Midterm Exam 1
Math 1321 - Accelerated Engineering Calc II
February 19, 2016

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules or caught cheating will be asked to leave immediately. Point values are in the square to the left of the question. **If there are any other issues, please ask the instructor.**

By signing below, you are acknowledging that you have read and agree to the above paragraph, as well as agree to abide University Honor Code:

Name: ______________________________

Signature: ____________________________

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Question	Points	Score
1	10	
2	10	
3	15	
4	15	
5	20	
6	10	
7	20	
Total:	100	

10 1. **Choose one the following series** and prove whether it diverges or converges. In either case, make sure to state the requirements of the test you're using and why they are satisfied. Also, make sure to clearly indicate which series you are answering for.

(1) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

(2) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$.

Solution: **This question was taken directly from a quiz.**

(1) The integral test considers, in general, a series

$$\sum_{j=k}^{\infty} a_j$$

and says that we can relate this to the integral of $f(x)$, where $f(j) = a_j$ *only* if $f(x)$ satisfies three requirements:

(a) $f(x)$ is continuous.

(b) $f(x)$ is positive valued.

(c) $f(x)$ is decreasing.

Here, we see that our choice of $f(x)$ is $f(x) = \frac{1}{x(\ln x)^2}$. We must check the three conditions listed above.

(a) For $x \geq 2$, the whole thing is continuous (as $x = 0$ is the only problematic point).

(b) Since $(\ln x)^2$ is always positive and $x \geq 2$, the whole quantity is always positive.

(c) For this quiz, it's perfectly okay to just say the function is decreasing because the numerator stays 1 and the denominator grows, however we could show this a little more precisely by considering

$$f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3},$$

which we see, for $x \geq 1/e^2$ is always negative, meaning the function is decreasing.

Thus, $f(x)$ satisfies all of the requirements of the integral test meaning that if

$$\int_2^{\infty} f(x)\,\mathrm{d}x = \int_2^{\infty} \frac{1}{x(\ln x)^2}\,\mathrm{d}x$$

converges or diverges, so does $\sum_{j=2}^{\infty} 1/(n \ln^2 n)$. To see whether this integral converges or diverges, consider, as in the hint, $u = \ln x$, so $\mathrm{d}u = \mathrm{d}x/x$, meaning we have

$$\int_2^{\infty} f(x)\,\mathrm{d}x = \int_{\ln 2}^{\infty} \frac{1}{u^2}\,\mathrm{d}u = \left[-\frac{1}{u}\right]_{u=\ln 2}^{u=\infty} = \lim_{t\to\infty}\left\{-\frac{1}{t}\right\} + \frac{1}{\ln 2} = \frac{1}{\ln 2},$$

so this integral indeed converges and therefore, by the integral test, as does the series.

(2) The alternating series test considers series of the form

$$\sum_{n=1}^{\infty}(-1)^n b_n \qquad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n-1} b_n,$$

where the terms b_n must satisfy the following conditions:

(a) b_n is decreasing, that is, $b_n \leq b_{n+1}$

(b) b_n decays to 0 or $\lim_{n\to\infty} b_n = 0$.

Thus, in our case, $b_n = \frac{n}{n^2+1}$ and we must check these two properties.

(a) It's pretty clear to see that n^2 grows much faster than n so $\lim_{n\to\infty} b_n = 0$. We could do this more precisely by taking

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim \frac{\frac{1}{n}}{1+\frac{1}{n}} = 0.$$

(b) Again, the above basically already shows its decreasing, but to see this more rigorously, we can consider $f(x) = x/(x^2+1)$ and

$$f'(x) = \frac{1-x^2}{(x^2+1)^2},$$

which means that so long as $x > 1$ (which it is), $f'(x) < 0$ meaning we have a decreasing function (and therefore terms of the series.)

Thus, the requirements of the alternating series test are satisfied, which means that this series **converges**. A very important nuance here: **the alternating series test cannot prove divergence, only convergence**, however the integral test can prove either.

10 2. Determine for which values of p the following series converges:

$$\sum_{n=1}^{\infty} \frac{n^p}{7+n^3}.$$

Hint: what other series does this look like for large n?

