximately constant with mass

$$M = \rho \Delta x.$$

3. We now use Newton's Second's Law, which says $F = Ma$, but we know M from the previous problem, $a = \frac{\partial^2 u}{\partial t^2}$ and the sum of the forces are just the vertical components of the tensions at the end of our slice. This results in

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x, t)\sin\theta(x+\Delta x) - T(x,t)\sin\theta(x),$$

which we can use the product rule to rewrite as

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left\{T(x,t)\sin\theta(x,t)\right\}. \tag{1}$$

Note that we also have

$$\frac{\partial u}{\partial x} = \text{slope of the string} = \tan\theta.$$

If we assume that θ is *small*, approximate $\cos\theta$ by the first term in its Taylor series to approximate $\tan\theta$ by a term that appears in (1).

That is, $\frac{\partial u}{\partial x} = \tan\theta \approx ?$.

Solution: From basic trigonomotry, we know

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

However, for θ small, say, $\theta \approx 0$ we know that $\cos\theta \approx 1$, so that we're left with

$$\frac{\partial u}{\partial x} = \tan\theta \approx \frac{\sin\theta}{1} = \sin\theta.$$

However, note that when we use this approximation, (1) now becomes

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left\{T(x,t)\frac{\partial u}{\partial x}\right\}. \tag{2}$$

4. Make the substitution for $\frac{\partial u}{\partial x}$ described in the previous problem to (1).

   If we assume that ρ is constant, and that our string is *perfectly elastic*, we can say that T is constant too. Rearrange your result to get the **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

   What is c? What units does it have?

Solution: We now consider (2) when $T = T_0$ and $\rho = \rho_0$, constants, which is then

$$\rho_0\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left\{T_0\frac{\partial u}{\partial x}\right\},$$

which simplifies to

$$\frac{\partial u}{\partial t} = \frac{T_0}{\rho_0}\frac{\partial^2 u}{\partial x^2} := c^2\frac{\partial^2 u}{\partial x^2}.$$

Notice then that $c = \sqrt{T_0/\rho_0}$. What units does this have? T_0 is a force so we can think fo this as mass times acceleration. ρ_0 is a density with units of mass per length. Consequently, T_0/ρ_0 has units of acceleration times distance, or distance^2/time^2. Thus, c is a **velocity**.

Finite Domain

We'll study the wave equation on some fixed domain, say, $[0, L]$ like we have the wave equation.

1. As with every PDE we've discussed, we need to prescribe boundary conditions, so suppose the ends of the string are fixed

$$u(0, t) = u(L, t) = 0.$$

Using separation of variables, show that the solution to this equation is

$$u(x, t) = \sum_{n=1} A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L}.$$

Solution: This is rather tedious but very straightforward using the tools we've utilized previously in the class. As usual, take the separation of variables form

$$u(x, t) = p(x)q(t)$$

and when we plug it in, note that we get the two ODEs

$$p'' = -\lambda p, \qquad \ddot{q} = -\lambda c^2 q.$$

Because the ends of our string are fixed, we have the boundary conditions pop up in one of the ODEs

$$p(0) = p(L) = 0$$

and consequently the boundary value problem

$$p'' = -\lambda p, \qquad p(0) = p(L) = 0.$$

We have solved this many times in our class and know the eigenvalues are $\lambda_n = (n\pi/L)^2$ and eigenfunctions are $p_n = \sin n\pi x/L$.

However, we have to use the q ODE, but it is effectively the same structure with (what we now know) are positive values of λ, meaning solutions are of the form of sin's and cos's, explicitly:

$$q_n = \cos \frac{n\pi ct}{L}, \qquad or \qquad q_n = \sin \frac{n\pi ct}{L}$$

Thus, we know the form of the solutions for a particular value of λ (or n) but we don't know which to include, so we sum all of them (by the principle of superposition) to get our full solution

$$u(x, t) = \sum_{n=1} A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L}.$$

2. Although it's somewhat apparent these are oscillations in space and time, we will make this more transparent. Consider the term

$$\sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L}, \tag{3}$$

and use the trig identity

$$\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b),$$

to rewrite (3) as something of the form

$$f(x+ct)+g(x-ct).$$

Solution: This is effectively a plug and chug question just with a, b being a little bit messy

$$\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}=\frac{1}{2}\cos\left(\frac{n\pi x}{L}-\frac{n\pi ct}{L}\right)-\frac{1}{2}\cos\left(\frac{n\pi x}{L}-\frac{n\pi ct}{L}\right)$$

and cleaning this up a bit we get

$$\frac{1}{2}\cos\left(\frac{n\pi(x-ct)}{L}\right)+\frac{1}{2}\cos\left(\frac{n\pi(x+ct)}{L}\right).$$

Although this is just one term in the full series, they all follow this structure so we'll try to figure out what the behavior is in subsequent questions.

