Math 3140 – PDEs HW 4

Due: July 18, 2016

1. Solve Laplace on the semi circle $\theta \in [0, \pi]$ and $r \leq a$.

   (a) $u = 0$ on the diameter ($u(r, 0) = u(r, \pi) = 0$) and $u(a, \theta) = g(\theta) = \theta(\pi - \theta)$ on the upper curve. Sketch a graph of the solution surface indicating critical features of the solution and its boundary conditions.

   Solution: We write Laplace's equation in polar coordinates (r, θ):

   $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

   As usual, we perform separation of variables with $u(r, \theta) = p(r)q(\theta)$ to yield the two ODES

   $$r^2 p'' + rp' = \lambda p, \qquad q'' = -\lambda q.$$

   Here, we see from the problem we have that $q(0) = q(\pi) = 0$, so our boundary value problem becomes

   $$q'' = -\lambda q, \qquad q(0) = q(\pi) = 0.$$

   We have seen this exactly BVP a zillion times (here, with $L = \pi$) and know it has eigenvalues $\lambda_n = (n\pi/L)^2 = n^2$ and eigenfunctions $\sin n\pi\theta/L = \sin n\theta$.

   We also discussed in class that the solutions to the p ODE are of the form $p_n = c_1 r^n + c_2 r^{-n}$ but since $r \to 0$ would cause r^{-n} to blow up, we only get r^n. Combining our solutions in superposition, we get

   $$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

   We now use the boundaries to figure out the unknown coefficients A_n by evaluating at $r = a$:

   $$u(a, \theta) = g(\theta) = \theta(\pi - \theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta.$$

   This is a typical Fourier series, where we can exploit the orthogonality of $\sin n\theta$ to get the coefficients

   $$\int_0^\pi g(\theta) \sin n\theta \, \mathrm{d}\theta = A_n a^n \int_0^\pi \sin n\theta \, \mathrm{d}\theta,$$

   which, we can evaluate to yield

   $$\frac{2[1 - (-1)^n]}{n^3} = A_n a^n \frac{\pi}{2}$$

   which we can extract the coefficients from

   $$A_n = \begin{cases} \frac{8}{\pi n^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

   (b) Insulated on the diameter ($u_\theta(r, 0) = u_\theta(r, \pi) = 0$) and $u(a, \theta) = g(\theta) = \cos(\theta)$ on the upper curve. Sketch a graph of the solution surface indicating critical features of the solution and its boundary conditions.

Solution: This is nearly identical to the previous problem except for the q ODE being slightly different:

$$q'' = -\lambda q, \qquad q'(0) = q'(\pi) = 0.$$

Thankfully, we have again solved this a ton of times and know the eigenvalues are $\lambda_n = n^2$ (including $n = 0$) and eigenfunctions $\cos n\theta$. Thus, our full solution looks like

$$u(r,\theta) = \sum_{n=0}^{\infty} B_n r^n \cos n\theta.$$

We must use $g(\theta)$, our boundary to determine the unknown coefficients:

$$u(a,\theta) = g(\theta) = \cos\theta = B_0 + \sum_{n=1}^{\infty} B_n a^n \cos n\theta.$$

However, we don't even need to use the orthogonality to see that $n = 1$ is the only cos that survives, meaning we get $B_1 = a^{-n}$ and therefore our full solution is

$$u(r,\theta) = r^n a^{-n} \cos\theta.$$

2. Suppose $u(r,\theta)$ is equilibrium temperature solution to Laplace's equation on a disk for radius $r \leq 2$. On $r = 2$ the temperature is $u(2,\theta) = \theta^2 - 2\pi\theta + 1$. What is the temperature at $r = 0$? What essential property of Laplace's solution determines the answer.

Solution: This is an immediate application of the **mean value property**, which says that the temperature at any point is the average of the temperatures on the circle around it. Thus, we just average the temperature on the edge of the disk to get the temperature in the middle:

$$u_0 = \frac{1}{2\pi}\int_0^{2\pi} \theta^2 - 2\pi\theta + 1 d\theta = 1 - \frac{2\pi^2}{3} \approx -5.58$$

3. In what situation does one use the *even extension* of $f(x)$ for a Fourier series? In what situation do we instead use a *odd extension*? *Hint:* what eigenfunctions are you trying to build your series with?

