# Final Exam
## Math 3140 - Vector Calc & PDEs
## August 5, 2016

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules or caught cheating will be asked to leave immediately. Point values are in the square to the left of the question. **If there are any other issues, please ask the instructor.**

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

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Signature: ______________________________

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## Solutions

| Question | Points | Score |
|---|---|---|
| 1 | 45 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 30 | |
| 5 | 25 | |
| 6 | 25 | |
| 7 | 25 | |
| 8 | 25 | |
| Total: | 215 | |

**Note:** There are 8 questions on the exam with 215 points available but the exam will be graded out of 200.

1. Consider the vector field in $\mathbb{R}^3$ described by

$$\mathbf{F} = \langle 0, -z^2, yz \rangle.$$

$7\frac{1}{2}$ (a) Does there exist a vector field $\mathbf{G}$ such that $\nabla \times \mathbf{G} = \mathbf{F}$? Why or why not?

$7\frac{1}{2}$ (b) Is $\mathbf{F}$ conservative? Why or why not?

15 (c) Let $S$ be the right half of the unit sphere, defined to be

$$S = \left\{(x, y, z) : \quad x^2 + y^2 + z^2 = 1, \quad x \geq 0\right\}.$$

**without using any theorems to transform it**, compute the flux integral

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

That is, compute it directly. *Two hints:*

i) the unit normal for a unit sphere is $\mathbf{n} = \langle x, y, z \rangle$.
ii) take the parameterization $x = \sin\phi\cos\theta, y = \sin\phi\sin\theta,\ z = \cos\phi$ with appropriate bounds

15 (d) Verify your answer to the previous question using **Stokes' theorem**. *Again, some hints:*

i) the boundary $\partial S$ is the unit circle in the $yz$ plane. parameterize this.
ii) recall (and use) $\sin^2 t + \cos^2 t = 1$.

**Solution: This question was taken directly from one of the exam 1 practice exams, labeled M427L (55200).**

(a) We know that $\operatorname{div}\operatorname{curl}\mathbf{G} = 0$ for any vector field $\mathbf{G}$. Thus, if $\mathbf{F} = \nabla \times \mathbf{G}$, that is, it comes from the curl of a vector field, then its divergence must be zero. However,

$$\nabla \cdot \mathbf{F} = 0 + 0 + y = y \neq 0,$$

so no, no such $\mathbf{G}$ exists.

(b) Although we have a considerable number of equivalent requirements for conservation in a vector field, the most natural here is checking the curl, where we know $\nabla \times \mathbf{F} = 0$ if it is indeed conservative. However, we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & -z^2 & yz \end{vmatrix} = \langle 3z, 0, 0 \rangle \neq \mathbf{0},$$

so no, this is not a conservative vector field.

(c) This might be a bit annoying of a problem, but with the hints, it's quite straightforward. Provided the unit normal hint, we know

$$d\mathbf{S} = \mathbf{n}\, dS,$$

so our integral becomes

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \mathrm{d}S \\ &= \iint_S \langle 3z, 0, 0 \rangle \cdot \langle x, y, z \rangle \, \mathrm{d}S \\ t &= \iint_S 3xz \, \mathrm{d}S.\end{aligned}$$

Now, we convert to the new parameterization, noting that that our bounds are $\theta \in [-\pi/2, \pi/2]$ since we only get the positive $x$ components and the standard $\phi \in [0, \pi]$. Thus, we get

$$\begin{aligned}\iint_S 3xz \, \mathrm{d}S &= \int_0^{\pi} \int_{-\pi/2}^{\pi/2} 3 \sin^2 \phi \cos \phi \cos \theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \int_0^{\pi} 3 \sin^2 \phi \cos \phi \left[ \int_{-\pi/2}^{\pi/2} \cos \theta \, \mathrm{d}\theta \right] \mathrm{d}\phi \\ &= 0,\end{aligned}$$

where we've used the fact that the integral over a symmetric region of $\cos \theta = 0$.

