Math 1321 – Midterm 2: April 4, 2016

This outline is not meant to be an end-all exhaustive study replacement, but rather a structured baseline for your studying. That is, this is a study *companion*.

The topics covered on the exam are roughly described below. I recommend you identify which topics are your own weaknesses and practice those without your notes until you are comfortable with the topic. This is **not** a cumulative exam. That is, nothing from exam 1 will be on this exam.

I'll be around as much as possible during the week of the exam and will happily answer any questions. I'm particularly likely to be in my office if you see nothing on my schedule, found here: `http://www.math.utah.edu/~miles/#contact`. Joaquin will also review in lab the Thursday before the exam.

Anything covered in lab, quiz, homework (even ungraded), and review practice is fair game, and I'll mostly draw from these sources when constructing the exam.

Chapter 10: Vector Functions

10.4: Velocity and acceleration

- Often useful to think of $\mathbf{v}(t)$, the velocity of an object as a vector function
- Now we have machinery to compute position $\mathbf{r}(t) = \mathbf{v}'(t)$ and acceleration $\mathbf{a}(t) = \int \mathbf{v}(t)dt$.
- Other relationships: $\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(u)\,du$ and $\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)du$.
- $\mathbf{F} = m\mathbf{a}$ still holds. Now we get a force vector.
- Projectile motion: acceleration due to gravity: $-mg\mathbf{j}$. We can use vector relationships to find that $\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0$. Be able to interpret extensions/variations of this.
- Useful to realize that acceleration is $\mathbf{a} = |\mathbf{v}'|\mathbf{T} + \kappa|\mathbf{v}|^2\mathbf{N}$, that is acceleration has a tangential and normal component (think slamming on your breaks vs sharply turning).

10.5: Parametric surfaces

- Most general of our surfaces. We now have things of the form: $\mathbf{r}(u,v) = x(u,v)\boldsymbol{i} + y(u,v)\boldsymbol{j} + z(u,v)\boldsymbol{k}$, which defines the surface.
- For instance, $x = 2\cos u, y = v, z = 2\sin u$ defines a cylinder.
- Plane equation: $\mathbf{r}(u,v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$, where $\mathbf{a}, \mathbf{b}$ are some vectors in the plane.
- Sphere equation: $x = a\sin\phi\cos\theta, y = a\sin\phi\sin\theta, z = a\cos\phi$, basically spherical.
- Surface of revolution: rotate $f(x)$ around x axis yields: $x = x,\ y = f(x)\cos\theta,\ z = f(x)\sin\theta$.
- Again, be able to recognize these and match equations/graphs.
- Some examples we've seen: torus, intersection of a plane/cylinder.

Chapter 11: Multivariate Functions

11.1: Functions of Several Variables

- Now looking at things of the form $z = f(x,y)$ or $F(x,y,z) = k$.
- Typically think of z as height.
- For $z = f(x,y)$, a level curve is when we set $z = k$ and draw in the x, y plane the corresponding curve. Think of slicing at a height k.
- Be able to identify functions by their level curves (or vice versa)

11.2: Limits, Continuity

- New idea: 2D limits. For limit to exist, must be same in all directions

- Consequence: easy to show limits do not exist, just need 2 different paths
- Common paths: $y = 0, x = 0, y = mx, y = x^2$. Alway check these first.
- Hard to show limits exist. Two examples: we used inequality $x^2 \leq x^2 + y^2$ and we also converted to polar. Know these examples.
- Continuous definition: the same as 1D. Limit = evaluation.

11.3: Partial Derivatives

- Partial derivative: for $z = f(x, y)$, $\partial_x f(x, y)$ describes the rate of change of $f(x, y)$ as JUST x is changed. We can think of y as fixed.
- Computationally: to compute ∂_x, we treat y as a constant and take a normal derivative of x.
- Geometrically: $\partial_x f(x, y)$ gives us the slope of the tangent line in the x direction at a point.
- Chain rule, implicit differentiation, product rule, etc. all still apply. Chain rule is investigated further in 11.5.
- Clairaut's theorem: for f_{xy} and f_{yx} both continuous, then $f_{xy} = f_{yx}$. That is, for most "nice" functions, we can take partial derivatives in any order we want. Also means that $f_{xyy} = f_{yxy} = f_{yyx}$ for instance.
- We can now define a partial differential equation. Example: Laplace's equation: $\partial_{xx} u + \partial_{yy} u = 0$. Can verify solutions by plugging in just like for regular differential equations.

