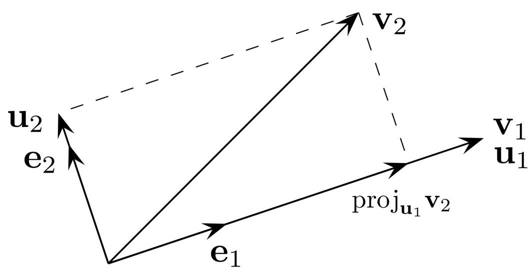
Name:____ Lab Score:_____/40

Answer each question in the area below. This assignment is due **one week** after the distribution of the lab, collected at the beginning of the next lab. Show all work and explain your reasoning. Due to the length of time allowed to complete the assignment, your work is expected to be clear and polished. If the work is at all ambiguous, it is considered incorrect.

1. In this problem, we'll introduce the **Gram-Schmidt process**, which takes in a finite, linear independent set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and generates an orthogonal set $S' = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$.



2 (a) Start with $\mathbf{v}_1 = \mathbf{u}_1$. The second element of your set $\mathbf{u}_2$ is then $\mathbf{v}_2$ but with the projection of $\mathbf{v}_2$ onto $\mathbf{u}_1$ removed. Write down the expression for this.

5 (b) Prove that the expression you wrote previously for $\mathbf{u}_2$ is indeed orthogonal to $\mathbf{u}_1$.

5 (c) Using the previous idea, how do we generate $\mathbf{u}_3$? And in general, $\mathbf{u}_j$?

2 (d) How can we construct an *orthonormal* set $\hat{S} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ from S'?

Solution:

(a) We call (either reconvince yourself or look up in the book) that the projection of **b** onto **a** is

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\,\frac{\mathbf{a}}{\|\mathbf{a}\|}$$

The Gram Schmit procedure then specifies that

$$\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}\mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 \tag{1}$$

(b) We now need to prove that (1) is indeed orthogonal to $\mathbf{u}_1$, as this procedure hopes to generate. Using the linearity of the dot product, we have

$$\begin{aligned}
\mathbf{u}_2\cdot\mathbf{u}_1 &= \left(\mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1\right)\cdot\mathbf{u}_1\\
&= \mathbf{v}_2\cdot\mathbf{u}_1 - \left(\frac{\mathbf{v}_2\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1\cdot\mathbf{u}_1\\
&= \mathbf{v}_2\cdot\mathbf{u}_1 - \left(\frac{\mathbf{v}_2\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\|\mathbf{u}_1\|^2\\
&= \mathbf{v}_2\cdot\mathbf{u}_1 - \mathbf{v}_2\cdot\mathbf{u}_1\\
&= 0.
\end{aligned}$$

(c) In general, the procedure is: take our newest vector v_j and subtract out the components of v_j that have already been used up, that is, all of the previous u_i. In other words,

$$\mathbf{u}_3 = \mathbf{v}_3 - \operatorname{proj}_{\mathbf{u}_2} \mathbf{v}_3 - \operatorname{proj}_{\mathbf{u}_1} \mathbf{v}_3.$$

From this, we can generalize

$$\mathbf{u}_j = \mathbf{v}_j - \sum_{k=1}^{j-1} \operatorname{proj}_{\mathbf{u}_k} \mathbf{v}_j.$$

(d) Ad-hoc, we have no guarantee that the collection $\{\mathbf{u}_j\}$ is *orthonormal*, only that it is orthogonal, however if we take

$$\mathbf{e}_j = \frac{\mathbf{u}_j}{\|\mathbf{u}_j\|},$$

which we know completely, then $\{\mathbf{e}_j\}$ is indeed an orthonormal set we generated from a linearly independent set of vectors.

2. Perform the Gram-Schmidt procedure on the set and verify your results are indeed orthogonal.

5 (a)

$$S = \{\mathbf{v}_1 = (3,2), \qquad \mathbf{v}_2 = (2,2)\}, \qquad \langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a} \cdot \mathbf{b}.$$

5 (b)

$$S = \{f_1(x) = 1, \qquad f_2(x) = x\}, \qquad \langle f, g \rangle = \int_0^1 f(x) g(x) \, \mathrm{d}x.$$

Solution: We use the aforementioned Gram-Schmidt procedure, which takes a set of linearly independent vectors S and transforms it into an orthogonal set $S' = \{u_j\}$ under the relationship

$$u_j = v_j - \sum_{k=1}^{j-1} \operatorname{proj}_{u_k} v_j = v_j - \sum_{k=1}^{j-1} \frac{\langle u_k, v_j \rangle}{\|u_k\|^2} u_k.$$

In particular, the first two vectors are

$$u_1 = v_1 \qquad u_2 = v_2 - \frac{\langle u_1, v_2 \rangle}{\|u_1\|^2} u_1.$$

(a) Here, the inner product we use is just the standard dot product. By construction, $u_1 = v_1$, but then we need to compte some other things

$$\|u_1\|^2 = \langle u_1, u_1 \rangle = 3(3) + 2(2) = 13,$$

and also

$$\langle u_1, v_2 \rangle = (3,2) \cdot (2,2) = 10.$$

Thus, our second vector is

$$u_2 = v_2 - \frac{\langle u_1, v_2 \rangle}{\|u_1\|^2} u_1 = (2,2) - \frac{10}{13}(3,2) = \left(-\frac{4}{13}, \frac{6}{13}\right).$$

We can verify this is indeed orthogonal to our first vector

$$\langle u_1, u_2 \rangle = 3\left(-\frac{4}{13}\right) + 2\left(\frac{6}{13}\right) = 0.$$

Thus, the procedure was successful.