(d) Here, we use Stokes' theorem, which states

$$\iint_S (\nabla \times \mathbf{F}) \, \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

However, we know the boundary of this region is a circle in the $yz$ plane, which is simply parameterized by

$$\mathbf{r}(t) = \langle 0, \cos t, \sin t \rangle, \qquad t \in [0, 2\pi].$$

We need the derivative of this, which is

$$\mathbf{r}'(t) = \langle 0, -\sin t, \cos t \rangle,$$

because we know $\mathrm{d}\mathbf{r} = \mathbf{r}' \, \mathrm{d}t$. Thus, our integral becomes

$$\begin{aligned}\int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \langle 0, -sin^2 t, \cos t \, \sin t \rangle \cdot \langle 0, -\sin t, \cos t \rangle \, \mathrm{d}t \\ &= \int_0^{2\pi} \sin t (\sin^2 t + \cos^2 t) \, \mathrm{d}t \\ &= \int_0^{2\pi} \sin t \, \mathrm{d}t \\ &= 0,\end{aligned}$$

since the integral of $\sin t$ over a whole period is zero. This matches our answer from the previous part, which is assuring.

20 2. Use Green's Theorem to evaluate the line integral:

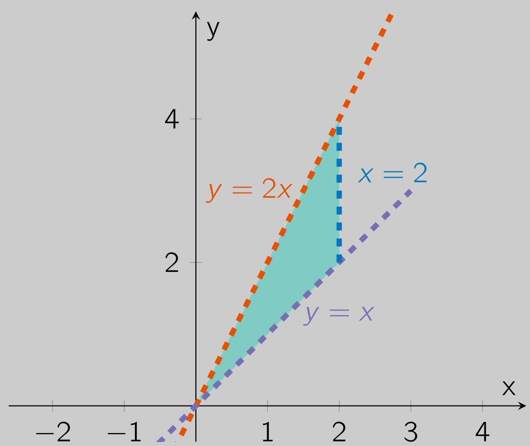
$$\oint_C xy^2\,dx + 2x^2y\,dy,$$

where $C$ is the positively oriented triangle with vertices $(0,0)$, $(2,2)$, $(2,4)$.

**Solution:** Green's Theorem (ignoring all the technical conditions), says that if $\mathbf{F} = \langle P, Q\rangle$, then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA.$$

Here, we see that $P = xy^2$ and $Q = 2x^2y$ meaning that $Q_x = 4xy$ and $P_y = 2xy$ and our integrand is $Q_x - P_y = 4xy - 2xy = 2xy$.




To know the bounds of integration, we must draw the region. We could set this up as either type 1 or type 2, but I think type 1 is easier here. Note that $x$ varies between 0 and 2 and then $y$ is bounded by the two sides of the triangle described by $y = 2x$ and $y = x$, thus:

$$\begin{aligned}\iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA &= \int_0^2 \int_x^{2x} 2xy\,dy\,dx\\ &= \int_0^2 \left[xy^2\right]_{y=x}^{y=2x} dx\\ &= \int_0^2 3x^3\,dx\\ &= \left[\frac{3}{4}x^4\right]_{x=0}^{x=2} = 12.\end{aligned}$$

3. Consider a fluid velocity field

$$\mathbf{V}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z\mathbf{k}$$

and the flux through a boundary $\partial\Omega$

$$J := \iint_{\partial\Omega} \mathbf{V} \cdot \mathrm{d}\mathbf{S},$$

where $\partial\Omega$ is the boundary of the (closed) cone

$$\Omega = \left\{(x, y, z) : \quad z^2 = x^2 + y^2, \quad 0 \leq z \leq 1\right\}.$$

10 (a) Using the Divergence Theorem, set up, but **do not evaluate** a volume integral representing the flux $J$.

10 (b) By converting to a different (appropriate) coordinate system, evaluate the integral you found in the previous part.

