Math 3140 – PDEs HW III

Due: July 6, 2016

1. **Subtracting nonhomogeneities:** Consider the linear heat equation PDE $u_t = u_{xx}$ with nonhomogenous boundary conditions:

$$u(0) = -10, \quad u(2) = 10.$$

   (a) Find the equilibrium solution $u_e(x)$.

   (b) Modify the above PDE boundary conditions so they are homogeneous, that is, make the boundary conditions equal to zero. Let $w(x,t)$ be the solution to this modified equation.

   (c) Verify $u(x,t) = w(x,t) + u_e(x)$ solves the original nonhomogeneous PDE (i.e., it solves the equation AND satisfies the boundary conditions) by plugging in $w + u_e$ and using the above property of w.

Solution:

(a) Although these boundaries are a bit uglier than we're used to dealing with, it is no problem. In equilibrium, we have

$$0 = \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

so after integrating twice we find

$$u = c_1 + c_2 x.$$

But we have the boundary conditions $u(0) = -10$ and $u(2) = 10$, so this means that $c_1 = -10$ and consequently

$$10 = -10 + c_2(2) \implies c_2 = 10,$$

meaning our full equilibrium temperature is the straight line $u = -10 + 10x$.

(b) If we have zero-temperature conditions, we know the solution to this PDE is

$$w = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t},$$

where here $L = 2$ and $k = 1$so we have

$$w = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2} e^{-(n\pi/2)^2 t}.$$

(c) As the problem states, let us verify that

$$u = w + u_e = -10 + 10x + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2} e^{-(n\pi/2)^2 t},$$

does indeed satisfy the PDE and the boundary conditions. We know at $x = 0$ and $x = 2$, the sum term is 0 and consequently our boundary conditions are right. Now, we note

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\{-10 + 10x\} + \frac{\partial}{\partial t}\left\{\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2} e^{-(n\pi/2)^2 t}\right\}\\ &= 0 + \frac{\partial}{\partial t}\left\{\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2} e^{-(n\pi/2)^2 t}\right\}\end{aligned}$$

but since this satisfies the homogeneous equation, this is just u_{xx} since $(-10+10x)_{xx} = 0$ also does not appear.

2. Solve the heat equation $u_t = ku_{xx}$ on $x \in [0, 2]$ with the boundary conditions

$$u(0) = 20, \quad u(2) = 0$$

and initial condition

$$u(x, 0) = \sin(\frac{\pi}{2}x) - 10x + 20.$$

Hint: see previous problem and find $u_e(x)$, $w(x, t)$ to construct your full solution $u(x, t)$.

Solution: We know the recipe for inhomogeneous boundary conditions is to combine the homogeneous solution with the non-homogeneous equilibrium. Thus, we first find $u_e(x)$ which satisfies

$$0 = \frac{\partial u_e}{\partial t} = \frac{\partial^2 u_e}{\partial x^2}, \qquad u(0) = 20, \qquad u(2) = 0,$$

which suggests that $u_e(x) = 20 - 10x$. Next, we know the solution to our homogeneous equation is

$$w = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2} e^{-(n\pi/2)^2 t},$$

and therefore the full solution is

$$u = u_e + w = 20 - 10x + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2} e^{-(n\pi/2)^2 t}.$$

We have, at $t = 0$, using the initial condition

$$u(x, 0) = \sin(\pi x/2) - 10x + 10 = 20 - 10x + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2}.$$

Thus, we see the only term that appears in the sum is $n = 1$, which means $B_1 = 1$ and all the other terms are zero. Our full solution is thus

$$u(x, t) = 20 - 10x + \sin(\pi x/2)e^{-(\pi/2)^2 t}.$$

3. Use separation of variables to find all solutions $u_n(x, t)$ for a bar of length $x \in [0, L]$ with no-flux at $x = 0$ and and zero-temperature at $x = L$.

Solution: The general p-equation solution is

$$p(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

We incorporate BCs to determine

$$p'(x) = 0 \implies B = 0$$

and

$$\cos(\sqrt{\lambda}L) = 0$$

so

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2L}, \quad n = 1, 2, 3, 4, \ldots$$

So the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi}{2L}x\right)e^{-k\left(\frac{(2n-1)\pi}{2L}x\right)t}$$

That is,

$$u_n = \cos\left(\frac{(2n-1)\pi}{2L}x\right)e^{-k\left(\frac{(2n-1)\pi}{2L}x\right)t}$$

If you want to go further, the IC:

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi}{2L}x\right)$$

where the cosine terms are presumably orthogonal and the Fourier coefficients are :

$$a_n = X\int_0^L f(x)\cos\left(\frac{(2n-1)\pi}{2L}x\right)dx$$
$$X = \frac{2}{\int_0^L 1 + \cos\left(\frac{(2n-1)2\pi}{2L}x\right)dx} = 2/L$$

4. Solve the heat equation $u_t = u_{\theta\theta}$ on the ring $\theta \in [0, 2\pi)$ for $u(\theta, t)$, with initial condition

$$u(\theta, 0) = 3 + \sin(\theta).$$

Solution: We actually said in class that you get $\sin nx/2$ and $\cos nx/2$ as eigenfunctions for this, but let's derive it.