(b) The procedure is exactly the same regardless of inner product. We still compute

$$u_1 = v_1 \qquad u_2 = v_2 - \frac{\langle u_1, v_2\rangle}{\|u_1\|^2} u_1.$$

where here the inner product is the integral. Computing these components

$$\|u_1\|^2 = \langle u_1, u_1\rangle = \int_0^1 1(1)\,\mathrm{d}x = 1.$$

And the inner product

$$\langle u_1, v_2\rangle = \int_0^1 x\,\mathrm{d}x = \frac{1}{2}.$$

Combining these parts, we get the second vector

$$u_2 = v_2 - \frac{\langle u_1, v_2\rangle}{\|u_1\|^2} u_1 = x - \frac{1}{2}(1) = x - \frac{1}{2}.$$

To verify our vectors our orthogonal we compute their inner product

$$\langle 1, x - \frac{1}{2}\rangle = \int_0^1 \left(x - \frac{1}{2}\right)\,\mathrm{d}x = 0.$$

6 3. Show that if $f(x)$ is an even function on the interval $x \in [-\pi, \pi]$, then the projection onto the Fourier basis $\cos(nx)$ and $\sin(nx)$, where n is a natural number, contains only cosine terms. That is, the sine projections are zero. Conversely if f is odd, show there's only sine terms.

Solution: We recall that using the orthogonality of $\sin nx$ and $\cos nx$, if we wanted to write $f(x)$ as the Fourier series

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$$

then we can explicitly find the coefficients using the orthogonal projection formula

$$b_n = \frac{\langle f, \sin nx\rangle}{\langle \sin nx, \sin nx\rangle}.$$

However, we note that if $f(x)$ is even, which is defined to be $f(-x) = f(x)$, then we can compute this integral and note that $\sin nx$ is odd, and an odd times an even function is still odd, resulting in

$$\begin{aligned} b_n &= \int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x \\ &= \int_0^{\pi} f(x)\sin(nx)\mathrm{d}x + \int_{-\pi}^{0} f(x)\sin(nx)\mathrm{d}x \\ & u = -x, \quad -\mathrm{d}u = \mathrm{d}x \implies \\ b_n &= \int_0^{\pi} f(x)\sin(nx)\mathrm{d}x - \int_0^{\pi} f(-u)\sin(nu)\mathrm{d}u \\ &= \int_0^{\pi} f(x)\sin(nx)\mathrm{d}x - \int_0^{\pi} f(u)\sin(nu)\mathrm{d}u = 0. \end{aligned}$$

If f is odd, defined to be $f(x) = -f(-x)$, we note that an odd times an odd function is even, and perform a nearly identical calculation.

4. Consider the square wave, described by

$$f(x) = \begin{cases} -1 & 0 \le x \le \frac{1}{2} \\ 1 & \frac{1}{2} < x \le 1 \end{cases}.$$

And consider the collection of functions

$$S_m = \{s_n(x) = \sin(2n\pi x)\}_{n=1}^{m}.$$

It's worth noting here that because $f(x)$ we can think of as an odd function, $\cos(2n\pi x)$ is not necessary by the previous problem.

7 (a) Write down an expression for c_n in the least squares approximation of

$$f(x) = c_1 s_1(x) + \cdots c_m s_m(x).$$

3 (b) Plot your least squares approximation for $m = 1, 2, 3, 4$ (or higher) against $f(x)$ using any plotting software. What do you see? Is this a good approximation?

Solution: By the orthogonality of $\sin 2n\pi x$, we can compute the coefficients c_j by the orthogonal projections formula

$$c_j = \frac{\langle f, s_j(x)\rangle}{\langle s_j, s_j\rangle}.$$

Let us first compute the denominator

$$\begin{aligned}
\langle s_j(x), s_j(x)\rangle &= \int_0^1 \sin(2\pi jx)\sin(2\pi jx)\,\mathrm{d}x \\
&= \int_0^1 \left(\frac{1}{2} - \frac{1}{2}\cos(4\pi jx)\right)\mathrm{d}x && \text{(trig identity)} \\
&= \frac{1}{2} - \left\{\frac{1}{8\pi j}\sin(4\pi jx)\right\}_{x=0}^{x=1} \\
&= \frac{1}{2} && \text{(since } \sin 0 = 0 \text{ and } \sin 4\pi = 0)
\end{aligned}$$

