Math 3140 – PDEs HW II

Due: June 29, 2016

1. Given the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

compute the following

(a) $\mathbf{u} \cdot \mathbf{v}$

(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$

(c) $||\mathbf{v}||^2$

(d) $||\mathbf{v} - \mathbf{w}||^2$

(e) $(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w})$

(f) Find constants a and b such that $\mathbf{u}$ is orthogonal to $a\mathbf{v} + b\mathbf{w}$

(g) Find constants a and b such that the distance between $\mathbf{u}$ and $a\mathbf{v} + b\mathbf{w}$ is as small as possible. That is, minimize $||\mathbf{u} - (a\mathbf{v} + b\mathbf{w})||^2$

Solution:

(a) The dot product is simply the sum of products of the components of vectors

$$\mathbf{u} \cdot \mathbf{v} = 1(-1) + 2(1) = 1.$$

(b) This just uses the distributive law (linearity) of the dot product

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = 1 + 2 = 3.$$

(c) Although we know how to compute the magnitude, it's nice to relate this to the dot product

$$\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = 2.$$

(d)

$$\|\mathbf{v} - \mathbf{w}\|^2 = (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = 9 + 1 = 10.$$

(e) This one cleans up nicely after we distribute everything

$$(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}^2 = \|\mathbf{v}\|^2 + \|vectw\|^2 = 2 + 4 = 6.$$

(f) We need to find a, b such that

$$\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = 0,$$

here we see that

$$a\mathbf{v} + b\mathbf{w} = \begin{bmatrix} -a + 2b \\ a \end{bmatrix},$$

so that

$$\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = -a + 2b + 2a = a + 2b = 0.$$

We could then choose say, $a = -2$ and $b = 1$ and this works.

(g) This problem is a little bit tricky. It seems like we should use the orthogonal projection formula, *however*, v, w are **not** orthogonal!

We can make use of the fact that $\mathbf{w}$ has a 0 component, meaning that $a = 2$ to get the 2 in the second component of $\mathbf{v}$ and then to make the first component correct we have $1 = -1(2) + b(2)$ which implies that $b = 3/2$.

Note that we know we can get the minimization to equal zero since we have a theorem that says *any two linearly indepdent vectors in $\mathbb{R}^2$ span $\mathbb{R}^2$. Thus, we can construct* $\mathbf{u}$ *from a linear combination of* $\mathbf{v}, \mathbf{w}$

2. Consider the functions

$$u(x) = 1 \quad v(x) = x \quad w(x) = x^2$$

defined on the interval $x \in [0, 1]$, which are members of the vector space of continuous functions with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Compute the following

(a) $\langle u, v \rangle$

(b) $\langle u, (v + w) \rangle$

(c) $||v||^2$

(d) $||v - w||^2$

(e) $\langle (v - w), (v + w) \rangle$

(f) Find constants a and b such that u is orthogonal to $av + bw$.

(g) Find the constant a such that the distance between w and au is as small as possible. That is, minimize $||w - au||^2$.

Solution: This whole problem should be similar to the previous just with a more general notion of inner product.

(a) With the defined inner product, we have

$$\langle u, v\rangle = \int_0^1 x\,\mathrm{d}x = \frac{1}{2}.$$

(b) Again we have the linearity of the inner product, so

$$\langle u, (\langle v + \langle w)\rangle = \langle u, \rangle v + \langle u, \rangle w = \int_0^1 x\,\mathrm{d}x + \int x^2\,\mathrm{d}x = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

(c) Here, the the notion of a norm (or magnitude) is *defined* by the inner product

$$\|v\|^2 = \langle v, v\rangle = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}.$$

(d) Again, we can compute $v - w$ and then tak ethe norm of it

$$\|v - w\|^2 = \langle v - w, v - w\rangle = \int_0^1 (x - x^2)^2\,\mathrm{d}x = \frac{1}{30}.$$

(e) This simplifies nicely in the same way the previous does

$$\langle v - w, v + w\rangle = \langle v, v\rangle + \langle v, w\rangle - \langle w, v\rangle + \langle w, w\rangle = \|v\|^2 + \|w\|^2 = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}.$$

(f) For these to be orthogonal, their inner product is zero, so

$$\langle u, av + bw\rangle = 0 = \int_0^1 \left\{ax + bx^2\right\}\,\mathrm{d}x = \frac{a}{2} + \frac{b}{3},$$

so if we say, chose, $a = -2$ and $b = 3$, we're done.