Solution: Notice, as $n \to \infty$, the 2 becomes insignificant and this looks like n^{p-3}. While you could use the limit comparison test, this is actually even simpler since you can compare directly:

$$\frac{n^p}{2+n^7} < \frac{n^p}{n^3} \text{ for large } n.$$

Thus, we can actually just use the direct comparison test, which states that if $b_n \leq a_n$ and $\sum a_n$ converges, then $\sum b_n$ converges. Here, our $\sum b_n$ is our original series and $a_n = n^{p-3}$. We can rewrite this in a more suggestive form:

$$\frac{n^p}{n^3} = \frac{1}{n^{3-p}}.$$

Notice, this is exactly a p series, meaning this converges when the exponent is > 1 or $3 - p > 1$, which implies that $p < 2$.

10 3. (a) Compute the Taylor series of

$$f(x) = \frac{1}{x} \qquad \text{around } a = 1.$$

5 (b) Compute the radius of convergence.

Solution: **This question was taken directly from a quiz.** The definition of a Taylor series around $x = a$ (the expansion point), is defined to be

$$\sum_{j=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

where $f^{(n)}$ denotes the nth derivative of the function $f(x)$. Thus, we need to compute the deriavtives of our function. Let's see a pattern:

$$f'(x) = -\frac{1}{x^2}, \qquad f''(x) = \frac{2}{x^3}, \qquad f'''(x) = -\frac{3 \cdot 2}{x^4}, \qquad f^{(4)}(x) = \frac{4 \cdot 3 \cdot 2 \cdot 1}{x^5},$$

and hopefully from this, the pattern is clear:

$$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}.$$

Using $a = 1$ for this problem, we have

$$f^{(n)}(1) = (-1)^n n!,$$

and using this in the defintion of the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{j=0}^{\infty} \frac{(-1)^n n!}{n!}(x-1)^n,$$

and noting that the factorials cancel, our final answer is then

$$\sum_{n=0}^{\infty} (-1)^n (x-1)^n = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots.$$

The solution we got to the previous problem is

$$\sum_{n=0}^{\infty} (-1)^n (x-1)^n = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots.$$

We now invoke the ratio test to determine for which x values this series converges. The ratio test says that if you have a series

$$\sum_{n=0}^{\infty} a_n \qquad \text{and} \qquad \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| := L$$

then the series converges for $L < 1$, (and as a reminder, diverges for $L > 1$ and if $L = 1$, we have no clue). Thus, in this case, $a_n = (-1)^n (x-1)^n$, so we have

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{(-1)^n (x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \right| = \lim_{n\to\infty} |x-1|,$$

thus our condition for convergence is

$$|x-1| < 1 \implies R = 1,$$

where R is the radius of convergence. Note that we don't know if this series converges for the end points, that is at $x = 0, x = 2$ but only the radius was asked for this problem. A separate test could be used for each of these cases if the interval of convergence was asked for.

______/15 pts

4. Consider the following two vectors:

$$\mathbf{u} = \langle 3, 2 \rangle, \qquad \mathbf{v} = \langle 2, 0 \rangle.$$

5 (a) Find an expression for the angle θ between $\mathbf{u}$ and $\mathbf{v}$ and draw graphical representation of these vectors on a standard (x, y) coordinate system.

Solution: **This is almost identical to a question from the quiz and homework.** We know that this is just an immediate application of the dot product. Specifically,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta,$$

where θ is the desired angle between the two vectors, meaning that

$$\theta = \cos^{-1} \left\{ \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \right\},$$

thus we just need to compute each of these. Our dot product can be obtained by taking the sum of the products of the components:

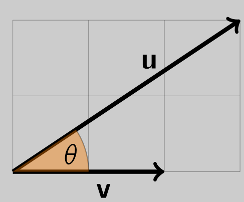
$$\mathbf{u} \cdot \mathbf{v} = \langle 3, 2 \rangle \cdot \langle 2, 0 \rangle = 3 \cdot 2 + 2 \cdot 0 = 6.$$

Also, the magnitudes can be computed from the distance formula:

$$\|\mathbf{u}\| = \sqrt{3^2 + 2^2} = \sqrt{13}, \qquad \|\mathbf{v}\| = \sqrt{2^2 + 0^2} = 2.$$

Thus, our θ is:

$$\theta = \cos^{-1} \left\{ \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \right\} = \cos^{-1} \left\{ \frac{6}{2\sqrt{13}} \right\}.$$



10 (b) Compute $\text{proj}_{\mathbf{v}} \mathbf{u}$ and draw the graphic representation of this operation and its result.

