Math 3140 – Exam I
June 17, 2016 – 10 AM

This outline is not meant to be an end-all exhaustive study replacement, but rather a structured baseline for your studying. That is, this is a study *companion*. If something is missing on the outline, that is not a guarantee it will absent from the exam! However, it's likely that if I forgot about it here, I forgot to put it on the exam.

On the exam, I will provide the more annoying formulas: spherical coordinates, statement of Green's,Stokes', Divergence Theorem, however, it will be on you to know how to apply them!

I'll be around as much as possible during the week before the exam and will happily answer any questions. On **Thursday, June 16**, we'll review during our normal class time, same location.

Anything covered in lab, quiz, homework (even ungraded), and review practice is fair game, and I'll mostly draw from these sources when constructing the exam.

Chapter 12: Multiple integrals

12.1: Double Integrals over Rectangles

- 1D integral idea: add up rectangles of with Δx and height $f(x_i)$ to get an area
- new idea: add up rectangular prisms of base area ΔA and height $f(x_i, y_i)$.
- base area $\Delta A = \Delta x \Delta y$, we can think about chopping our surface in "uniform grid" where these are constant
- limit of infinitely small (and many) grids = double integral
- we write this as $\iint_D f(x,y)\,\mathrm{d}A$. That is, the double integral over some region D.

12.2: Iterated Integrals

- How do we actually compute $\iint_D f(x,y)\,dA$?
- If we think of $\int \int f(x,y)\,dx\,dy$ we can interpret this as $\int \left[\int f(x,y)\,dx\right]\,dy$
- In other words: we compute inside integral, and then outside integral.
- Actual computational technique: like partial derivatives, when computing $\int dx$, treat y as a constant and vice versa
- Fubini's theorem says: if f is continuous and D is a rectangle, we can switch the order of dx, dy. We can't do this in general!

12.3: General Regions

- previous section: D = rectangle, but really D can be anything.
- type 1 region: $a \le x \le b$, that is x is just bound between two numbers (left, right boundaries) and then the top/bottom boundaries can be more interesting: $g_1(x) \le y \le g_2(x)$.
- if type 1, always integrate: $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$
- type 2: similar idea. $c \le y \le d$, that is, top and bottom chop off at just a number, but then left/right boundaries are interesting: $g_1(y) \le x \le g_2(y)$.
- always integrate type 2 as: $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$.

12.4: Double Integrals in Polar Coordinates

- polar relations $r^2 = x^2 + y^2$ where r is the radius from the origin, and $x = r\cos\theta$, $y = r\sin\theta$, where θ is measured counter clockwise from the x axis

- sometimes it's easier to integrate in polar (when your region is a circle or similar)
- big idea: if we take Δr and $\Delta\theta$ to be constants, the grid we get is no longer uniform!
- to correct for the non-uniform grid, we get the following transformation: $\iint f(x,y)\,dA = \iint f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$
- note: we pick up an extra factor of r whenever you transform integral into polar
- the radius can depend on the angle in interesting way: $\int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$.

12.5: Applications

- applications of 2D integrals basically the same as 1D. We want to add up SOMETHING over a region D now instead of a range of numbers (interval)
- very common "type" of integral: weighted average. something like $\int xf(x)\,dx / \int f(x)\,dx$. This basically computes the "average" value of x with weights $f(x)$. continuous version of a weighted sum.
- mass of an object: $m = \iint \rho(x,y)\,dA$ where ρ is the density.
- center of mass: $\bar{x} = \frac{1}{m}\iint x\rho(x,y)\,dA$ and $\bar{y} = \frac{1}{m}\iint y\rho(x,y)\,dA$.
- note that center of mass is just a weighted average!
- $f(x)$ can describe a probability density function. we need this if there are infinitely many possible outcomes. a single outcome has probability 0. Think: how many numbers can you choose between 0 and 1, for instance? What's the probability you choose exactly 0.5?
- probability of ANY event occurring = 1, so $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
- $Prob(a \le X \le b) = \int f(x)\,dx$ or in 2D: $Prob((X,Y) \in D) = \iint_D f(x,y)\,dA$
- $f(x,y)$ = joint probability density function. Basically the probability density of a value of both some event X occurring and some event Y.
- Mean (average) of a random variable: weighted average! $\mu = \int xf(x)\,dx$ If you have a joint density: $\mu_x = \iint_D xf(x,y)\,dA$. Basically: average value of x.
- Know how to do the movie theater waiting time example done in class. I really like this problem!