11.4: Tangent Planes, Linear Approximations, Differentials

- Partial derivatives f_x, f_y basically define two vectors that generate the tangent plane, giving us the formula: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
- Can think of the tangent plane as a linear approximation around (x_0, y_0, z_0).
- Technical note: tangent plane only exists if f_x, f_y are continuous.
- Linearization $L(x, y)$ defined the same way: $L(x, y) = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
- Theorem: if f_x, f_y exist and are continuous near (a, b), then $f(x, y)$ is differentiable near (a, b).
- Differentials: roughly same idea as linear approximation, just different way of interpreting
- Rough idea of differentials: absolute change in z as you take $x \to x + \mathrm{d}x$ vs. the change you get by approximating via the tangent line (or plane). Ultimately, $\mathrm{d}z = f_x \mathrm{d}x + f_y \mathrm{d}y$. Good approximation.
- If we have a parametric surface $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, then we just compute $\mathbf{r}_u$ and $\mathbf{r}_v$, the partial derivatives, which give us two vectors that generate the normal vector for the tangent plane. That is, $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$. If this cross product is 0, the tangent plane does not exist.

11.5: Chain Rule

- Case 1: $z = f(x, y), x = x(t), y = y(t)$, then $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$.
- Case 2: $z = f(x, y), x = x(s, t), y = y(s, t)$, then $\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$. Similar can be defined for s.
- Useful tool: draw tree. Top of tree = thing you're taking the derivative of, then draw lines to everything its a function of, and then draw lines to everything those are a function of. To get full chain rule: start at top and move down to variable you're taking the derivative with respect to.
- For example: $\frac{\partial z}{\partial t}$, you would start at z and end at t. There are 2 paths so we get 2 terms. Each line corresponds to multiplication of some derivative.

- Second partials often involve using chain rule more than once. Know how to do this.
- Implicit differentiation gives us that if we have $F(x, y) = 0$, then $\frac{dy}{dz} = -\frac{F_x}{F_y}$.
- Similarly, $F(x, y, z) = 0$ implies $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ or $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

11.6: Directional Derivatives, Gradient

- ∂_x, ∂_y are rates of change in the direction of x, y, but we can consider any arbitrary direction $\mathbf{u}$, where $\mathbf{u} = \langle a, b \rangle$ is a unit vector.
- Gives us: directional derivative $D_u f(x, y)$. Know limit definition, but easier definition: $D_{\boldsymbol{u}} f = f_x a + f_y b$.
- Define the gradient $\nabla f = \langle f_x, f_y \rangle$. That is, the gradient is a vector function that contains the partial derivatives at a point.
- Convenient form: $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u}$.
- Big idea #1: at a point, the direction $\boldsymbol{u}$ that maximizes the directional derivative $D_{\boldsymbol{u}} f$ is in the direction of ∇f (since it is a vector) and the maximum value is $|\nabla f|$. Know why.
- Big idea #2: consider some surface $F(x, y, z) = k$, and define $\delta x = \langle x - x_0, y - y_0, z - z_0 \rangle$. Then the tangent plane at a point is $\nabla F \cdot \delta x = 0$.
- In the case we have that $z = f(x, y)$, this reduce to our same tangent plane equation, but useful because more general. Therefore: generally, the gradient points outward (normal) to any surface.
- Thus, at any level curve, ∇f gives us the direction of fastest rate of change, and is also perpendicular to any level curve.

11.7: Maxima and Minima

- Local minima/maxima: largest or smallest value near some point. Global maxima/minima: largest/smallest in domain.
- Same idea as calc 1: if (a, b) is a local minima or maxima, then $f_x(a, b) = 0$ AND $f_y(a, b) = 0$.
- We call points where $f_x = 0$ and $f_y = 0$ critical points. Not necessarily min or max.
- Again, similarity to calc 1: need to check 2nd derivative to tell if min/max.
- Define $D = f_{xx} f_{yy} - f_{xy}^2$. (Think of as a determinant of a matrix)
- Test says that if: (1) $D > 0, f_{xx} > 0$ then local min, (2) $D > 0, f_{xx} < 0$, local max, (3) $D < 0$, neither (called a saddle point).
- Note: if $D = 0$, no conclusion can be made.
- Thus, general procedure: find all critical points and then test D for each.
- Distance from (x, y, z) to a point (x_0, y_0, z_0): $d = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0^2)}$. Often easier (and valid) to minimize d^2.
- Closed set: includes boundary. Open set: does not.
- Extreme (absolute/global min or max): must occur for any closed bounded set.
- Recipe for finding global/min or max: find critical numbers, check value of f at critical numbers, check value of f on boundaries. Key idea: max/min MUST occur on boundary or at a critical number.

