Name:______________________________ Quiz Score:_____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

[5] 1. The linearized Korteweg-deVries (KdV) equation is described by

$$\frac{\partial u}{\partial t} = k\frac{\partial^3 u}{\partial x^3}, \qquad u(x,0) = f(x).$$

Solve for $\tilde{U}(\omega, t)$, the Fourier transform of the solution $u(x,t)$, but **do not invert back to** $u(x,t)$**.**

Solution: I threw a random PDE here we haven't seen before just to illustrate the versatility of the Fourier transform. To begin, we Fourier transform both sides of the PDE with respect to space (that is, we take $x \to \omega$).

As with all the other PDEs we've looked at, since we are only transforming space, a derivative in time persists through the transform. Meaning that, denoting the transformed solution

$$\mathcal{F}[u(x,t)] = \tilde{U}(\omega, t),$$

we have that

$$\mathcal{F}\left[\frac{\partial}{\partial t}u\right] = \frac{\partial}{\partial t}\tilde{U}.$$

The right hand side is a little bit trickier. However, we use the relationship provided that a derivative in space x turns into a multiplicative factor $-i\omega$:

$$\mathcal{F}\left[\frac{\partial}{\partial x}u\right] = (-i\omega)\tilde{U},$$

which means if we take a derivative three times, we just apply three of the multiplicative factors

$$\mathcal{F}\left[\frac{\partial^3}{\partial x^3}u\right] = (-i\omega)^3\tilde{U} = i\omega^3\tilde{U},$$

where we recall that $i^2 = -1$. Also note that the factor of k in the PDE is just a constant so we can pull it in and out of the transform, resulting in the fully transformed PDE

$$\frac{\partial \tilde{U}}{\partial t} = ik\omega^3\tilde{U}.$$

Note this is an *ordinary differential equation* in t for $\tilde{U}$, hence we can rearrange to (where prime denotes time derivative)

$$\frac{\tilde{U}'}{\tilde{U}} = ik\omega^3,$$

which we can integrate and exponentiate to attain

$$\tilde{U}(\omega, t) = c(\omega)e^{ik\omega^3 t}. \qquad (1)$$

Note that the integration constant $c(\omega)$ depends on ω because we integrated with respect to t which knows nothing about the other variable. What is $c(\omega)$? Well, note that at $t = 0$, we have

$$u(x, 0) = f(x)$$

but we can transform both sides of this equation

$$\mathcal{F}[u(x, 0)] = \tilde{U}(\omega, 0) \qquad = \qquad \mathcal{F}[f(x)] = \tilde{F}(\omega)$$

but we know our solution at $t = 0$ from (1), which says

$$\tilde{U}(\omega, 0) = c(\omega)e^{ik\omega^3 t}\Big|_{t=0} = c(\omega) = \tilde{F}(\omega),$$

which means the identity of $c(\omega)$ is simply the Fourier transform of our initial condition. Thus we have fully solved the PDE in ω description

$$\tilde{U}(\omega, t) = \tilde{F}(\omega)e^{i\omega^3 kt},$$

all of which is known. This is a bit trickier to invert back to $u(x, t)$ than the examples in class, but that was not required for this quiz.

4 2. (a) Compute the inverse Fourier transform of

$$\tilde{G}(\omega) = e^{-\alpha|\omega|}.$$

1 (b) What must be true of α for this to make any sense?

Solution: This is simply just applying the definition of the (inverse) Fourier transform, which says

$$g(x) = \int_{-\infty}^{\infty} \tilde{G}(\omega)e^{-i\omega x}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} e^{-\alpha|\omega|}e^{-i\omega x}\,\mathrm{d}\omega.$$

This looks a bit ugly at first, but the obvious trick is to split it from $(-\infty, 0)$ and $(0, \infty)$, where we have

$$\tilde{G}(\omega) = e^{-\alpha|\omega|} = \begin{cases} e^{-\alpha\omega} & \omega > 0 \\ e^{\alpha\omega} & \omega < 0. \end{cases}$$

Thus, the integral becomes

$$\int_{-\infty}^{\infty} e^{-\alpha|\omega|}e^{-i\omega x}\,\mathrm{d}\omega = \int_{-\infty}^{0} e^{\alpha\omega}e^{-i\omega x}\,\mathrm{d}\omega + \int_{0}^{\infty} e^{-\alpha\omega}e^{-i\omega x}\,\mathrm{d}\omega.$$

The first integral is then

$$\begin{aligned} \int_{-\infty}^{0} e^{\alpha\omega}e^{-i\omega x}\,\mathrm{d}\omega &= \int_{-\infty}^{0} e^{(\alpha - ix)\omega}\,\mathrm{d}\omega \\ &= \left[\frac{e^{\omega(\alpha - ix)}}{\alpha - ix}\right]_{\omega=-\infty}^{\omega=0} \\ &= \frac{1}{\alpha - ix}. \end{aligned}$$

Note that during this analysis, we needed $\alpha < 0$ for the integral to converge. This makes sense since unless α is negative. This in general illustrates the issue that both $\tilde{G}(\omega)$ and $g(x)$ need to decay for Fourier analysis to make sense.

Similary, for the second integral, we get

$$\begin{aligned}\int_0^{\infty} e^{-\alpha\omega} e^{-i\omega x}\,\mathrm{d}\omega &= \int_0^{\infty} e^{(-\alpha - ix)\omega}\,\mathrm{d}\omega \\ &= \left[\frac{e^{\omega(-\alpha-ix)}}{-\alpha - ix}\right]_{\omega=0}^{\omega=\infty} \\ &= \frac{1}{\alpha + ix}.\end{aligned}$$

Thus, the full transform is

$$g(x) = \frac{1}{\alpha - ix} + \frac{1}{\alpha + ix}.$$

This looks bizarre. We got a complex valued function? No! Simplify by finding a common denominator.

$$g(x) = \frac{1}{\alpha - ix} + \frac{1}{\alpha + ix} = \frac{2\alpha}{\alpha^2 + x^2},$$

a very nice function.

Useful Fourier Properties:

$$f(x) = \int_{-\infty}^{\infty} \tilde{F}(\omega) e^{-i\omega x}\,\mathrm{d}\omega, \qquad \tilde{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\,\mathrm{d}x.$$

$$\mathcal{F}\left[\frac{\partial f}{\partial x}\right] = (-i\omega)\tilde{F}(\omega).$$