Solution: Recall that $\text{proj}_{\mathbf{v}} \mathbf{u}$ means the projection of the vector $\mathbf{u}$ onto the vector $\mathbf{v}$. That is, we want to compute the component of $\mathbf{u}$ that is in the direction of $\mathbf{v}$. This corresponds to the following picture:
The length of the vector we want, denoted $\text{comp}_{\mathbf{v}} \mathbf{u}$ is the following:

$$\text{comp}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|v\|} = \frac{6}{2} = 3.$$

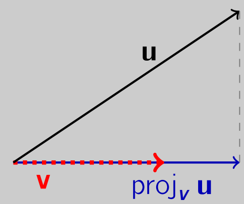
Thus, we want a vector of length $\text{comp}_{\mathbf{v}} \mathbf{u}$ in the direction of $\mathbf{v}$, so we just compute the unit vector in the direction of $\mathbf{v}$:

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2} \langle 2, 0 \rangle = \langle 1, 0 \rangle.$$

Therefore, our total projection is:

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \text{comp}_{\mathbf{v}} \mathbf{u} \frac{\mathbf{v}}{\|\mathbf{v}\|} = \langle 3, 0 \rangle.$$

Notice, this makes sense. Since $\mathbf{v}$ is just in the x direction, the projection of $\mathbf{u}$ onto $\mathbf{v}$ is just the x component of $\mathbf{u}$.



5. Consider the two following points:

$$P_1 = (1, 1, 1) \qquad P_2 = (-1, 1, -2).$$

10 (a) Find a vector representation of the line that passes through both P_1, P_2.

Solution: We first need to find the vector that points in the direction of the line through P_1 and P_2, which is just the displacement vector between these two points:

$$\overrightarrow{P_1P_2} = \langle P_2 - P_1 \rangle = \langle -2, 0, -3 \rangle.$$

Note, this is our $\mathbf{v}$ for our line. The equation of a line is then

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t,$$

where $\mathbf{r}_0 = \overrightarrow{OP_1} = \langle 1, 1, 1 \rangle$, so we have:

$$\mathbf{r} = \langle x, y, z \rangle = \langle 1, 1, 1 \rangle + \langle -2, 0, -3 \rangle t = \langle 1 - 2t, 1, 1 - 3t \rangle.$$

10 (b) Consider the following vector:

$$\mathbf{v} = \langle 1, 1, 1 \rangle.$$

Find the equation of the plane that contains the line described in part (a) that also is in the direction of the vector $\mathbf{v}$.

Solution: We want a plane that is in the direction of the line from the previous problem, which I'll call $\mathbf{v}_0 = \langle -2, -2, 1 \rangle$ and now also $\mathbf{v} = \langle 1, 1, 1 \rangle$. We need the normal vector to the plane, which is orthogonal to both of these and therefore can be obtained via the cross product:

$$\mathbf{n} = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -3 \\ 1 & 1 & 1 \end{vmatrix} = \langle 3, -1, -2 \rangle.$$

For a plane, we need a point and a normal vector, so we can just choose a point on the line, say $P_1 = (1, 2, 3)$, which gives us that our plane equation is:

$$\mathbf{n} \cdot \overrightarrow{PP_1} = 0,$$

which is the same as:

$$\langle 3, -1, -2 \rangle \cdot \langle x - 1, y - 1, z - 1 \rangle = 3(x - 1) - (y - 1) - 2(z - 1) = 0.$$

10 6. A missle was launched to hit a mobile target. Suppose that a missle is moving with the following trajectory:

$$\mathbf{r}_1 = \left\langle t, t^2, t^3 \right\rangle,$$

and the target trajectory is

$$\mathbf{r}_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle.$$

Does the missle hit the target? If so, when/where does the collision occur?

Solution: **This was modified slightly (to be easier) but otherwise directly from the lab.** For the two objects to collide, they must be at the same position at the same time. In other words, if we set their x components equal, we see that this only occurs when

$$t = 1 + 2t \implies t = -1.$$

However, are they at the same y location at this time? We check the y components of each of the trajectories:

$$t^2\big|_{t=-1} \stackrel{?}{=} 1 + 6t\big|_{t=-1}$$

Plugging in the value of t, we find

$$t^2\big|_{t=-1} = 1 \neq 1 + 6t\big|_{t=-1} = -5.$$

Thus, the two objects *do not* collide.