(g) Here, because we *do* have a mutually orthogonal basis (since one function is trivially orthogonal to the others), we can use the orthogonal projection formula

$$a = \frac{\langle u, w\rangle}{\langle u, u\rangle} = \frac{1/3}{1} = 1/3.$$

3. Show that $p_1 = 1$ and $p_2 = e^x - e + 1$ are orthogonal on the interval $x \in [0, 1]$ by direct computation using the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

Solution: This is just a straightforward application of the inner product, so we compute

$$\langle p_1, p_2 \rangle = \int_0^1 e^x - e + 1 = \{e^x + x - ex\}_{x=0}^{x=1} = 0.$$

4. Show that $f(x) = x$ on the unit interval is orthogonally projected onto $p_1(x) = 1$ and $p_2(x) = e^x - e + 1$ to be

$$x \approx -1/2 + e^x/(e-1).$$

Use a technology to plot the function and its orthogonal projection approximation.

Solution: Since we're trying to represent

$$f \approx \sum_j c_j \phi_j.$$

by mutually orthogonal ϕ_j (by the previous problem), we can simply use the orthogonal projection formula

$$c_j = \frac{\langle f, v_j \rangle}{\langle v_j, v_j \rangle}.$$

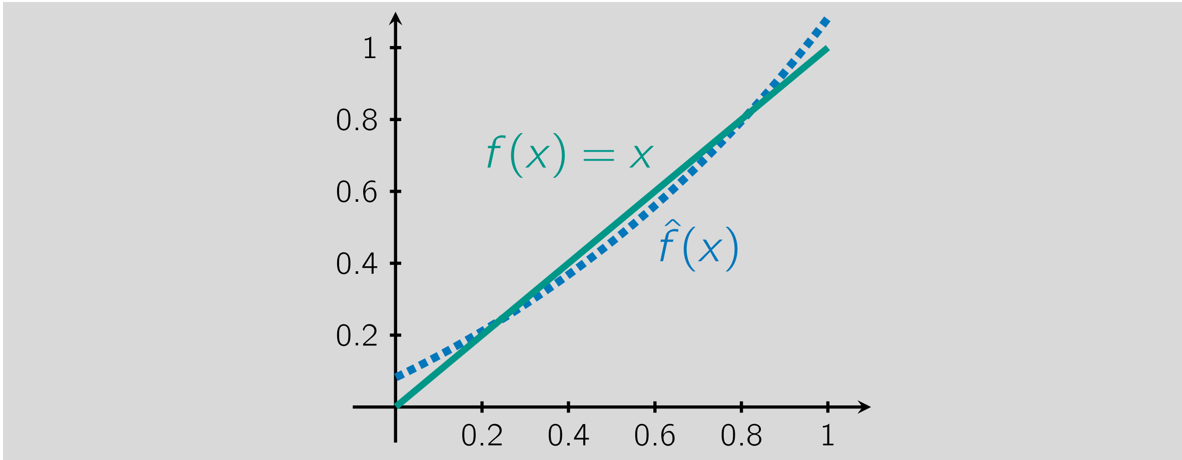
$$
\begin{aligned}
c_1 &= \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} = \frac{1}{2} \\
c_2 &= \frac{\langle x, e^x - e + 1\rangle}{\langle e^x - e + 1, e^x - e + 1\rangle}
\end{aligned}
$$

$$
\begin{aligned}
\text{where} \quad \langle x, e^x - e + 1\rangle &= \frac{(1-e)}{2} + \int_0^1 x e^x dx \\
&= \frac{(1-e)}{2} + 1
\end{aligned}
$$

