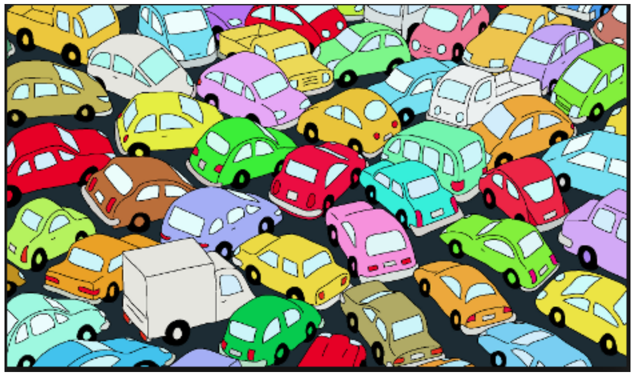
Math 3140 Project - Traffic Dynamics



From afar, intense traffic on a road can be considered as a fluid flow. This project will aim to use numerical and mathematical techniques to model this type of fluid flow. We will start with a numerical solution of a partial differential equation involving traffic density on a circular track. Then we will apply the method of characteristics to analyze traffic density time evolutions. Finally we will study a discontinuous initial condition that describes the traffic density at a traffic light. All of these situations can be described by means of macroscopic variables such as density of cars ρ, their average speed v, and their flux q. These three properties are related as follows:

$$q = v\rho$$

To construct a model for the evolution of ρ through time, we need to make some assumptions:

- There is only one lane and overtaking is not allowed.
- There are no exits or entrances for the road
- The average car speed is not constant and depends on the density of cars *only*:

$$v = v(\rho)$$
$$\frac{dv}{d\rho} \leq 0$$

1. Phase 1: **Circular Track Numerical Solution**
   In this problem, we are modeling the propagation of traffic traveling at constant speed on a circular track given that the initial density profile is $\sin(2\pi x)$. The following partial differential equation describes the set-up

$$\rho_t + 0.1\rho_x = 0$$
$$\rho(0, t) = \rho(1, t)$$
$$\rho(x, 0) = \sin(2\pi x).$$

A particular finite difference scheme to numerically solve the above PDE is known as the *Lax-Friedrichs* scheme. It approximates solutions by discretizing time and space. It is given by the following:

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j-1}^n + u_{j+1}^n)}{k} + c\Big(\frac{u_{j+1}^n - u_{j-1}^n}{2h}\Big) = 0$$

where the subscripts indicate a discrete point in space, the superscripts indicate point in time, k is the time step, h is the spatial step, and c is the speed of propagation ($c = 0.1$).

(a) Show that this scheme can be rearranged to be written in a more easily programmable format:

$$u_j^{n+1} = -ck\Big(\frac{u_{j+1}^n - u_{j-1}^n}{2h}\Big) + \frac{1}{2}(u_{j-1}^n + u_{j+1}^n)$$

This scheme is first order accurate in time and second order accurate in space. It is numerically stable if and only if

$$\Big|\frac{ck}{h}\Big| \leq 1$$

(b) Write a computer program that simulates the PDE using the Lax-Friedrichs scheme. To begin, notice that u_j^{n+1} is a linear function of u_{j-1}^n and u_{j+1}^n. This means that we can create a matrix that maps a vector $\langle u_0^n, \cdots, u_k^n\rangle$ to $\langle u_0^{n+1}, \cdots, u_k^{n+1}\rangle$. Our computer program will repeatedly apply this matrix to a certain initial density distribution in order to find the density distribution at a future time. Plot the density distribution at several times to get an idea as to what a *traveling wave* means. Make note of the periodic boundary conditions, and take care to properly encode these into your program (take extra care to make sure your matrix is correct). Interpret what this computed solution is saying about the evolution of the density profile.

(c) Choose new values of k and h so that the solution is not stable. Use your computer program to plot solutions for different values of time with these new step sizes. What happens to the solutions?

2. Phase 2: **Method of characteristics**

Suppose we wish to solve a first order partial differential equation of the form

$$u_t + [f(u)]_x = 0 \quad -\infty < x < \infty \quad t > 0 \tag{1}$$

with $u(x, 0) = g(x)$. Observe that equation (1) can be re-written as

$$u_t + f'(u)u_x = 0 \tag{2}$$

We look for curves in the $x - t$ plane parameterized by some variable s along which the value of u is a constant, i.e. we look for curves along which the initial value of u is propagated. Once we have found these curves, called *characteristics*, we can fill out the entire $x - t$ plane with them and thus know the value of u at each point (x, t).