**Solution:** **This question was taken directly from one of the exam 1 practice exams, labeled M427L (55200).**

(a) The divergence theorem says that

$$\iint_{\partial\Omega} \mathbf{V} \cdot \mathrm{d}\mathbf{S} = \iiint_{\Omega} (\nabla \cdot \mathbf{V})\, \mathrm{d}V.$$

Here, we can compute the divergence

$$\nabla \cdot \mathbf{V} = 3x^2 + 3y^2 + 1,$$

so our answer becomes

$$\iint_{\Omega} (3x^2 + 3y^2 + 1)\, \mathrm{d}V.$$

(b) The natural choice here is cylindrical coordinates. So take the normal transformation and recall that the infinitesimal element is $\mathrm{d}V = r\mathrm{d}r\,\mathrm{d}\theta\mathrm{d}z$. The height of the object is described by

$$z^2 = x^2 + y^2 = r^2 \qquad \Longrightarrow \qquad z = r,$$

so our integral becomes

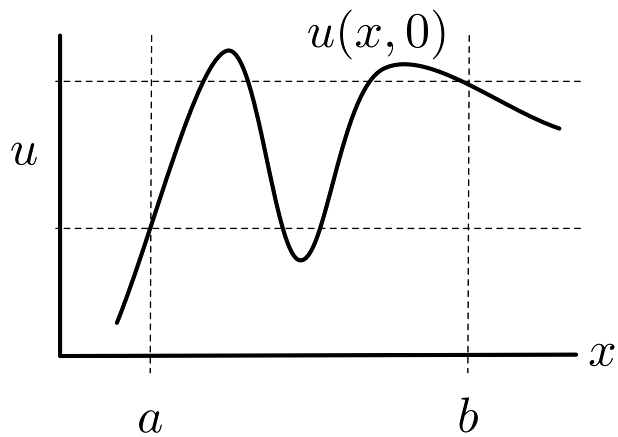
$$\begin{aligned}
\int_0^1 \int_0^{2\pi} \int_0^z (3r^2 + 1) r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z &= 2\pi \int_0^1 \int_0^z (3r^2 + 1) r\, \mathrm{d}r\, \mathrm{d}z \\
&= 2\pi \int_0^1 \left(\frac{z^2}{2} + \frac{3z^4}{4}\right) \mathrm{d}z \\
&= \frac{19}{30}\pi.
\end{aligned}$$

4. Consider the diffusion/heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

10 (a) Explain the notions of *flux* and *conservation*. What expression (and law) determine the flux for the heat equation? Call the flux $\phi(x, t)$.

10 (b) Consider a snapshot of the initial condition ($t = 0$) shown below.




Given the flux rule $\phi$ you found above, evaluate the following by drawing an arrow with appropriate direction and magnitude (with justification):

i) $\phi(a, 0)$

ii) $\phi(b, 0)$

10 (c) Let $U(t) = \int_a^b u\, dx$ be the total amount of chemical (or heat) in our region. Is $\frac{dU}{dt}(0)$ positive or negative? Why?

**Solution:** **This question was taken (and slightly modified) from the final exam practice.**

(a) Flux is basically just a rule for how much stuff flows through a particular point. We utilized this heavily with the idea of conservation (basically, you can't lose or gain most things without explicitly accounting for it) to derive most of our PDEs. In the heat equation, Fourier's Law told us that the flux for the heat equation is

$$\phi(x, t) = -\frac{\partial u}{\partial x}$$

which basically says that heat flows from high temperature to low temperature (or up the gradient).

(b) Here, we know by our flux rule, the gradient of heat $u_x$ tells us the flux. Note that $u_x(a, 0) > 0$. Thus, this means that

$$\phi(a, 0) = -u_x(a, 0) < 0.$$

Thus, since $\phi$ is negative, and we've established that right is the positive direction, heat is flowing to the left, which makes sense since this is the direction of high to low.

For the next point, $u_x(b, 0) < 0$ meaning that

$$\phi(b, 0) = -u_x(b, 0) > 0$$

or that heat is flowing to the right here. Note that this is **smaller** in magnitude than the previous since $u_x$ is smaller (flatter line = smaller slope).

(c) From the previous two parts, we can conclude that heat is flowing OUT from both sides, hence $\mathrm{d}U/\mathrm{d}t < 0$ regardless of the magnitude of each.

