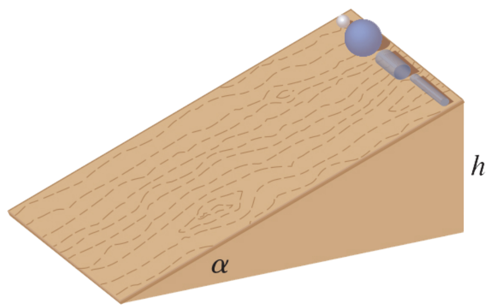
Name:____ Lab Score:_____/40

Answer each question in the area below. This assignment is due **one week** after the distribution of the lab, collected at the beginning of the next lab. Show all work and explain your reasoning. Due to the length of time allowed to complete the assignment, your work is expected to be clear and polished. If the work is at all ambiguous, it is considered incorrect.

Consider dropping four different types of objects down an incline. The first two objects are balls: one solid (think bowling ball) and another hollow (tennis ball). The next two objects are cylinders: one solid and one hollow. Which of these objects would we bet on (not in Utah) to win a race down the incline?



Consider each of these objects to have a mass m and radius r and a moment of inertia (about the axis of rotation) of I. The vertical drop of the incline is h and is at an angle α. We can immediately observe, that for each object

$$\text{potential energy at top} = mgh$$

and then, at the bottom, has two sources of kinetic energy, where it rolls with velocity v and angular velocity $v = \omega r$, resulting in

$$\text{kinetic energy at bottom} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

Thus, from the conservation of energy, we have

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

3 1. Show that

$$v^2 = \frac{2gh}{1+I^\star}, \qquad \text{where } I^\star := \frac{I}{mr^2}.$$

Solution: We start with the conservation of energy and use the fact that $v = \omega r$, so $\omega = v/r$, resulting in

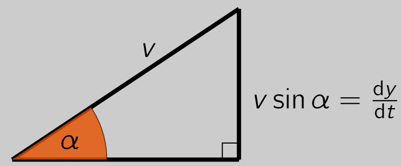
$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2$$
$$mgh = \frac{1}{2}\left(m + \frac{I}{r^2}\right)v^2$$
$$\frac{2mgh}{m + \frac{I}{r^2}} = v^2$$
$$\frac{2gh}{1 + \frac{I}{mr^2}} = v^2.$$

3 2. Let $y(t) :=$ the vertical distance traveled at time t. Part 1 allows us to conclude that $v^2 = 2gy/(1+I^\star)$ at any time t. Use this result to show that y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{\frac{2g}{1+I^\star}}\sqrt{y}\sin\alpha.$$

Hint: what is the vertical component of the speed?

Solution: Consider the geomtery of the velocity shown below.



In other words, we have that the vertical component of the speed is $v\sin\alpha$, which is the rate of change of the height, or $\mathrm{d}y/\mathrm{d}t$. Thus, we get the relationship

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v\sin\alpha = \sqrt{\frac{2gy}{1+I^\star}}\sin\alpha = \sqrt{\frac{2g}{1+I^\star}}\sqrt{y}\sin\alpha.$$

4 3. By solving the differential equation in part 2, conclude that the total travel time is

$$T = \sqrt{\frac{2h(1+I^\star)}{g\sin^2\alpha}}.$$

Solution: Here, we have a simple (if you remember ODEs) application of separation of variables. Rearranging the ODE so that y's and t's are exclusively on one side

$$\frac{\mathrm{d}y}{\sqrt{y}} = \sqrt{\frac{2g}{1+I^\star}}\mathrm{d}t.$$

And integrating, we get

$$2\sqrt{y} = \sqrt{\frac{2g}{1+I^\star}}\sin\alpha t + C.$$

Here, we consider that $y(0) = 0$ since y represents the vertical distance moved (which is none to begin with), so $C = 0$.We are now interested in finding the time T where $y = h$, so plugging this in, we get

$$2\sqrt{h} = \sqrt{\frac{2g}{1+I^\star}}\sin\alpha T \qquad \Longrightarrow \qquad T = \sqrt{\frac{2h(1+I^\star)}{g\sin^2\alpha}}.$$

2 4. From the previous part, what can you say about the relationship between an objects effective moment of inertia, $I^\star$ and how it performs in the race?