3. We now interpret what something of the form $f(x-ct)$ means. Consider $f(x)$ as some function. What does $f(x-1)$ look like? $f(x-2)$? $f(x-a)$?

   Now imagine that a depends on time, and say increases at a constant rate $a=ct$, so we get $f(x-ct)$. What would you call this behavior?

Solution: We now consider what $f(x-ct)$ looks like. Consider $f(x)$ to be a box. We can think of this as our starting configuration, at $t=0$. Then, $f(x-1)$ is simply the box shifted right by 1 unit.

By the same logic $f(x-2)$ is simply the function shifted to the right two units. Thus $f(x-ct)$ is our initial condition shifted to the right ct units. What can we think of ct as? If something moves at a constant rate c, the position it is after t units of time is ct. Thus, $f(x-ct)$ is a rightward **traveling wave**.

4. How does $f(x+ct)$ differ from $f(x-ct)$?

Solution: By the same logic, $f(x+ct)$ simply moves left at a velocity c. Or we could think of this as the same traveling wave as previous just with a new velocity of $-c$ instead of $+c$.

5. To summarize, solutions of the wave equation with fixed ends are of the form

$$u(x,t)=A(x-ct)+B(x+ct),$$

   which means they consist of two of *something* directed oppositely. What is the *something*?

Solution: As we've established, really the wave equation propagates traveling waves. Note that at $t=0$, our initial condition is just $A(x)+B(x)$ and in some way, the wave equation splits these apart and propagates them at a constant speed c in the left and right directions.

Infinite Domain

We can now extend the possible values of x to anywhere from $[0, \infty)$ or even $(-\infty, \infty)$ and the PDE still makes sense. We'll now study behavior of the wave equation in this case.

Consider the wave equation on the infinite domain with initial position and velocity

$$u_{tt} = c^2 u_{xx}, \qquad u(x, 0) = u_0(x), \qquad u_t(x, 0) = v_0(x),$$

then **D'Alembert's Formula** says solutions can be computed by

$$u(x, t) = A(x - ct) + B(x + ct) = \frac{1}{2}\left[u_0(x - ct) + u_0(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\,\mathrm{d}\xi.$$

1. Use D'Alembert's formula to compute the solution to the wave equation on $(-\infty, \infty)$ for

$$u_{tt} = u_{xx}, \qquad u_0 = \frac{1}{2}e^{-x^2}, \qquad v_0 = 4.$$

Solution: Here, we're taking $c = 1$ so we expect the initial conditions u_0 to travel at some rate c. However, this is not quite the case since they have some initial velocity which is described by v_0. The magic of D'Alembert's solution is that it tells us exactly how to combine these two:

$$u(x, t) = \frac{1}{2}\left[u_0(x - ct) + u_0(x + ct)\right] + \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)\,\mathrm{d}\xi = \frac{1}{4}\left[e^{-(x-t)^2} + e^{-(x+t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t} 4\,\mathrm{d}\xi,$$

note, however, the integral is fairly easy to compute, which is

$$\int_{x-t}^{x+t} 4\,\mathrm{d}\xi = 4\xi\Big\}_{\xi=x-t}^{\xi=x+t} = 8t.$$

Thus, the full solution is then

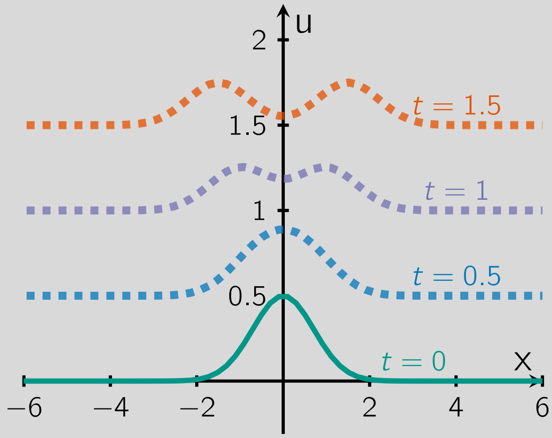
$$u(x, t) = \frac{1}{4}\left[e^{-(x-t)^2} + e^{-(x+t)^2}\right] + 4t.$$

It was really that easy to construct a full solution to the PDE.

As a note: D'Alembert's does indeed work for finite domains but it's trickier to consider stuff absorbing into or bouncing off the endpoints.

2. For the previous problem, draw or plot some snapshots for various values of t, say, $t = 0, 1, 2$. How would you describe the solution?