10 5. (a) Consider the three functions defined for the domain $x \in [0, 1]$

$$p_0(x) = 1, \qquad p_1(x) = \begin{cases} 1, & x \le 1/2, \\ -1, & x > 1/2. \end{cases}, \qquad p_2(x) = \begin{cases} 0, & x < 1/2, \\ 1 & x \ge 1/2. \end{cases}$$

Which pair of these functions is orthogonal (using the standard inner product)? *Hint:* it may be useful to plot these.

15 (b) Using the two orthogonal functions you found from above, denoted $\phi_1, \phi_2$, find the coefficients $c_1, c_2$ that best approximate

$$f(x) = \sin(2\pi x)$$

by the series

$$f \sim c_1\phi_1 + c_2\phi_2.$$

**Solution: This question was taken verbatim from the final exam practice.**

(a) The inner product for this case is defined to be

$$\langle a(x), b(x)\rangle := \int_0^1 a(x)b(x)\,\mathrm{d}x.$$

Hence, we just see which two of these functions inner product to zero. However, it's fairly easy to see visually which this is. For instance, $\int p_0p_2\,\mathrm{d}x$ does not add up to zero, nor does $\int p_1p_2\,\mathrm{d}x$, however $p_0p_1$ most definitely does.

(b) Now we know that $\{p_0, p_1\}$ form an orthogonal set of functions, so we can use the orthogonal projection formula to explicitly obtain the coefficients, which say

$$c_j = \frac{\langle f, \phi_j\rangle}{\langle \phi_j, \phi_j\rangle}.$$

Here, we have to compute a few inner products for this. Let's first take $c_1$, which corresponds to $\phi_1 = p_0$, which gives us

$$\langle f, p_0\rangle = \int_0^1 \sin(n\pi x)\,\mathrm{d}x = 0.$$

Hence, we know that $c_1 = 0$. This makes sense since $\sin 2\pi x$ has no constant offset. The next coefficient, $c_2$, corresponds to $\phi_2 = p_1$, which we can compute some inner products for:

$$\langle f, p_1 = \int_0^1 \sin(n\pi x)p_1(x)\,\mathrm{d}x = \int_0^{1/2} \sin(n\pi x) - \int_{1/2}^1 \sin(n\pi x)\,\mathrm{d}x = \frac{2}{\pi}.$$

We need the normalizing factor

$$\langle p_1, p_1\rangle = \int_0^1 p_1^2(x)\,\mathrm{d}x = 1.$$

Thus, the total coefficient is

$$c_2 = \frac{\langle f, \phi_2\rangle}{\langle \phi_2, \phi_2\rangle} = \frac{2/\pi}{1} = \frac{2}{\pi}.$$

Thus, the approximation is

$$f \sim \frac{2}{\pi} p_2(x),$$

which clearly isn't a good approximation, but we know it's the best possible with these building blocks.

25 6. Let $u(x,t)$ represent the population of fishes in a river described by $x \in [0,1]$. The population of fish can do two things: diffuse (move) around and die at a rate proportional to how many fishes there are. Consequently, the PDE that describes their spatial population is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u.$$

Fish (obviously) can't swim in or out of the river, so take the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0.$$

Using separation of variables, find the solution to the PDE subject to the initial conditions

$$u(x,0) = 2 + \cos \pi x.$$

*Hint:* use separation of variables so that the spatial ODE eigenvalue problem you get is the same as the classical heat equation.

**Solution:** **This is exactly the problem from quiz 6.**

As with all separation of variables problems, we start with the guess

$$u(x,t) = p(x)q(t).$$

Plugging this into our PDE, we get

$$p\dot{q} = p''q + pq, \qquad \dot{q} := \frac{\mathrm{d}q}{\mathrm{d}t}, \qquad p' := \frac{\mathrm{d}p}{\mathrm{d}x}.$$

Thus, we can separate this into

$$\frac{\dot{q}}{q} = \frac{p'' + p}{p} = \frac{p''}{p} + 1,$$

but the hint suggests we want the spatial ($p$) ODE to be the same as the heat equation, were we had $p''/p$. Thus, move the 1 over to yield

$$\frac{\dot{q}}{q} - 1 = \frac{p''}{p} = -\lambda,$$

where we know that if $f(t) = g(x)$ then they must both be equal to some constant, which we can call $-\lambda$.