$$
\begin{aligned}
\text{and} \quad \langle e^x - e + 1, e^x - e + 1\rangle &= \int_0^1 e^{2x} + 2e^x(1-e) + (1-e)^2 dx \\
&= \frac{e^2-1}{2} + 2(e-1)(1-e) + (1-e)^2 \\
&= \frac{e^2-1}{2} - (1-e)^2 \\
&= \frac{e^2-1}{2} - (1 - 2e + e^2) \\
&= \frac{e^2 - 1 - 2(1 - 2e + e^2)}{2} \\
&= \frac{-e^2 + 4e - 3}{2} \\
= \frac{1}{2}(e-3)(1-e) \implies c_2 &= \frac{\frac{(1-e)}{2} + 1}{\frac{1}{2}(e-3)(1-e)} \\
&= \frac{3-e}{(e-3)(1-e)} \\
&= \frac{1}{e-1}
\end{aligned}
$$

Put together

$$
\begin{aligned}
x \approx \hat{f}(x) &= c_1 + c_2(e^x - e + 1) \\
&= \frac{1}{2} + \frac{e^x - e + 1}{e-1} \\
&= -\frac{1}{2} + \frac{e^x}{e-1}
\end{aligned}
$$



5. On the interval $x \in [-1, 1]$, find the orthogonal projection of the function $f(x) = x^2$ for the orthogonal basis functions $\phi_0(x) = 1$ and $\phi_1(x) = x$. That is compute the function approximation $\hat{f}(x) = c_0\phi_0(x) + c_1\phi_1(x)$. Compute the error $||f - (c_0\phi_0 + c_1\phi_1)||$ given c_0 and c_1 are the projection coefficients. Use a technology to plot the function and its orthogonal projection approximation.

Solution: The procedure here is exactly the same as the previous. We're approximating

$$f(x) \approx \sum_j c_j\phi_j,$$

where ϕ_j are mutually orthogonal, which means we have the immediate result

$$c_j = \frac{\langle f, \phi_j}{\langle \phi_j, \phi_j \rangle}.$$

Thus, let's first do this for $j = 0$, which corresponds to $\phi_0 = 1$, and therefore

$$c_0 = \frac{\langle f, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle} = \frac{\int_{-1}^{1} x^2 \, \mathrm{d}x}{\int_{-1}^{1} 1 \, \mathrm{d}x} = \frac{2/3}{2} = \frac{1}{3}.$$

Next, for $j = 1$, which corresponds to $\phi_1 = x$

$$c_1 = \frac{\langle f, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} = \frac{\int_{-1}^{1} x^3 \, \mathrm{d}x}{\int_{-1}^{1} x^2 \, \mathrm{d}x} = \frac{0}{2/3} = 0.$$

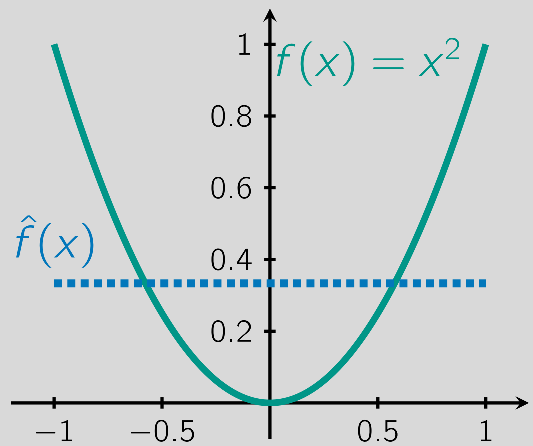
Why does this make sense? $\phi_1 = x$ is an odd function, but we're trying to represent an even function $f(x) = x^2$. Thus, there is no contribution. The approximation is then simply

$$\hat{f}(x) = c_0\phi_0 = \frac{1}{3}.$$

The error is

$$\|f - \hat{f}\| = \left\| x^2 - \frac{1}{3} \right\| = \sqrt{\left\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \right\rangle} = \sqrt{\int_{-1}^{1} \left\{ x^2 - \frac{1}{3} \right\} \mathrm{d}x} = \sqrt{0} = 0.$$

This is an interesting issue. It says basically, we've nailed it as far as the norm of the error, but this obviously isn't a perfect solution as we've accomplished this for kinda stupid reasons.