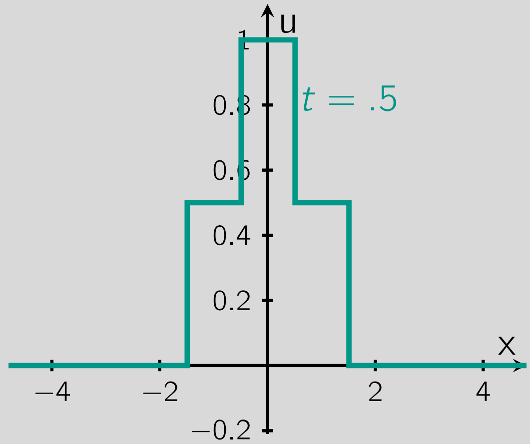
Solution: Basically: this solution propagates two bell curves left and right and also increases upward constantly due to the constant initial velocity.



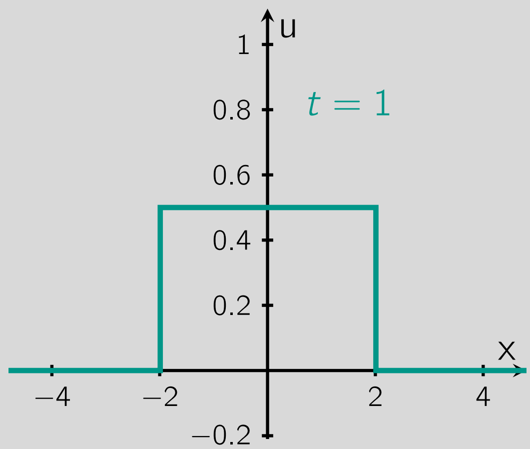
3. To give more intuition for D'Alembert's formula, plot solutions to the wave equation for $c = 1$, $v_0 = 0$ and
$$u_0(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$
That is, just an initial displacement but no initial velocity.

Solution: Here, the initial velocity plays no effect, so our solution is of the form
$$u(x,t) = \frac{1}{2}\left[u_0(x-t) + u_0(x+t)\right].$$
There are a few ways we could think of this. Consider for now, only $u_0(x-t)$. This means that the function is still a box, but a box between
$$u_0(x-t) = \begin{cases} 1 & -1 < x - t < 1 \\ 0 & \text{otherwise,} \end{cases}$$
which we can rearrange to to
$$u_0(x-t) = \begin{cases} 1 & -1 + t < x < 1 + t \\ 0 & \text{otherwise.} \end{cases}$$
Thus the region that this contributes to stays a constant size but moves as t changes. A similar argument could be made for $u_0(x+t)$ except that it moves to the left.

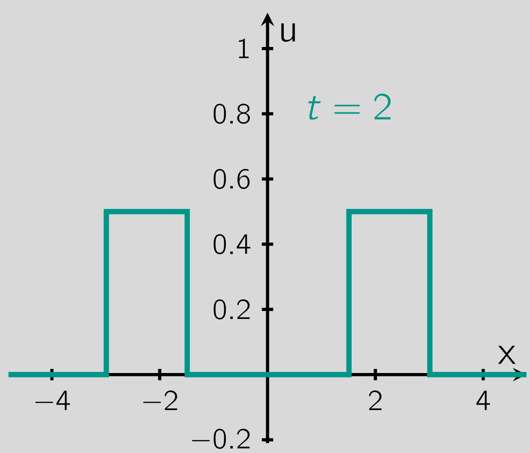
For instance, we could consider the snapshot at $t = 0.5$. We then get
$$u(x, 0.5) = \frac{1}{2}\left[u_0(x-0.5) + u_0(x+0.5)\right],$$
where
$$u_0(x-0.5) = \begin{cases} 1 & -0.5 < x < 1.5 \\ 0 & \text{otherwise,} \end{cases} \qquad u_0(x-0.5) = \begin{cases} 1 & -1.5 < x < 0.5 \\ 0 & \text{otherwise,} \end{cases}$$
In other words, our solution is
$$u(x, 0.5) = \begin{cases} 1/2 & -1.5 < x < -0.5 \\ 1 & -0.5 < x < 0.5 \\ 1/2 & 0.5 < x < 1.5. \end{cases}$$
Thus, we start to see the waves emerging from the left and the right.



At $t = 1$, they are exactly touching for the last time:



and at $t = 2$ we see they are clearly separated:



4. Consider now, the opposite, $c = 1$, $u_0 = 0$ and

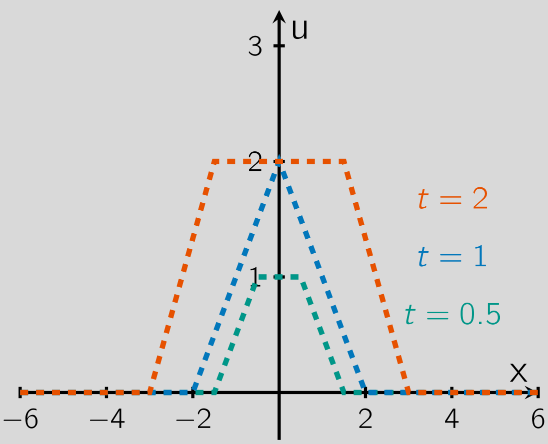
$$v_0(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

What do solutions to they wave equation look like in this case? Draw a few snapshots.