Thus, we're left with the spatial eigenvalue problem

$$p''(x) = -\lambda p, \qquad p'(0) = p'(1) = 0.$$

We know, from class, this is exactly the same eigenvalue problem as the no-flux heat equation, which has eigenvalues and eigenfunctions **(from the formula sheet )**

$$\lambda_n = \left(\frac{\pi n}{L}\right)^2, \qquad p_n(x) = \cos\frac{\pi n x}{L},$$

but here we have $L = 1$, so we've got the eigenvalues and eigenfunctions

$$\lambda_n = (\pi n)^2, \qquad p_n(x) = \cos(\pi n x).$$

We should note that we also have $\lambda = 0$, with $p_0(x) =$ constant as the corresponding eigenfunction. Now we turn to the $q$ ODE, which says

$$\dot{q} - q = -\lambda q,$$

which, rearranged, yields

$$\frac{\dot{q}}{q} = -(\lambda + 1),$$

which has solutions

$$q_n(t) = e^{-(\lambda_n+1)t} = e^{-\left[(\pi n)^2+1\right]t}.$$

Notice that this form makes sense. The only difference from our heat equation solution is that this one decays a little bit faster. This makes sense since we added a decay term to the PDE.

Now we know our eigenfunctions, so we combine them in a linear combination (because we currently don't know which $n$'s to include), so we see our general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n q_n(t) p_n(x) = A_0 e^{-t} + \sum_{n=1}^{\infty} A_n \cos(n\pi x) e^{-\left[(\pi n)^2+1\right]t}.$$

Note that I've pulled out the $n = 0$ case to make it more explicit, but this wasn't completely necessary. Now, to determine the $A_n$'s we need to use the initial condition. The initial condition says

$$u(x,0) = 2 + \cos\pi x,$$

but this has to be equal to our solution at $t = 0$, which yields

$$2 + \cos\pi x = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = A_0 + A_1\cos(\pi x) + A_2\cos(2\pi x) + \cdots.$$

It's clear from this that $A_0 = 2, A_1 = 1$ and $A_{\geq 2} = 0$. Putting this back into our general (time dependent) solution, yields the final solution to the PDE

$$u(x,t) = 2e^{-t} + \cos(\pi x) e^{-\left[\pi^2+1\right]t}.$$

Note that as $t \to \infty$, this decays to 0, which makes sense since no fish are fluxing into the river but they are dying at some rate.

7. Consider the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad x \in (-\infty, \infty)$$

10 (a) There is a difference in the speed at which the wave equation and heat equation propagate information. **Explain this difference**. It might be useful to include ideas like: *domain of influence* and *domain of dependence*.

15 (b) Consider the wave equation with $c = 1$ subject to the initial conditions

$$u(x,0) = u_0(x) = \begin{cases} 1 - x^2, & -1 < x < 1 \\ 0 & \text{otherwise,} \end{cases}, \qquad u_t(x,0) = v_0(x) = 0.$$

Find an expression for the solution at $t = 0.5$. That is, what is $u(x, 0.5)$?

**Plot some snapshots of this PDE to illustrate its behavior**.

**Solution:** This question is almost verbatim from quiz 7.

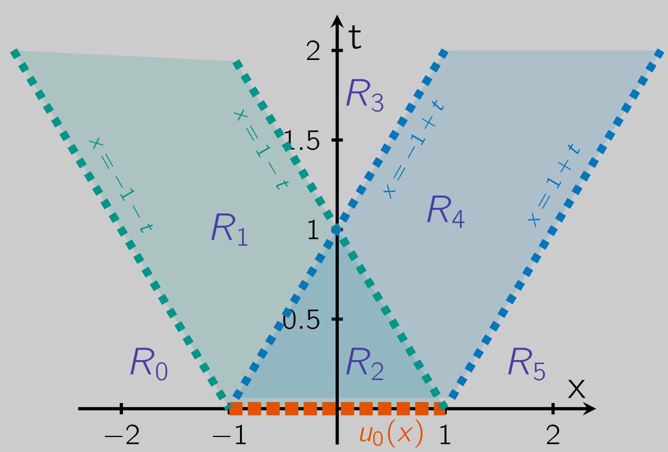
(a)