Solution: From the above expression, it's clear that the larger $I^\star$ is, the longer it takes to complete hte race. In other words, the smallest $I^\star$ wins. Profoundly, each of these objects has a constant $I^\star$. That is, the mass and radius of the object doesn't even matter!

7 5. For a cylinder rotating around the z axis with density $\rho(x, y, z)$, the moment of inertia I can be computed by

$$I = \iiint_E (x^2 + y^2)\, \rho(x, y, z)\, \mathrm{d}V. \tag{1}$$

For the solid cylinder, using (1), show that $I^\star = \frac{1}{2}$, supposing that the cylinder has mass m, radius r and length ℓ.

Hint: what is the (constant) density here?

Solution: Here, we have an object with a constant density that can be defined as

$$\hat{\rho} = \frac{\text{mass}}{\text{volume}} = \frac{m}{\pi r^2 \ell},$$

where I've switched to the $\hat{}$ notation to avoid confusion later. Now, we simply need to compute (1) in this case, recalling that we have a cylinder of radius r and height ℓ, meaning cylindrical coordinates make the most sense.

$$\begin{aligned} I &= \iiint_E (x^2 + y^2)\, \hat{\rho}(x, y, z)\, \mathrm{d}V \\ &= \int_0^\ell \int_0^{2\pi} \int_0^r R^2 \hat{\rho}\, R\, \mathrm{d}R\, \mathrm{d}\theta \mathrm{d}z \\ &= \hat{\rho} \int_0^\ell \int_0^{2\pi} \int_0^r R^3\, \mathrm{d}R\, \mathrm{d}\theta \mathrm{d}z \\ &= \frac{m}{\pi r^2 \ell} 2\pi\ell \left[\frac{1}{4} R^4\right]_{R=0}^{R=r} = \frac{mr^2}{2}. \end{aligned}$$

Note that we've used R instead of the traditional r for cylindrical coordinates because r is already the radius of the cylinder. Now, piecing this together, we get

$$I^\star = \frac{I}{mr^2} = \frac{\frac{mr^2}{2}}{\frac{1}{mr^2}} = \frac{1}{2}.$$

7 6. Show that $I^\star = 1$ for the hollow cylinder using (1).

Hint: where is all of the mass (density) located for the hollow cylinder? Do you need 3 dimensions here?

Solution: There are two solution techniques here.

1. The first technique is to assume that we have a semi-hollow cylinder, with inner radius a and outer radius r and then consider the limit that it becomes hollow, that is $a \to r$.

   In this case, we again have
   $$\hat{\rho} = \frac{\text{mass}}{\text{volume}} = \frac{m}{\pi\ell(r^2 - a^2)},$$

and the integration step is nearly identical:

$$\begin{aligned} I &= \iiint_E (x^2+y^2)\,\hat{\rho}(x,y,z)\,\mathrm{d}V \\ &= \int_0^\ell \int_0^{2\pi} \int_0^r R^2 \hat{\rho}\, R\,\mathrm{d}R\,\mathrm{d}\theta \mathrm{d}z \\ &= \hat{\rho} \int_0^\ell \int_0^{2\pi} \int_0^r R^3\,\mathrm{d}R\,\mathrm{d}\theta \mathrm{d}z \\ &= \frac{m}{\pi\ell(r^2-a^2)} 2\pi\ell \left[\frac{1}{4}R^4\right]_{R=a}^{R=r} = \frac{m(r^4-a^4)}{2(r^2-a^2)}. \end{aligned}$$

Now, we want to take the limit, of this expression as $a \to r$

$$\lim_{a\to r} \frac{m(r^4-a^4)}{2(r^2-a^2)} = \lim_{a\to r} \frac{m}{2}\frac{r^4-a^4}{r^2-a^2} = \frac{m}{2}\cdot\frac{0}{0}.$$