Now, we compute the numerator

$$\begin{aligned}
\langle f, s_j(x)\rangle &= \int_0^1 f(x)\sin j\pi x\,\mathrm{d}x \\
&= \int_0^{1/2} (-1)\sin j\pi x\,\mathrm{d}x + \int_{1/2}^1 (1)\sin j\pi x\,\mathrm{d}x && \text{(definition of } f(x)) \\
&= \left\{\frac{\cos(2\pi jx}{2\pi j}\right\}_{x=0}^{x=1/2} + \left\{-\frac{\cos(2\pi jx}{2\pi j}\right\}_{x=1/2}^{x=1} \\
&= \frac{1}{2\pi j}\{\cos(\pi j) - \cos(0) - \cos(2\pi j) + \cos(\pi j)\} \\
&= \frac{1}{2\pi j}\{2\cos(\pi j) - 2\} && \text{(since } \cos(2\pi j) = 1) \\
&= \frac{1}{\pi j}\{\cos(\pi j) - 1\}.
\end{aligned}$$

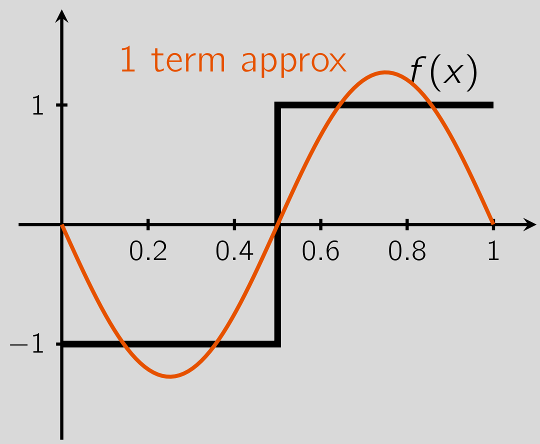
Now, a somewhat interesting issue is what is $\cos \pi j$? It's not constant, since $\cos 0 = 1, \cos \pi = -1, \cos 2\pi = 1$, but we see that it follows the pattern

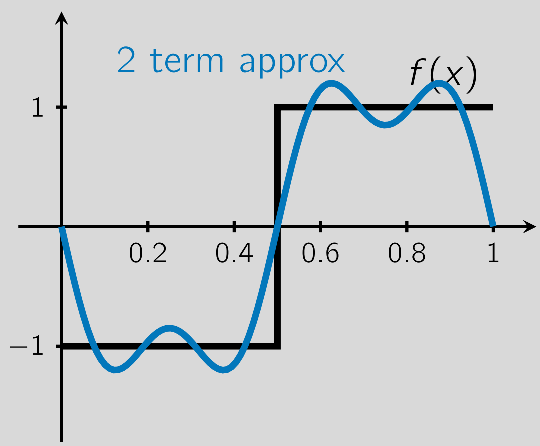
$$\cos \pi j = (-1)^j$$

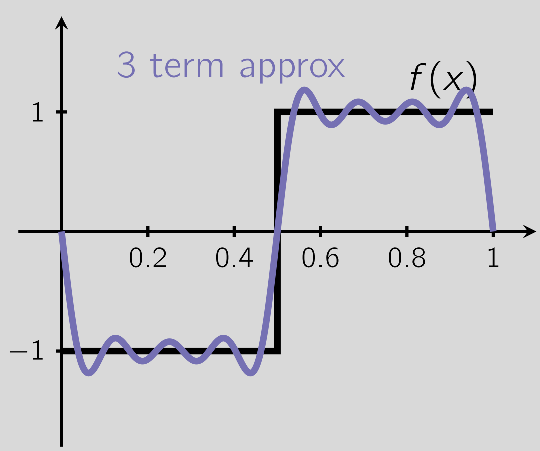
meaning that our full coefficient is then

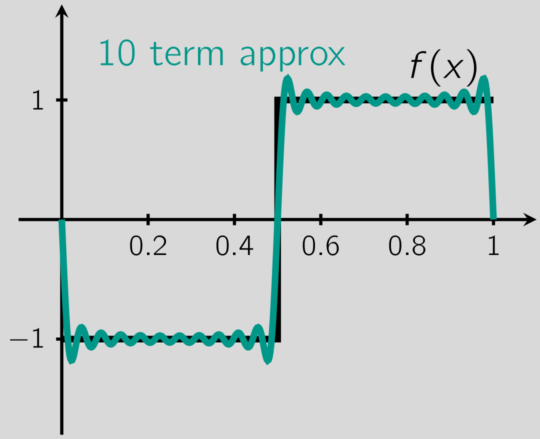
$$c_j = \frac{\langle f, s_j(x) \rangle}{\langle s_j, s_j \rangle} = \frac{2}{\pi j} \left\{ (-1)^j - 1 \right\} = \begin{cases} 0 & j \text{ even} \\ -\frac{4}{\pi j} & j \text{ odd} \end{cases}$$

Thus, we can now plot this after truncating after a few terms. I probably should have asked for more than 5 to see the desired effect, but it is shown below.









From this, we can draw a conclusion: **adding more terms does not necessarily resolve discontinuities**. In fact, in some sense, adding more terms makes the approximation "worse". This is a general property of Fourier series that can be intuited as: approximating a non-smooth function by a series of smooth ones is hard. This particular plot illustrates the **Gibbs Phenomenon**.