6. Project the function $f(x) = x$ onto the Fourier sine basis $\sin(nx)$ on the interval $x \in [-\pi, \pi]$, where n is a natural number:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(nx).$$

That is, compute B_n for all n values.

Solution: We've established that $\{s_n(x) \coloneqq \sin nx\}$ are mutually orthogonal but we could verify this again, noting that

$$\int_{-\pi}^{\pi} \sin nx \, \sin mx \, \mathrm{d}x = \begin{cases} 0 & m \neq n \\ \pi & m = n. \end{cases}$$

Thus, we can use the orthogonal projection formula

$$c_n = \frac{\langle f, s_n \rangle}{\langle s_n, s_n \rangle} = \frac{\int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x}{\int_{-\pi}^{\pi} \sin^2 n \, \mathrm{d}x} = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, \mathrm{d}x.$$

Using some integration by parts magic, we get

$$\int_{-\pi}^{\pi} x \sin nx \, \mathrm{d}x = \frac{-2n\pi \cos n\pi + 2 \sin n\pi}{n^2} = \{$$

We note that $\sin n\pi = 0$, so we only get $\cos n\pi$, which simply alternates sign:

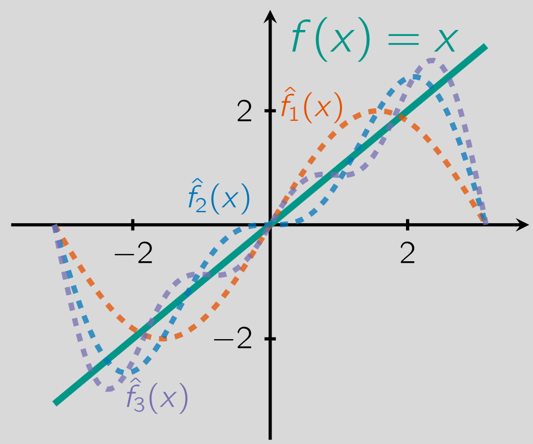
$$\cos n\pi = (-1)^n.$$

Piecing this together, we get

$$c_n = \frac{2}{n}(-1)^{n+1}.$$

Thus, our approximation is

$$f(x) = x \approx \hat{f}(x) = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x + \cdots .$$



7. Classify the following PDEs and companion boundary conditions as (a) linear, non-linear. If linear, (b), specify if the PDE is homogeneous or non homogeneous.
   (a) $u_t = u_{xx},\ u(0) = 0,\ u_x(L) = 0.$
   (b) $u_t = u_{xx} - x^2 u,\ u(0) = 0,\ u(L) = 0.$
   (c) $u_t = u_{xx} - x^2,\ u(0) = 0,\ u(L) = 0.$
   (d) $u_t = u_{xx} + u,\ u_x(0) = 0,\ u_x(L) = 0.$
   (e) $u_t = u_{xx},\ u + u_x(0) = 0,\ u - u_x(L) = 0.$
   (f) $u_t = u_{xx},\ u + u_x(0) = 0,\ u - u_x(L) = 0.$

Solution:

(a) $u_t = u_{xx},\ u(0) = 0,\ u(L) = 0.$ linear homogeneous

(b) $u_t = u_{xx} - x^2 u,\ u(0) = 0,\ u(L) = 0.$ linear homogeneous

(c) $u_t = u_{xx} - x^2,\ u(0) = 0,\ u(L) = 0.$ Linear nonhomogenous

(d) $u_t = u_{xx} - u^2$, $u(0) = 0$, $u(L) = 0$. nonlinear

(e) $u_t = u_{xx} + u$, $u_x(0) = 0$, $u_x(L) = 0$. linear homogeneous

(f) $u_t = u_{xx}$, $u(0) + u_x(0) = 0$, $u(L) - u_x(L) = 0$. linear homogeneous

(g) $u_t = u_{xx}$, $u(0) = 10$, $u = 0$.linear nonhomogeneous

(h) $u_t = u_{xx}$, $u(0) + u_x(0) = 1$, $u(L) - u_x(L) = -1$. linear nonhomogeneous