Math 3140: Checklist of PDEs knowledge

1. ☐ Core concept of PDE-vector calculus: If R is a region in space and $u(x, y, z, t)$ is some conserved quantity (e.g., mass, energy, current, etc ...), then the quantity $H(R)$ of of that stuff in R is given by the triple integral
$$H(R) = \iiint_R u \, dV.$$
To derive a PDE, what mathematical computation do we relate the time rate of change of $H(R) = dH/dt$ to?

2. ☐ The continuity equation $u_t = -\nabla \cdot \vec{\phi}$, where $\vec{\phi}$ is a vector flux field.

3. ☐ The continuity equation in 1D: $u_t = -\frac{\partial}{\partial x}[\phi]$, where $\phi(x, t)$ is the 1D flux.

4. ☐ Given a flux rule and an initial condition $u(x, t)$, either in the form of a graph or a function, be able to determine if the quantity $H = \int_a^b u dx$ is growing, shrinking, or staying the same instantaneously.

5. ☐ Know the definition of the inner product for both vectors in $\mathbb{R}^n$ and for functions $f(x)$ on intervals $[a, b]$.

6. ☐ Know the definition of the magnitude of a vector $||\vec{v}||$ or function $||f||$ in an inner product space in terms of the inner product.

7. ☐ How can you determine if two functions f and g in an inner product space are orthogonal? That is, what defines orthogonality in an inner product space?

8. ☐ Know that the functions $\{\cos([n\pi/L]x), \sin([n\pi/L]x)\}_{n=1}^{\infty} \cup \{1\}$ are orthogonal.

9. ☐ Know the definition of the orthogonal projection.

10. ☐ Suppose we have a function $g(x)$ and a set of orthogonal functions $\{p_i(x)\}_{i=1}^{n}$. What choice of coefficients a_i guarantee the minimum possible error $||g(x) - \sum_{i=1}^{n} a_i p_i(x)||$ between $g(x)$ and the set of all functions in $span(\{p_i(x)\}_{i=1}^{n})$?

11. ☐ Know how to derive the heat equation in a 1D bar of length L.

12. ☐ Know the three main types of boundary conditions and their physical meaning: Dirichlet (zero temp), Neumann (zero flux), Robin (newton's law of cooling).

13. ☐ Know how to find the equilibrium solution $u_{eq}(x)$ to the heat equation on a bar given any combination boundary conditions (see above).

14. ☐ Know the method of separation of variables for the heat equation on a bar, and the Laplace equation on a rectangle.

15. ☐ Things you can assume: If $u(x, t) = p(x)q(t)$ the resulting eigenvalue problem $p'' = -\lambda p$ For zero-temp or zero-flux conditions on $[0, L]$ has solutions $\lambda_n = (2\pi/L)^2$, $\{\sin([n\pi/L]x)\}_{n=1}^{\infty}$, and $\{\cos([n\pi/L]x)\}_{n=0}^{\infty}$, respectively. That is, you do not need to derive these results on the exam (we've done it so many times already!) but can use these results to solve problems.

16. ☐ In all other boundary condition cases other than above, I may request that you derive eigenvalues and solutions to $p'' = -\lambda p$ (or some other p-equation related to the given PDE; e.g., the uninsulated bar equation) that satisfy the boundary conditions.

17. ☐ Know the definition of a homogenous PDE, including homogeneous boundary conditions. If two distinct u and w solve the homogeneous problem, does $u + w$? What if they solve a non-homogeneous problem?

18. ☐ Be able to write down a general solution to the heat equation or related PDE and know how to specify a particular solution given an initial condition $u(x, 0) = f(x)$. In some instances, the specification involves computation of finitely many Fourier coefficients (because the rest are zero), and in other cases there are infinitely many non-zero coefficients, which may need to be computed via orthogonal projection integrations. In this latter case, be prepared to compute the coefficients a_n as functions of n.

19. ☐ Be able to solve Laplace's equation $\nabla^2 u = 0$ with non-homogeneous boundary conditions on the rectangle $[0, L] \times [0, H]$.

20. ☐ Be able to solve Laplace's equation $\nabla^2 u = 0$ with a non-homogeneous boundary condition on the disk $u(r = R, \theta) = f(\theta)$.

21. ☐ Know the three properties of solutions $u(x, y)$ of Laplace's equation. If given a set of statements about a function $u(x, y)$, be able to use the properties to make proper inferences regarding u: Zero net flux, maximum principle, mean value property.

22. ☐ Know how to find the solution of the wave equation and related PDEs involving a u_{tt} acceleration term using separation of variables.

23. ☐ Know the integral definition of Fourier transform and the inverse Fourier transform, know what types of functions $f(x)$ are transformable, be able to compute transforms for explicit function definitions, as well as use a transform table.

24. ☐ Know the definition and meaning of the energy spectral density $ESD(\omega) = F(\omega)\overline{F(\omega)}$ and be able to compute it given a $F(\omega)$.

25. ☐ Know the action of the Fourier transform on spatial (x) derivatives and temporal (t) derivatives, and the delta $\delta(x)$.

26. ☐ Know the shift theorem and the convolution theorem:

$$\mathcal{F}(f(x-\beta)) = F(\omega)e^{i\beta\omega}$$

$$\mathcal{F}^{-1}(F(\omega)G(\omega)) = (f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi$$

27. ☐ Know the heat/diffusion kernel $g_t(x)$ and how to use it to solve the heat/diffusion PDE on the real line.

$$g_t(x) = \frac{2\pi}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}}$$

28. ☐ Know D'Lambert's solution to the wave equation on the real line $u_{tt} = c^2 u_{xx}$, with initial position $f(x)$ and initial velocity $g(x)$:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi.$$

Specifically, if given a particular x_0-point, be able to derive and sketch the graph $u(x_0, t)$ as a function of time; conversely, given a specific point in time t_0, be able to derive and sketch the graph of $u(x, t_0)$ as a function of space x.