12.6: Surface Area

- Recall we can describe a surface by $\vec{r} = \langle x(u,v), y(u,v), z(u,v)\rangle$ where, as we iterate over the values of u, v $\vec{r}$ points to the surface of our object.
- u, v do not need to mean anything! they just keep track of tracing out the whole surface.
- Also recall: $\vec{r}_u = \partial\vec{r}/\partial u$ points in SOME tangent direction on the surface. So does $\vec{r}_v = \partial\vec{r}/\partial v$
- How do we compute surface area? Simplest idea for integrals: chop stuff up into uniform grid! here we have $\Delta u, \Delta v$, slicing the u, v plane in uniform grid: this places SOME slicing on our surface of an object
- thus we need to just compute the surface area of one slice S_{ij} which we can approximate to be $\|\vec{r}_u \times \vec{r}_v\|$, that is, area of parallelogram made by those two vectors
- result: $SA = \iint \|\vec{r}_u \times \vec{r}_v\|\,du\,dv$.
- special case: if "top" of the object is described by $f(x,y)$, we can take the parametrization $x = x, y = y, z = f(x,y)$, this results in the formula: $SA = \iint \sqrt{1 + f_x^2 + f_y^2}\,dA$.

12.7: Triple Integrals

- We can chop 3D space up into boxes of dimension $\Delta x, \Delta y, \Delta z$ with volume $\Delta V = \Delta x \Delta y \Delta z$
- We can then add up some property of each of these boxes to yield the triple integral: $\iiint f(x,y,z)\,dV$.
- Fubini's theorem still applies.
- $\iiint 1\,dV$ = volume of the box
- Ideas of type 1, 2 integrals in 2D still apply. One possibility "type 1 in 3D": weird top, so $x, y \in D$ but $u_1(x,y) \le z \le u_2(x,y)$. Integrate z first here.

- In the above, D can be type 1 or type 2 2D region, telling you which order to integrate dx, dy.
- Type 2 in 3D: weird x side: integrate x first. Type 3: weird y side. Integrate it first.
- Mass of an object with density $\rho(x,y,z) \to m = \iiint \rho\, dV$. $\bar{x} = \frac{1}{m}\iiint x\rho\, dV$ and so on.

12.8: Triple Integrals in Spherical, Cylindrical

- Cylindrical: $x = r\cos\theta, y = r\sin\theta, z = z$, where r, θ satisfy polar coordinates. basically polar + z coordinate
- Integrals in cylindrical: $\iiint f(x,y,z)\, dV = \iiint f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$
- note the addition of the r like in polar!
- spherical relations: $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$, $\rho^2 = x^2+y^2+z^2$, where ϕ is measured from the z axis (downward) and θ is measured from the x axis
- integrals in spherical: $\iiint f(x,y,z)\, dV = \iiint f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\, \sin\phi,\ d\rho\, d\theta\, d\phi$
- note the addition of $\rho^2\sin\phi$, same idea! non-uniform grid.
- know basic objects in each of these coordinate systems: $z = r$ is a cone in cylindrical, $\rho = a$ is a sphere, $\theta = a$ is a plane, $\phi = b$ is a cone

Chapter 13: Vector Calculus

13.1: Vector Fields

- think of fluid flow or wind: at every point (x,y) we have a vector $\mathbf{F}(x,y)$ associated with it with a magnitude and direction.
- another example: force field (say, due to gravity)
- know how to draw/recognize these
- important idea: if f is a scalar function (gives a number out), ∇f is a vector field
- HOWEVER: if $\mathbf{F}$ is a vector field, its not necessarily true that some f exists such that $\mathbf{F} = \nabla f$
- in the special case that we can find f such that $\mathbf{F} = \nabla f$, then $\mathbf{F}$ is said to be **conservative** and f is called the **potential function**