(b) In the case that $v_0 = 0$ (and $c = 1$ from the PDE), we see that D'Alembert's reduces to

$$u(x,t) = \frac{1}{2}\left[u_0(x-t) + u_0(x+t)\right].$$

We could try to grind out what this works out to be from the definition of $u_0$, but it's easier to take a step back and think about what the formula tells us: the solution is the (average) of two waves, one of which moves to the left with velocity 1 and the other moves to the right with velocity 1.

Thus, since we only start with information from $[-1, 1]$, we just need to track how this "window" moves, which is entirely told to us by the two wave solutions. This is what I attempt to depict below.




There's a lot going on in this picture, but let's try to digest it. We know that one of our waves travels left. That is the region in green. It starts at $[-1, 1]$ and travels with a constant rate (slope) 1 to the left.

The same can be said for the blue region: this is our rightward traveling wave, with again (slope = $c$) velocity 1.

From this, it's clear that 5 regions emerge. We need to just figure out what goes on in each of these regions to fully construct our solution.

We can first identify what goes on in regions $R_0, R_3, R_5$ quite easily. Since our solution is non-zero in the blue and green, we can safely say

$$u(x,t) = 0, \qquad \text{for } (x,t) \in \{R_0, R_3, R_5\}.$$

We could write down explicit descriptions of these regions if we wanted but this is fine enough for me.

In $R_1$, we see that this corresponds to just our leftmost traveling wave, meaning here we have

$$u(x,t) = u_0(x-t) = 1-(x-t)^2, \text{ for } \qquad (x,t) \in R_1.$$

Similarly, for $R_4$, we have

$$u(x,t) = u_0(x+t) = 1-(x+t)^2, \text{ for } \qquad (x,t) \in R_4.$$

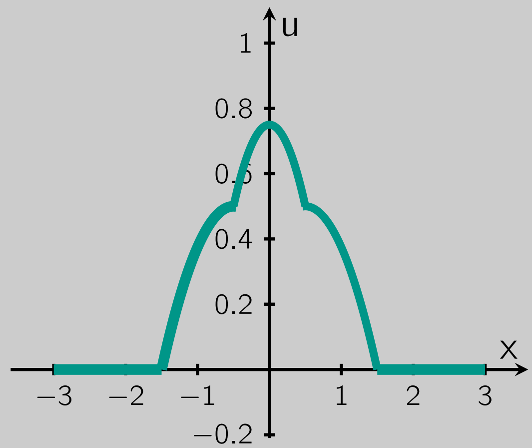
Lastly, in $R_2$, we see that the waves haven't separated yet, so we get the superposition of the two

$$u(x,t) = \frac{1}{2}\left[u_0(x+t) - u_0(x-t)\right] = \frac{1}{2}\left[\left\{1-(x+t)^2\right\} + \left\{1-(x-t)^2\right\}\right] \text{ for } (x,t) \in R_2.$$

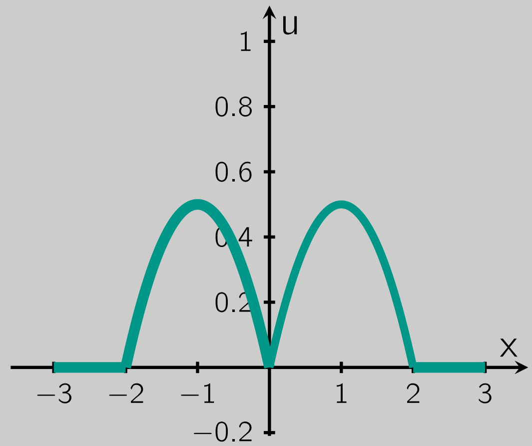
**For this particular problem,** at $t = 0.5$, we see that we get five different modes of behavior: two zeroes ($R_0, R_5$), two half-waves ($R_1, R_4$), and a superposition of those waves ($R_2$). Plugging in $t = 0.5$ to the previous expressions, we get