The key thing to note is that because we have a ring, we have periodic boundary conditions, say

$$u(0,t) = u(2\pi, t) \qquad \text{and} \qquad u_\theta(0,t) = u_\theta(2\pi, t).$$

We then take the separation of variables ansatz

$$u = p(\theta)q(t),$$

which when we plug it in, find the standard

$$\frac{p''}{p} = \frac{\dot{q}}{q} = -\lambda.$$

Our first ODE is then

$$\frac{\mathrm{d}^2 p}{\mathrm{d}\theta^2} = -\lambda p$$

subject to the conditions $p(0) = p(2\pi)$ and $p'(0) = p'(2\pi)$. We now consider all the possibilities of λ. If $\lambda > 0$, we get

$$p = c_1 \cos\sqrt{\lambda}\theta + c_2 \sin\sqrt{\lambda}\theta$$

and our initial conditions provide

$$p(0) = c_1 \cos\sqrt{\lambda}0 + c_2 \sin\sqrt{\lambda}0 = p(2\pi) = c_1 \cos\sqrt{\lambda}2\pi + c_2 \sin\sqrt{\lambda}2\pi,$$

but because $\cos 0 = 1$ and $\sin 0 = 0$, we get

$$c_1 \cos\sqrt{\lambda}2\pi + c_2 \sin\sqrt{\lambda}2\pi = 0.$$

However, we also have the condition $p'(0) = p'(2\pi)$, which ultimately provides us with

$$c_1\sqrt{\lambda}\sin\sqrt{\lambda}2\pi = 0,$$

which means that $\sqrt{\lambda}2\pi = n\pi$, or $\sqrt{\lambda} = n/2$ or $\lambda = n^2/4$. We see that c_1, c_2 really satisfy no other constraints, and consequently we get both eigenfunctions

$$p_n(\theta) = A_n \cos\frac{n\theta}{2} + B_n \sin\frac{n\theta}{2},$$

we can plug $\sqrt{\lambda} = n/2$ into the q ode to get

$$q_n(t) = c_n e^{-(n/2)^2 t}.$$

We also see that if $\lambda = 0$, our p ODE becomes $p'' = 0$, but $p = c_0$, a constant, does indeed satisfy our boundaries, so we get the $\lambda = 0$ eigenvalue. Our full solution is consequently

$$u(\theta, t) = a_0 + \sum_{n=1}\left\{a_n \cos\frac{n\theta}{2} + b_n \sin\frac{n\theta}{2}\right\} e^{-(n/2)^2 t}.$$

We need this to satisfy our boundary conditions at $t = 0$, so we get

$$u(\theta, 0) = 3 + \sin\theta = a_0 + \sum_{n=1}\left\{a_n \cos\frac{n\theta}{2} + b_n \sin\frac{n\theta}{2}\right\},$$

so we see that $a_0 = 3$, $b_2 = 1$ and all the other $b_j, a_j = 0$. Thus, our full solution is

$$u(\theta, t) = 3 + \sin\theta e^{-t}.$$

5. Non insulated rod on $x \in [0, \pi]$: The PDE $u_t = u_{xx} - u$ models heat transfer in through a rod whose radial surface is not insulated and is in contact with its environment with temperature $T = 0$. The end points are at a fixed temperature $u(0) = u(\pi) = 0$. The term $-u$ represents the heat loss/gain related to Newton's law of cooling. Let $u(x, t) = p(x)q(t)$. Perform the separation of variables, find all eigenfunctions and eigenvalues. Then express the general solution as a Fourier series. Suppose the initial condition is

$$u(x, 0) = f(x) = \begin{cases} x, & x \in [0, \pi/2] \\ \pi - x, & x \in (\pi/2, \pi] \end{cases}.$$

Specify the Fourier coefficients exactly and represent your solution as an infinite series.