13.2: Line Integrals

- main idea: add stuff up along a line. really, 1D integral but what is base? *arc length*
- think: f describes height of a fence along some base curve C and you want to know total area of fence.
- $\int_C f(x,y)\, \mathrm{d}s$ = adding up scalar f along a curve C. effectively weighing f by how far along the curve you've moved
- huge idea: first step of computing line integrals is always parameterizing your curve C!
- if $\mathbf{r}(t)$ parameterizes C then $\mathrm{d}s = \|\mathbf{r}'\|\, \mathrm{d}t$. example: if $x(t), y(t)$ describes your curve $\mathrm{d}s = \sqrt{(x')^2+(y')^2}\, \mathrm{d}t$
- two new types of **scalar** line integrals: $\int_C \mathrm{d}x$ or $\int_C \mathrm{d}y$, basically the same as previous except you only add up when you move in x or y
- in either case: $\mathrm{d}x = x'\mathrm{d}t$ or $\mathrm{d}y = y'\mathrm{d}t$.
- previous line integrals: adding up **scalar** f. What if we want to add up **vector F**?
- think: non-constant vector work along a trajectory. we only want to add up forces applied in the direction of motion: that is, we'll want to dot product with the tangent vector
- $W = \int_C \mathbf{F}\cdot\mathbf{T}\, \mathrm{d}s$ but since $\mathbf{T} = \mathbf{r}'/\|\mathbf{r}'\|$ and $\mathrm{d}s = \|\mathbf{r}'\|\mathrm{d}t$, this reduces to: $\int \mathbf{F}\cdot\mathbf{r}'\, \mathrm{d}t$
- we abbreviate this shorthand: $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int \mathbf{F}(\mathbf{r})\cdot\mathbf{r}'\, \mathrm{d}t$.
- note this is fundamentally different than first line integral! we're adding up a vector, not a scalar
- orientation of C matters: $\int_{-C} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ where $-C$ implies you flip the direction of the curve

- if $\mathbf{F} = \langle P, Q, R \rangle$ then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$. That is, we can reduce **vector** line integral to component-wise **scalar** line integrals

13.3: Fundamental Theorem of Line Integrals

- regular fundamental theorem: $\int_a^b F'(x)\,dx = F(b) - F(a)$. basically: integral of derivative is equal to the difference of the boundary values. does such an analog exist for line integrals?
- Fundamental Theorem of Line Integrals: $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. exact same idea! only evaluate endpoints
- In other words, if $\mathbf{F} = \nabla f$ = conservative vector field, line integral only depends on end points. in general, this is **not** true! path matters.
- many equivalent statements to $\mathbf{F}$ being conservative. first: line integrals are independent of path in a simply connected domain (no holes)
- also equivalent to $\mathbf{F}$ being conservative: any closed loop line integral is 0 (again, no holes)
- if $\mathbf{F} = \langle P, Q \rangle$ by Clairaut's theorem, if it is conservative then $P_y = Q_x$. This is an easy check! opposite is true so long as your region has no holes.
- if $\mathbf{F}$ is conservative, we derived the conservation of energy, hence the name

13.4: Green's Theorem

- positive orientation of a curve = counter-clockwise. negative orientation = clockwise
- if we have some region D bounded by a positively oriented curve C and $\mathbf{F} = \langle P, Q \rangle$ then Green's Theorem says: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_D Q_x - P_y\,dA$.
- that is: we transform an integral around the boundary to an integral of some derivatives over the region. basically the fundamental theorem of calculus for 2D integrals.
- this is a useful, practical tool! if line integral around boundary is too hard, try area or vice versa.
- cute trick: compute the area of an object by measuring its perimeter. this is what planimters do.
- although we stated for no holes and simply connected domains: you can do cute tricks to split into statements where Green's theorem applies.
- example: split a donut (with a hole) into two half donuts where Green's theorem applies

13.5: Curl and Divergence

- for this section, think of $\mathbf{F}$ as a velocity field for a fluid.
- macro circulation: how a ball would move in the fluid. micro circulation: how the ball would ROTATE in the fluid
- curl measures how the fluid is able to do torque on the ball, forcing it to rotate
- curl is a **vector-valued** quantity. The direction of the vector is the axis of rotation of the ball and the length of the vector is how quickly the ball spins
- imagine a ball of fluid: would $\mathbf{F}$ cause your fluid to expand or contract? this is what divergence measures.
- if divergence is positive: fluid expands. divergence negative: contracts. divergence zero: neither.
- mathematically: $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$. not quite a vector but we can think of it as one, then:
- $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$ and $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$, where you take the dot and cross product normally, applying the derivatives
- conservative vector fields: closed loop line integral is zero. thus, micro circulation (swirly-ness) is zero, and therefore: $\operatorname{curl} \nabla f = 0$.
- thus, we have another equivalent statement for conservative vector fields: they are curl-free
- another useful fact: $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$. think about symbolically: $\nabla \cdot (\nabla \times \mathbf{F})$. cross product gives you thing orthogonal to ∇ so dot product must be 0. this is a good way to check if $\mathbf{F}$ is the curl of another field