Solution: Here, this is a bit trickier to analyze. It corresponds to the string not having any displacement initial but an initial velocity only in some region. Regardless, D'Alembert's tells us that the solution is

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)\,\mathrm{d}\xi.$$

This integral is non-trivial to compute until we stop and think about what v_0 is: a constant function while $\xi \in [-1,1]$ and then zero otherwise. We know the anti-derivative of 1 is just ξ, so we really just need to check if $\xi \in [-1,1]$. This is a bit of a nightmare to do in general, but let's plot just a few snapshots like we did in the previous problem.



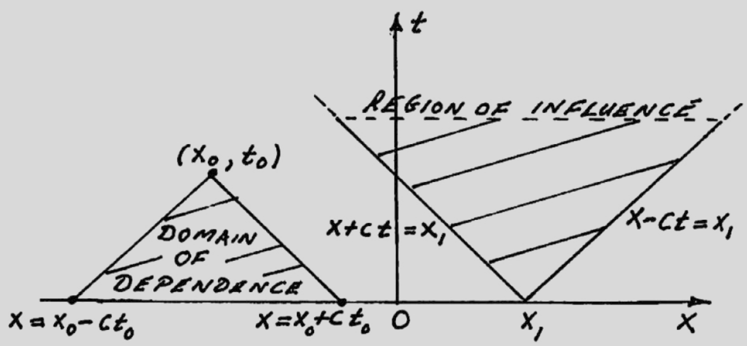
And this pattern continues with this flat pyramid shape growing in size. Note that the action happens on the boundaries on an interval of length 2. This makes perfect sense since our initial velocity only has a region of length 2 of behavior and this is propagating to the left and right.

Domain of Dependence

1. Consider a point x_1 at $t = 0$. We know that the information of the initial condition at this point travels at waves described by $x_1 - ct$ and $x_1 + ct$. Draw this point and the lines describing the propagation of information in an (x, t) plane. Shade in the region that would be non-zero in D'Alembert's solution. This is called the **domain of influence** of an initial point x_1.

Solution: From D'Alembert's solution, we see two things: our initial conditions propagate right and left at speeds c, meaning that if we examine a single value at $x = x_1$ at t_0, this value propagates along the lines $x_0 + ct$ and $x_1 - ct$. This forms a region where the integral term of D'Alembert's also may contribute, resulting in this wedge being the **only region of influence** of the initial point x_1.

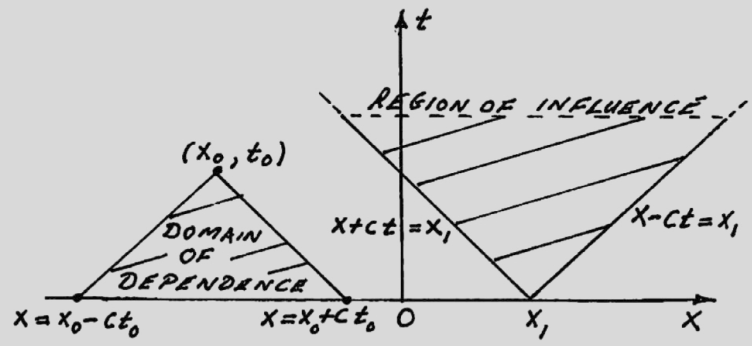
Shamelessly stolen picture:



2. We now consider the alternative. Say, some point (x, t). We want to know the **domain of dependence**. That is, given the previous question about how the influence of an initial point has: what is the domain of **dependence** a later point has? That is for a non-initial point (x, t), sketch the region of points that influence it. *Hint:* think about the "worse cases" of the previous problem. Say, where is the worst place you can start for a left traveling and right traveling wave to JUST hit (x, t)? It should look like a flip of your answer to the previous problem.

Solution:

This problem is the opposite. Say we observe some point on our string x_0 at some later time t_0. What is the range of initial points that could influence it? Well, the reasoning is still the same. Basically try to draw a bunch of domains of influence of different starting points and ask which hit this point. This forms a wedge shape called the **domain of dependence**, forming the region that entirely determines the behavior at this point.



3. The heat equation has *infinite speed of propagation of information*. That is, instantaneously, temperature in the bar influences all other other points. Is this still true for the wave equation? How quickly does the wave equation propagate information?

Solution: From these previous two problems, we see that if we start at $t = 0$ with at some collection of x values in our initial condition $f(x)$, this information propagates at most to $x + ct$ or $x - ct$. In other words, the PDE propagates the initial condition with speed c, which is indeed just a property of the PDE itself, not of any particular initial conditions.