11.8: Lagrange Multipliers

- Recipe for constrained optimization. That is, we want to maximize $f(x, y)$ constrained to $g(x, y) = k$.
- Equivalently: we find the largest c such that the level curve $f(x, y) = c$ intersects the level curve $g(x, y) = k$. This occurs when the level curves have a common tangent line!

- Common tangent line = same normal direction = parallel gradient vectors!
- Key idea: $\nabla f = \lambda \nabla g$ for some λ. Thus, we just need to find λ. This is called a Lagrange multiplier.
- General recipe: find all x, y, z, λ such that this is true, compare f on values of x, y, z.
- Component equations: $f_x = \lambda g_x, f_y = \lambda g_y, f_z = \lambda g_z, g = k$.
- If we have two constraints: that is $g = k, h = c$, we are looking for $f(x, y, z)$ that lies in the intersection of these two level surfaces. By gradient properties, ∇f is orthogonal to this point, and so are $\nabla g, \nabla h$. Thus, we just need to find λ, μ such that: $\nabla f = \lambda \nabla g + \mu \nabla h$. Same idea.

Chapter 12: Multiple integrals

12.1: Double Integrals over Rectangles

- 1D integral idea: add up rectangles of with Δx and height $f(x_i)$ to get an area
- new idea: add up rectangular prisms of base area ΔA and height $f(x_i, y_i)$.
- base area $\Delta A = \Delta x \Delta y$, we can think about chopping our surface in "uniform grid" where these are constant
- limit of infinitely small (and many) grids = double integral
- we write this as $\iint_D f(x, y)\,\mathrm{d}A$. That is, the double integral over some region D.

12.2: Iterated Integrals

- How do we actually compute $\iint_D f(x, y) dA$?
- If we think of $\int \int f(x, y) dx\, dy$ we can interpret this as $\int \left[\int f(x, y)\, dx\right]\, dy$
- In other words: we compute inside integral, and then outside integral.
- Actual computational technique: like partial derivatives, when computing $\int dx$, treat y as a constant and vice versa
- Fubini's theorem says: if f is continuous and D is a rectangle, we can switch the order of dx, dy. We can't do this in general!

12.3: General Regions

- previous section: D = rectangle, but really D can be anything.
- type 1 region: $a \leq x \leq b$, that is x is just bound between two numbers (left, right boundaries) and then the top/bottom boundaries can be more interesting: $g_1(x) \leq y \leq g_2(x)$.
- if type 1, always integrate: $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy\, dx$
- type 2: similar idea. $c \leq y \leq d$, that is, top and bottom chop off at just a number, but then left/right boundaries are interesting: $g_1(y) \leq x \leq g_2(y)$.
- always integrate type 2 as: $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx\, dy$.

12.4: Double Integrals in Polar Coordinates

- polar relations $r^2 = x^2 + y^2$ where r is the radius from the origin, and $x = r\cos\theta$, $y = r\sin\theta$, where θ is measured counter clockwise from the x axis
- sometimes it's easier to integrate in polar (when your region is a circle or similar)
- big idea: if we take Δr and $\Delta\theta$ to be constants, the grid we get is no longer uniform!
- to correct for the non-uniform grid, we get the following transformation: $\iint f(x, y)\, dA = \iint f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$.
- note: we pick up an extra factor of r whenever you transform integral into polar
- the radius can depend on the angle in interesting way: $\int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$.