Let Γ be a curve in the $x - t$ plane parameterized s: $x = x(s), t = t(s)$. Computing the total derivative of $u = u(x(s), t(s))$ with respect to s, we obtain by chain rule:

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} \tag{3}$$

Hence, if we set

$$\frac{dt}{ds} = 1 \quad \frac{dx}{ds} = f'(u) \quad \frac{du}{ds} = 0$$

equations (2) and (3) are the same. We have reduced the problem of solving a partial differential equation to solving a relatively simple system of ordinary differential equations. We need initial conditions for t, x, and u with respect to s. Since t always starts from 0, we have $t(0) = 0$. Our initial condition for x depends on where we choose to begin. Hence, we set $x(0) = x_0$, from some arbitrary value of x_0. Based on the initial condition for u given above in (1), we have that $u(x(0), t(0)) = u(x_0, 0) = g(x_0)$. Solving the system, we obtain

$$t = s \tag{4}$$
$$x = f'(u)s + x_0 \tag{5}$$
$$u = g(x_0) \tag{6}$$

In particular, along Γ, t and s are equivalent and $u = g(x_0)$, a constant value. We thus have:

$$x = f'(g(x_0))t + x_0 \tag{7}$$

Solving for x_0 in (7) and substituting into (6), we obtain the solution:

$$u(x, t) = g(x - f'(g(x_0))t) \tag{8}$$

Equation (7) is the *characteristic* along which u is a constant. In this case, it is a straight line in the $x - t$ plane with slope $f'(g(x_0))$. Equation (8) is the solution to the partial differential equation provided we begin at the point $x = x_0$. Physically, (8) describes a *traveling wave* solution, i.e. it is a wave propagating with speed $f'(g(x_0))$ in the positive x direction. We can construct a characteristic for any x_0 and thus obtain the solution for u throughout the $x - t$ plane.

(a) Solve the partial differential equation

$$u_t + cu_x = 0 \quad u(x, 0) = \sin(x)$$

using the method of characteristics.

i. Calculate $f'(g(x_0))$ of equation (7) for this differential equation and initial conditions.

ii. What does this value of $f'(g(x_0))$ tell us about the physical properties of our traffic model?

(b) Plot (computer) the characteristics for $c = 0.1$ in the $x-t$-plane for several different x_0.

3. Phase 3: **The Green Light Problem**

Suppose that bumper-to-bumper traffic is standing at a red light, placed at $x = 0$, while the road ahead is empty. Accordingly, the initial density profile is

$$g(x) = \begin{cases} \rho_m & x \leq 0 \\ 0 & x > 0 \end{cases}$$

where ρ_m is the bumper-to-bumper car density.

At time $t = 0$ the traffic light turns green. We would like to describe the car density evolution for $t > 0$. Consider the following model for the evolution of the car density:

$$\rho_t + [q(\rho)]_x = 0$$

where ρ is defined above and

$$q(\rho) = v_m \rho \Big(1 - \frac{\rho}{\rho_m}\Big)$$

where v_m is the maximum possible speed the cars are allowed to travel (i.e. we are assuming an ideal world where everyone obeys the speed limit!).

(a) Find $q'(\rho) = dq/d\rho$.

(b) Graph $x = q'(g(x_0))t + x_0$ for several different values of x_0, in the $x - t$ plane.

(c) Do the characteristics fill the entire $x - t$ plane? Given any point in the $x - t$ plane, are we able to determine the traffic density?

(d) The issues in part (a) arise from the fact that the initial data is discontinuous. A natural remedy is to approximate the initial data by a continuous (not necessarily everywhere differentiable) profile g_ϵ that satisfies

$$\lim_{\epsilon \to 0} g_\epsilon = g$$

and solve the problem with g_ϵ as the prescribed initial data. Construct an appropriate g_ϵ and construct the characteristics for the ϵ-solution. Sketch the characteristics in the $x - t$ plane. Obtain the solution to the partial differential equation in the case that g_ϵ is the initial data.

(e) Construct the solution to the original problem based on what you derived in (b).

(f) In the $x-t$ plane, draw a sample trajectory beginning at some point $x_0 < 0$. Describe what the path means with regards to the evolution of a car density in the aftermath of a red light turning green.