Solution: We know that Fourier series are defined for a full interval $[-L, L]$ and include **both** sine and cosine terms. However, sometimes we want to **only** use cosine or sine terms (think about our solutions to the heat equation for various boundaries).

Thus, if we say, wanted to construct a solution to the heat equation on $[0, L]$ with zero-temperature boundaries, we would want a Fourier sine series. We could think of this as a cut-out of a full Fourier series on $[-L, L]$. How could we cook up a series that all of the cosine terms die? Well, cosine is an even function and sine is odd, so if we do an **odd extension**, all of the cosine terms die and we get a Fourier sine series.

By the same logic, if we say, had the heat equation on $[0, L]$ with no flux boundaries, we'd want to use a cosine series. We could think of this as a cut-out of a full Fourier series on $[-L, L]$ where all the sine terms die. Since sine is an odd function, this means we can do an **even extension** and all of the terms would naturally die.

In both cases, we use the liberty of doing whatever we want outside of the region of $[0, L]$ because our final series matches in this region of interest.

4. Consider the function

$$f(x) = \begin{cases} x & x > 0 \\ -x - x^2 & x \leq 0, \end{cases}$$

over the interval $[-1, 1]$.

(a) Compute the Fourier *sine series* of $f(x)$. Sketch the result.

(b) Compute the Fourier *cosine series* of $f(x)$. Sketch the result.

(c) Compute the full Fourier series of $f(x)$. You guessed it: sketch the result.

(d) If you did this correctly, you should get 3 different series. Why are they different?

Solution:

(a) For the sine series, we only consider the definition of $f(x)$ from $[0, L]$ or in this case, $[0, 1]$. From there, we imagine it is indeed a full Fourier series on $[-1, 1]$ but we construct the function we want to compute the full Fourier series by taking the **odd extension** of $f(x)$ onto the interval $[-1, 1]$. Thus, the odd extension of this is really:

$$f_o(x) = x$$

for $x \in [-1, 1]$. The coefficients are then

$$B_n = 2 \int_0^1 x \sin n\pi x \, \mathrm{d}x = \frac{2}{n\pi}(-1)^{n+1}.$$

Thus, our representation is

$$f_o \sim \sum_{n=1}^{\infty} B_n \sin n\pi x,$$

which indeed matches our original function on $[0, 1]$. The plot of this looks like a sawtooth.

(b) For the cosine series, we only consider the definition of $f(x)$ from $[0, L]$ or in this case, $[0, 1]$. This is similar to before, but now we consider the even extension to get the cosine series, so which in this case is

$$f_e(x) = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

from which we can construct the coefficients through the relationship we established in class

$$A_0 = \int_0^1 x\,\mathrm{d}x = \frac{1}{2}, \qquad A_n = 2\int_0^1 x\cos n\pi x\,\mathrm{d}x = 2\frac{-1+(-1)^n}{n^2\pi^2},$$

with our representation as

$$f_e(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x.$$

The plot of this indeed matches $f(x)$ on $[0,1]$ but now looks like a repeated V shape.

(c) For the full series, we consider the full interval $[-L, L]$ or in this case, $[-1, 1]$. We represent this by the series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x$$

from which we can compute each (somewhat annoyingly)

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x = \frac{1}{2}\int_0^1 x\,\mathrm{d}x + \frac{1}{2}\int_{-1}^{0} -x - x^2\,\mathrm{d}x = \frac{2}{6}.$$

The other coefficients follow from the same type of integrals, which you must split up:

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,\mathrm{d}x, \qquad b_n = \int_{-1}^{1} f(x)\sin n\pi x\,\mathrm{d}x.$$

This gets really messy so don't worry if you didn't get it. I don't want to do it either.

This series looks like a bunch of (periodic) copies of our original $f(x)$.

(d) The point of this question is: all three of these series match on $[0, 1]$, which is perfectly fine if we were solving the heat equation on this interval, but by changing the behavior out of this interval, we can actually get 3 different series completely.