Name:________________________________ Quiz Score:______/10

> Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

[5] 1. Compute the line integral

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}, \qquad \text{where} \qquad \mathbf{F}(x,y) = x^2\mathbf{i} - xy\mathbf{j},$$

and C is the quarter-circle in the top-right half-plane, oriented counter-clockwise.

Solution: As is the first step in all line integral problems, we have to parameterize the curve C. We know the parameterization of a **FULL** circle is:

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}, \qquad 0 \le t \le 2\pi,$$

however, we only have a quarter circle. Thus, we take the modification

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}, \qquad 0 \le t \le \pi/2.$$

We can compute the derivative of this

$$\mathbf{r}'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j},$$

and we know the relationship between the infinitesimals is

$$\mathrm{d}\mathbf{r} = \mathbf{r}'\,\mathrm{d}t = [-\sin(t)\mathbf{i} + \cos(t)\mathbf{j}]\mathrm{d}t,$$

so our integral becomes

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r})\cdot\mathbf{r}'\,\mathrm{d}t = \int_0^{\frac{\pi}{2}} \langle \cos^2(t), -\cos(t)\sin(t)\rangle\cdot\langle -\sin(t),\cos(t)\rangle\,\mathrm{d}t,$$

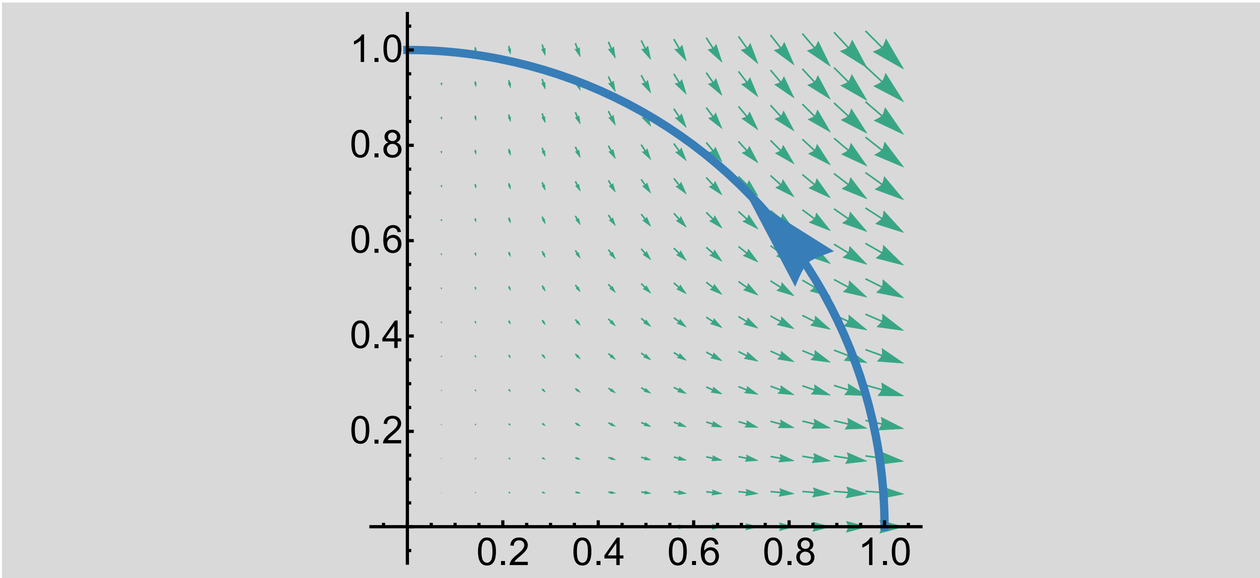
where, the dot product evaluates to

$$\langle \cos^2(t), -\cos(t)\sin(t)\rangle\cdot\langle -\sin(t),\cos(t)\rangle = -2\cos^2(t)\sin(t).$$

so our integral becomes a u-substitution, where $u = \cos(t)$

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -2\int_0^{\frac{\pi}{2}} \cos^2(t)\sin(t)\,\mathrm{d}t = \frac{2}{3}\left[\cos^3(t)\right]_{t=0}^{t=\pi/2} = -\frac{2}{3}.$$

Note it makes perfect sense we get a negative answer when we notice the line integral goes against the "circulation" of the vector field:


1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0

5 2. Use Green's Theorem to evaluate the line integral:

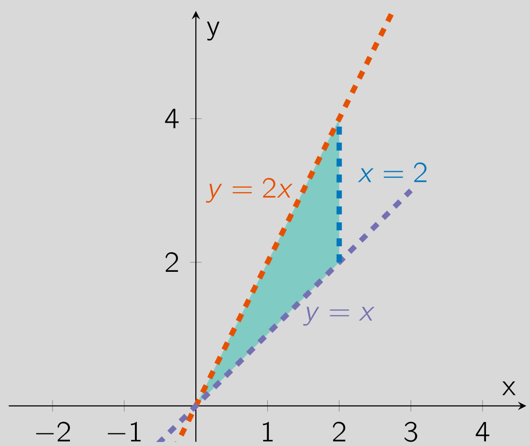
$$\oint_C xy^2\,\mathrm{d}x + 2x^2y\,\mathrm{d}y,$$

where C is the positively oriented triangle with vertices $(0,0)$, $(2,2)$, $(2,4)$.

Solution: Green's Theorem (ignoring all the technical conditions), says that if $\mathbf{F} = \langle P, Q\rangle$, then

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]\mathrm{d}A.$$

Here, we see that $P = xy^2$ and $Q = 2x^2y$ meaning that $Q_x = 4xy$ and $P_y = 2xy$ and our integrand is $Q_x - P_y = 4xy - 2xy = 2xy$.



To know the bounds of integration, we must draw the region. We could set this up as either type 1 or type 2, but I think type 1 is easier here. Note that x varies between 0 and 2 and then y is bounded by the two sides of the triangle described by $y = 2x$ and $y = x$, thus:

$$\begin{aligned}
\iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]\mathrm{d}A &= \int_0^2 \int_x^{2x} 2xy\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 \left[xy^2\right]_{y=x}^{y=2x}\mathrm{d}x \\
&= \int_0^2 3x^3\,\mathrm{d}x \\
&= \left[\frac{3}{4}x^4\right]_{x=0}^{x=2} = 12.
\end{aligned}$$