Solution: Yet again, we use a standard separation of variables argument, taking $u = p(x)q(t)$, which leads to

$$p\dot{q} = p''q - pq,$$

which we can rearrange to get

$$\frac{\dot{q}}{q} = \frac{p'' - p}{p} = -\lambda.$$

However, this is a pretty annoying way to do this problem, so notice we can move the 1 over to get

$$\frac{\dot{q}}{q} + 1 = \frac{p''}{p} = -\lambda.$$

The nice thing about this is that our p ODE is exactly the same and we consequently get, with $L = \pi$,

$$\lambda_n(x) = \frac{n^2}{,} \qquad p_n(x) = \sin nx.$$

We now turn to the q ODE, which yields

$$\frac{\dot{q}}{q} = -(\lambda + 1),$$

or that

$$q_n(t) = c_n e^{-(n^2+1)t},$$

which gives us our solution

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin nx e^{-(n^2+1)t}.$$

We see the effect of the $-u$ term on the PDE is that our modes simply decay faster. We can then compute the coefficients

$$\begin{aligned} B_n &= \frac{\int_0^\pi f(x) \sin nx \, \mathrm{d}x}{\pi/2} \\ &= \frac{2}{\pi} \left\{ \int_0^{\pi/2} x \sin(nx) \, \mathrm{d}x + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, \mathrm{d}x \right\} \\ &= \frac{2}{n} (1 - \cos \frac{n\pi}{2}). \end{aligned}$$

6. Solve $\nabla^2 u = 0$ with the following boundary conditions on the rectangle $[0, L] \times [0, H]$:
$u_x(0, y) = g(y) = \sin(\frac{\pi}{H} y)$, $u_x(L, y) = 0$, $u(x, 0) = 0$ and $u(x, H) = 0$.
Use technology to render a graph of the solution.

Solution: Let $u(x, y) = p(x)q(y)$ and then

$$\begin{aligned} p''q + q''p &= 0 \\ p''/p = -q''/q &= \lambda \\ q'' &= -\lambda q \\ q_n(y) &= \sin\left(\frac{n\pi}{H} y\right) \\ \lambda = \left(\frac{n\pi}{H}\right)^2, \quad n &= 1, 2, 3, \ldots \end{aligned}$$

Now for the p-equation

$$\begin{aligned} p'' &= \lambda p \\ p(x) &= Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ u_x(L, y) = 0, \quad u_x(0, y) = g(y) &\implies \\ p'(x) &= \sqrt{\lambda}(Ae^{\sqrt{\lambda}x} - Be^{-\sqrt{\lambda}x}) \\ p'(L) = 0, \quad p'(0) \neq 0 \implies Ae^{\frac{n\pi L}{H}} - Be^{-\frac{n\pi L}{H}x} &= 0 \\ \implies Ae^{\frac{n\pi L}{H}} &= Be^{-\frac{n\pi L}{H}x} \\ B &= Ae^{\frac{2n\pi L}{H}} \\ \implies p_n(x) &= e^{\frac{n\pi}{H}x} + e^{\frac{2n\pi L}{H}} e^{-\frac{n\pi}{H}x} \end{aligned}$$

Put it all together:

$$\begin{aligned} u_n(x, y) &= \left(e^{\frac{n\pi}{H}x} + e^{\frac{2n\pi L}{H}} e^{-\frac{n\pi}{H}x}\right) \sin\left(\frac{n\pi}{H} y\right) \\ u(x, y) &= \sum_{n=1}^{\infty} b_n \left(e^{\frac{n\pi}{H}x} + e^{\frac{2n\pi L}{H}} e^{-\frac{n\pi}{H}x}\right) \sin\left(\frac{n\pi}{H} y\right) \end{aligned}$$

Apply boundary condition $g(y)$:

$$\begin{aligned} \frac{\partial u}{\partial x}(0, y) = g(y) &= \sum_{n=1}^{\infty} b_n \frac{n\pi}{H}\left(1 - e^{\frac{2n\pi L}{H}}\right) \sin\left(\frac{n\pi}{H} y\right) \\ &= \sin(\pi/Hy) \\ \implies b_1 \frac{\pi}{H}\left(1 - e^{\frac{2\pi L}{H}}\right) &= 1 \\ \implies b_1 &= \frac{1}{\frac{\pi}{H}\left(1 - e^{\frac{2\pi L}{H}}\right)} \\ b_n &= 0 \quad \text{otherwise} \end{aligned}$$

The full solution:

$$u(x,y) = \frac{1}{\frac{\pi}{H}\left(1 - e^{\frac{2\pi L}{H}}\right)}\left(e^{\frac{\pi}{H}x} + e^{\frac{2\pi L}{H}} e^{-\frac{\pi}{H}x}\right) \sin\left(\frac{\pi}{H}y\right)$$