$$u(x,0.5) = \begin{cases} 0, & x \in (-\infty,-1.5) \bigcup (1.5,\infty) \\ 1-(x-0.5)^2, & x \in (-1.5,-0.5) \\ \frac{1}{2}\left[\left\{1-(x+0.5)^2\right\} + \left\{1-(x-0.5)^2\right\}\right], & x \in [-0.5, 0.5] \\ 1-(x+t)^2, & x \in (0.5, 1.5). \end{cases}$$

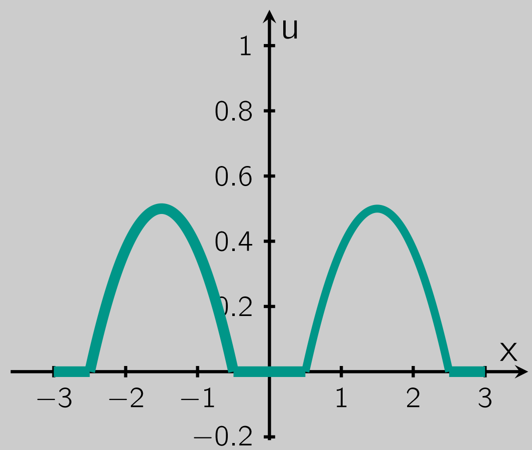
We can then plot this for a few snapshots of $t$. Let's first include one that incorporates $R_3$, so say $t = 0.5$. Note that to plot it for a single $t$ slice, we get 3 different scenarios for the different $x$ values, depending if it's in $R_1, R_2, R_4$, but we know the behavior in each.



Next, we see at $t = 1$, this is the exact moment the two waves split apart




And then say, at $t = 1.5$ and beyond they are clearly distinct




and they continue to travel apart.

20 8. (a) Solve the advection-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$$

for $x \in (-\infty, \infty)$ with the initial condition $u(x, 0) = \delta(x)$. Use the Fourier transform.

5 (b) Interpret your answer. Why does this physically make sense with what the PDE is describing?

**Solution:** **This question was taken directly from a homework assignment.**

We first Fourier transform both sides of the equation, noting that $t$ derivatives move freely in and out of the transform and each derivative picks up a factor of $(-i\omega)$, which brings us to our standard ODE in $t$ which we can integrate to find

$$\begin{aligned} \mathcal{F}(u_t = u_{xx} + u_x) = U_t &= -\omega^2 U - i\omega U \\ U_t &= -(\omega^2 + i\omega)U \\ U(\omega, t) &= A(\omega)e^{-(\omega^2+i\omega)t} \qquad \text{where} \qquad U(\omega, 0) = A(\omega) \end{aligned}$$

In other words, $A(\omega)$ is the Fourier transform of our initial condition, which means we must compute this

$$\mathcal{F}[\delta(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)e^{i\omega x}\, \mathrm{d}\omega = \frac{1}{2\pi}e^{i\omega 0} = \frac{1}{2\pi}.$$

Note that we've used the property of the $\delta$ function that

$$\int_{-\infty}^{\infty} \delta(x-a)g(x)\, \mathrm{d}x = g(a),$$

where here the shift was zero so $a = 0$. To summarize, the Fourier transform of the $\delta$ function is a constant. This is a nice fact to know. Basically to produce an instantaneous pulse, we need all frequencies. We can now write our full solution as

$$U(\omega, t) = e^{-i\omega t}\frac{1}{2\pi}e^{-\omega^2 t}$$

How do we invert this? Note that we can utilize the shift theorem, which says

$$\mathcal{F}^{-1}\left[e^{i\omega x_0}G(\omega)\right] = g(x - x_0).$$

Thus, we just need to shift the inverse Fourier transform of the Gaussian part, which is

$$\mathcal{F}^{-1}\left[\frac{1}{2\pi}e^{-\omega^2 t}\right] = \frac{1}{\sqrt{4\pi t}}e^{-\frac{-(x)^2}{4t}}$$

and now piecing this with the shift theorem, we finally get

$$u(x, t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{-(x+t)^2}{4t}}$$

Think about why this solution makes perfect sense. It's basically a Gaussian with moving mean $\mu = t$. Why is this the mean? We have an advection term with velocity $v = 1$. The diffusion term gives us a bell curve and the advection term pushes it to the right at speed $v = 1$, which is exactly what the solution says.