12.5: Applications

- applications of 2D integrals basically the same as 1D. We want to add up SOMETHING over a region D now instead of a range of numbers (interval)
- very common "type" of integral: weighted average. something like $\int xf(x)\,dx / \int f(x)\,dx$. This basically computes the "average" value of x with weights $f(x)$. continuous version of a weighted sum.
- mass of an object: $m = \iint \rho(x,y)\,dA$ where ρ is the density.
- center of mass: $\bar{x} = \frac{1}{m}\iint x\rho(x,y)\,dA$ and $\bar{y} = \frac{1}{m}\iint y\rho(x,y)\,dA$.
- note that center of mass is just a weighted average!
- $f(x)$ can describe a probability density function. we need this if there are infinitely many possible outcomes. a single outcome has probability 0. Think: how many numbers can you choose between 0 and 1, for instance? What's the probability you choose exactly 0.5?
- probability of ANY event occurring = 1, so $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
- $Prob(a \le X \le b) = \int f(x)\,dx$ or in 2D: $Prob((X,Y) \in D) = \iint_D f(x,y)\,dA$
- $f(x,y)$ = joint probability density function. Basically the probability density of a value of both some event X occurring and some event Y.
- Mean (average) of a random variable: weighted average! $\mu = \int xf(x)\,dx$ If you have a joint density: $\mu_x = \iint_D xf(x,y)\,dA$. Basically: average value of x.
- Know how to do the movie theater waiting time example done in class. I really like this problem!

12.6: Surface Area

- Recall we can describe a surface by $\vec{r} = \langle x(u,v), y(u,v), z(u,v)\rangle$ where, as we iterate over the values of u,v $\vec{r}$ points to the surface of our object.
- u,v do not need to mean anything! they just keep track of tracing out the whole surface.
- Also recall: $\vec{r}_u = \partial\vec{r}/\partial u$ points in SOME tangent direction on the surface. So does $\vec{r}_v = \partial\vec{r}/\partial v$
- How do we compute surface area? Simplest idea for integrals: chop stuff up into uniform grid! here we have $\Delta u, \Delta v$, slicing the u,v plane in uniform grid: this places SOME slicing on our surface of an object
- thus we need to just compute the surface area of one slice S_{ij} which we can approximate to be $\|\vec{r}_u \times \vec{r}_v\|$, that is, area of parallelogram made by those two vectors
- result: $SA = \iint \|\vec{r}_u \times \vec{r}_v\|\,du\,dv$.
- special case: if "top" of the object is described by $f(x,y)$, we can take the parametrization $x = x, y = y, z = f(x,y)$, this results in the formula: $SA = \iint \sqrt{1 + f_x^2 + f_y^2}\,dA$.

12.7: Triple Integrals

- We can chop 3D space up into boxes of dimension $\Delta x, \Delta y, \Delta z$ with volume $\Delta V = \Delta x \Delta y \Delta z$
- We can then add up some property of each of these boxes to yield the triple integral: $\iiint f(x,y,z)\,dV$.
- Fubini's theorem still applies.
- $\iiint 1\,dV$ = volume of the box
- Ideas of type 1, 2 integrals in 2D still apply. One possibility "type 1 in 3D": weird top, so $x,y \in D$ but $u_1(x,y) \le z \le u_2(x,y)$. Integrate z first here.
- In the above, D can be type 1 or type 2 2D region, telling you which order to integrate dx, dy.
- Type 2 in 3D: weird x side: integrate x first. Type 3: weird y side. Integrate it first.

- Mass of an object with density $\rho(x, y, z) \to m = \iiint \rho \, dV$. $\bar{x} = \frac{1}{m} \iiint x\rho \, dV$ and so on.

12.8: Triple Integrals in Spherical, Cylindrical

- Cylindrical: $x = r\cos\theta, y = r\sin\theta, z = z$, where r, θ satisfy polar coordinates. basically polar + z coordinate
- Integrals in cylindrical: $\iiint f(x, y, z) \, dV = \iiint f(r\cos\theta, r\sin\theta, z) \, r \, dz \, dr \, d\theta$
- note the addition of the r like in polar!
- spherical relations: $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$, $\rho^2 = x^2 + y^2 + z^2$, where ϕ is measured from the z axis (downward) and θ is measured from the x axis
- integrals in spherical: $\iiint f(x, y, z) \, dV = \iiint f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \, \rho^2 \sin\phi, \, d\rho \, d\theta \, d\phi$
- note the addition of $\rho^2 \sin\phi$, same idea! non-uniform grid.
- know basic objects in each of these coordinate systems: $z = r$ is a cone in cylindrical, $\rho = a$ is a sphere, $\theta = a$ is a plane, $\phi = b$ is a cone