- equipped with this, we can restate Green's theorem in two new ways: $\oint_C \mathbf{F}\,\mathrm{d}\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,\mathrm{d}A$, that is, the weird derivatives are actually just the curl in disguise
- different flavor: instead of measuring $\mathbf{F}$ against the tangent of C, we can measure it against the normal, $\mathbf{n}$ and get: $\oint_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s = \iint_D \nabla \cdot \mathbf{F}\,\mathrm{d}A$
- Stokes' and Divergence theorem generalize each of these statements

13.6: Surface Integrals

- Line integral: for a 2D blob, we're adding stuff up on the boundary. What is the analog for 3D blobs? surface integral! boundary = surface.
- for a scalar f and surface parameterized by $\mathbf{r}(u,v)$: $\iint_S f\,\mathrm{d}S = \iint_D f(\mathbf{r})\|\mathbf{r}_u \times \mathbf{r}_v\|\,\mathrm{d}A$
- notice how similar this is to $\int_C f\,\mathrm{d}s = \int_C f(\mathbf{r})\,\|\mathbf{r}'\|\mathrm{d}t$ for line integrals
- again, we want to generalize to adding up a vector $\mathbf{F}$. motivation: think of trying to understand how much fluid is flowing out of a surface? this is the idea of **flux**.
- if fluid is flowing out of the surface: we need to compare the direction with respect to directly out of the surface, or the normal vector $\mathbf{n}$.
- result: for vector $\mathbf{F}$ surface integral: $\iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S$ which is abbreviated $\iint_S \mathbf{F}\,\mathrm{d}\mathbf{S}$
- one technical note: we do care if the surface is "orientable", that is, we can decide which direction is "out". we'll call positive orientation when $\mathbf{r}$ and $\mathbf{n}$ go the same direction (out)
- how to actually compute? parameterize!
- super useful fact: $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v / \|\mathbf{r}_u \times \mathbf{r}_v\|$, using this, we get the computational formula: $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,\mathrm{d}A$. use this to actually compute!
- if $z = g(x,y)$, a special case of our surface, the formula simplifies (usefully) to: $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D (-Pg_x - Qg_y + R)\,\mathrm{d}A$
- Gauss's law: net charge enclosed by a surface is $Q = \varepsilon_0 \iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$

13.7: Stokes' Theorem

- Remember the statement of Green's theorem: $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$. Can we generalize this to higher dimension surfaces?
- Stokes' theorem: $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$.
- Remember, $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\mathrm{d}s$ and $\iint \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S$, so this effectively says: line integral around the tangential component of our boundary is equal to the flux of the curl out of our surface
- Green's theorem is a special case where $\mathbf{n} = \mathbf{k}$
- Stokes' theorem works for non-closed surfaces. Think sheets and similar.
- practical use: hard line integrals → easy surface integrals or vice versa

13.9: Divergence Theorem

- Recall the statement of Green's theorem that said: $\int_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s = \iint_D \nabla \cdot \mathbf{F}\,\mathrm{d}A$.
- We know that line integrals can be generalized to surface integrals and area integrals can be generalized to volume integrals, so does this still work? Yes!
- Divergence Theorem: $\iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iiint_E \nabla \cdot \mathbf{F}\,\mathrm{d}A$
- only works for simple solid regions (no holes) and positive orientation (outward)
- practical use: hard flux (surface) integrals → easier volume integrals (or vice versa)

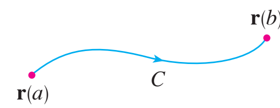
Chapter 13 summary:

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

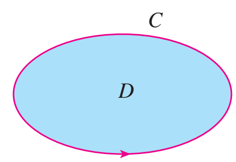
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



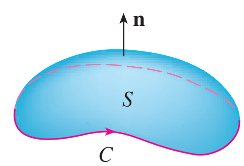
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$



Stokes' Theorem

$$\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \text{div}\, \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$