## Bonus Questions

9. **(University History)**

Ted Bundy, the infamous serial killer, briefly attended the University of Utah after being accepted not once, but *twice*. What did he study at the U?

**Solution:** A nice quotation from his wikipedia page:

> In August 1974 Bundy received a second acceptance from the **University of Utah Law School** and moved to Salt Lake City, leaving Kloepfer in Seattle. While he called Kloepfer often, he dated 'at least a dozen' other women. As he studied the first-year law curriculum a second time, 'he was devastated to find out that the other students had something, some intellectual capacity, that he did not. He found the classes completely incomprehensible. 'It was a great disappointment to me,' he said.

https://en.wikipedia.org/wiki/Ted_Bundy

10. **(Pokémon)**

The pokémon Drowzee (shown to the right) is based on what animal species known in Japanese folklore to eat dreams and nightmares?

**Solution:** Tapirs.

11. **(Math)** Using the ingredients of conservation and a physical law, *derive the heat equation* in a rod of length $[0, L]$. As a possibly useful reminder: $H = c\rho u$, where $H$ is the energy stored with a specific heat $c$ and mass density $\rho$.

**Solution:** This is just directly from the book. Look it up.

# Potentially Useful Information

### Cylindrical Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta,$$
$$z = z, \qquad r^2 = x^2 + y^2$$

### Spherical Coordinates

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta,$$
$$z = \rho\cos\phi, \qquad \rho^2 = x^2 + y^2 + z^2$$

### Surfaces

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}, \qquad \mathrm{d}S = \|\mathbf{r}_u \times \mathbf{r}_v\|\mathrm{d}A$$

### Cryptic hint when $\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle$

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle.$$

### Green's Theorem

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,\mathrm{d}A$$
$$\oint_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \iint_D \nabla\cdot\mathbf{F}\,\mathrm{d}A$$

### Stokes' Theorem

$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathrm{d}\mathbf{S}$$

### Divergence Theorem

$$\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_E \nabla\cdot\mathbf{F}\,\mathrm{d}V$$

### Heat Equation (0 temp)

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$
$$\phi_n = \sin\frac{n\pi x}{L}.$$

### Exponential/Trig Relation

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

### Fourier Transform

$$f(x) = \int_{-\infty}^{\infty} \tilde{F}(\omega)e^{-i\omega x}\,\mathrm{d}\omega,$$
$$\tilde{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,\mathrm{d}x.$$

### Derivative Property

$$\mathcal{F}\left[\frac{\partial f}{\partial x}\right] = (-i\omega)\tilde{F}(\omega).$$

### Shift Property

$$\mathcal{F}\left[e^{i\omega x_0}\tilde{F}(\omega)\right] = f(x - x_0).$$

### Convolution Property

$$\mathcal{F}^{-1}\left[F(\omega)G(\omega)\right] = f(x) * g(x)$$
$$f(x) * g(x) := \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,\mathrm{d}\xi.$$

### Heat Kernel

$$\mathcal{F}^{-1}\left[e^{-k\omega^2 t}\right] = \frac{1}{\sqrt{4\pi k t}}e^{-\frac{x^2}{4kt}}.$$

### Delta Transform

$$\mathcal{F}\left[\delta(x)\right] = \frac{1}{2\pi}.$$

### D'Alembert's Solution

$$u(x,t) = \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\,\mathrm{d}\xi.$$

### Heat Equation (0 flux)

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, \ldots$$
$$\phi_n = \cos\frac{n\pi x}{L}.$$

### Orthogonal Projection

$$c_j = \langle f, \phi_j\rangle / \langle \phi_j, \phi_j\rangle.$$