Thus, we need to take some derivatives with respect to a and use L'Hôpital's rule, resulting in

$$\lim_{a\to r} \frac{m(r^4-a^4)}{2(r^2-a^2)} = \lim_{a\to r} \frac{m}{2}\cdot\frac{-4a^3}{-2a} = mr^2.$$

Thus, we can conclude

$$I^\star = \frac{I}{mr^2} = \frac{mr^2}{mr^2} = 1.$$

2. The second technique is to consider the fact that if the cylinder is completely hollow, it no longer has a volume, only an area. Thus, the density now is defined to be

$$\hat{\rho} = \frac{\text{mass}}{\text{surface area}} = \frac{m}{2\pi r\ell}.$$

Therefore, we simply need to integrate over the surface area, which becomes a 2D integral now:

$$I = \iint_E (x^2+y^2)\hat{\rho}\,\mathrm{d}A = \hat{\rho}\int_0^\ell \int_0^{2\pi} r^2\,\mathrm{d}\theta\,\mathrm{d}z = \frac{m}{2\pi\ell r}\pi r^2 \ell = mr^2,$$

so again, we get the same result that $I^\star = 1$.

7 7. Calculate $I^\star$ for a partly hollow ball with inner radius a and an outer radius r. Express your answer in terms of $b := a/r$.

Hint: to figure out ρ here, consider the volume of this ball as a big ball minus a little ball.

Solution: Although, we could in theory do this problem the same way we did the hollow cylinder (with a surface area integral instead of a volume), but this way is considerably easier. Yet again, we have

$$\hat{\rho} = \frac{\text{mass}}{\text{volume}} = \frac{m}{\frac{4}{3}\pi(r^3-a^3)},$$

where we've used the formula for a sphere $V = \frac{4}{3}\pi r^3$. Now, we simply use (1), but in spherical coordinates (since we have a sphere). Noting that

$$x^2+y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = \rho^2\sin^2\phi.$$

Using this fact, we have

$$\begin{aligned}\iiint_E (x^2+y^2)\hat{\rho}\,\mathrm{d}V &= \int_0^{\pi}\int_0^{2\pi}\int_a^r \hat{\rho}\,\rho^2\sin^2\phi\,\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi\\ &= \hat{\rho}\int_0^{\pi}\int_0^{2\pi}\int_a^r \rho^2\sin^2\phi\,\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi\\ &= \hat{\rho}\cdot 2\pi\left[-\frac{(2+\sin^2\phi)}{3}\right]_{\phi=0}^{\phi=\pi}\cdot\left[\frac{\rho^5}{5}\right]_{\rho=a}^{\rho=r}\\ &= \hat{\rho}\cdot 2\pi\cdot\frac{4}{3}\cdot\frac{r^5-a^5}{5}\\ &= \frac{m}{\frac{4}{3}(r^3-a^3)}\cdot 2\pi\cdot\frac{r^5-a^5}{5} = \frac{2m(r^5-a^5)}{(r^3-a^3)}.\end{aligned}$$

5 8. Using the previous answer, what is $I^\star$ for a solid ball? A hollow ball?

Solution: Here, the solid ball corresponds to $a \to 0$, which is easy to read off that

$$I = \left[\frac{2m(r^5-a^5)}{(r^3-a^3)}\right]_{a=0} = \frac{2}{5}mr^2,$$

which means that $I^\star = 2/5$.

For the hollow sphere, we need to take $a \to r$, but this again leads us to a L'Hôpital limit, so we have to take a derivative with respect to a

$$I = \left[\frac{2m(r^5-a^5)}{(r^3-a^3)}\right]_{a\to r} = \frac{2m}{\cdot}\frac{-5a^4}{-3a^2} = \frac{2}{3}mr^2,$$

so we conclude $I^\star = 2/3$.

2 9. Among the solid and hollow balls and cylinders, which wins the race?

Solution: We see that, after comparing each of the $I^\star$ values, the results of the race are:

1. solid ball ($I^\star = 2/5$)
2. solid cylinder ($I^\star = 1/2$)
3. hollow ball ($I^\star = 2/3$)
4. hollow cylinder ($I^\star = 1$)