7. Consider the following vector-valued function:

$$\mathbf{r}(t) = \langle 2t, t^2, -\frac{1}{3}t^3 \rangle.$$

10 (a) Compute the unit tangent vector, $\mathbf{T}(t)$. *Hint:* you can factor $4 + 4t^2 + t^4 = (2 + t^2)^2$.

10 (b) Compute the curvature, $\kappa(t)$.

Solution: **This was modified slightly (to be easier) but otherwise directly from the homework.**

(a) We know the unit tangent vector $\mathbf{T}(t)$ is defined to be

$$\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

thus we must compute the vector derivative, $\mathbf{r}'(t)$, which in this case, is

$$\mathbf{r}'(t) = \langle 2, 2t, -t^2 \rangle.$$

Which has magnitude, using the algebra hint provided.

$$\|\mathbf{r}'(t)\| = \sqrt{2^2 + (2t)^2 + (-t^2)^2} = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2 + t^2)^2} = 2 + t^2.$$

Thus, our answer is

$$\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{2 + t^2}\langle 2, 2t, -t^2 \rangle.$$

(b) We can use two different formulas for the curvature, κ. Although we defined the curvature to be

$$\kappa := \left\| \frac{d\mathbf{T}}{ds} \right\|,$$

where s denotes the arc-length, this computation would be insane, so we have two other equivalent formulations:

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \qquad \text{or} \qquad \kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'\|^3}$$

Although I tend to think the second is more universally useful, we already have $\mathbf{T}$ from the previous part, so either is acceptable here. To do the first, we see

$$\mathbf{T}'(t) = \frac{1}{(2 + t^2)^2}\langle -4t, 4 - 2t^2, -4t \rangle,$$

and therefore, after some mildly gruesome algebra

$$\|\mathbf{T}'(t)\| = \frac{\sqrt{16t^2 + 16 - 16t^2 + 4t^4 + 16t^2}}{(2 + t^2)^2} = \frac{\sqrt{4(t^4 + 4t^2 + 4)}}{(2 + t^2)^2} = \frac{2}{2 + t^2}.$$

Thus, we find

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2}{(2 + t^2)^2}.$$

Computing κ using the other formula, we first compute

$$\mathbf{r}''(t) = \langle 0, 2, -2t \rangle \qquad \text{and therefore } \mathbf{r}'(t) \times \mathbf{r}''(t) = \langle -2t^2, 4t, 4 \rangle.$$

Computing the magnitude of this, we get

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{4t^4 + 16t^2 + 16} = \sqrt{4(t^2 + 4t^2 + 1)} = 2(t^2 + 2).$$

Using the formula for κ, we find

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'\|^3} = \frac{2(t^2 + 2)}{(t^2 + 2)^3} = \frac{2}{(2 + t^2)^2}.$$

If you've survived reading this whole solution, you see that the two techniques give us the same answer, which is assuring.

Bonus Questions

8. **(Current Events)** This week, Kanye West released an album titled *The Life of Pablo*. In an interview, Kanye reportedly claimed the album was inspired by two famous Pablos: one of which is Pablo Picasso. Who is the other Pablo, often referred to as *Don Pablo*, for his escapades as a Colombian drug lord resulting in being the wealthiest criminal in history?

Solution: Pablo Escobar. `https://en.wikipedia.org/wiki/Pablo_Escobar`

9. **(Joaquin's Homeland)** Chile is only one of two countries in all of South America that does not border Brazil. What is the other?

Solution: Ecuador.

10. **(Math)** Prove that if $\|\mathbf{r}(t)\| = c$, where c is a constant, then $\mathbf{r}(t) \perp \mathbf{r}'(t)$ for all t.

Solution: This proof was done in class and is also in the book. We start with supposing that $\|\mathbf{r}\| = c$ and square both sides, yielding $\|\mathbf{r}\|^2 = c^2$. We now make the observation that the left hand side is simply the dot product. That is, $\|\mathbf{r}\|^2 = \mathbf{r} \cdot \mathbf{r}$. Now, we take a derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{r}(t) \cdot \mathbf{r}(t)\} \qquad = \qquad \frac{\mathrm{d}}{\mathrm{d}t}\{c^2\} = 0.$$

Thus, we have $\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{r} \cdot \mathbf{r}\} = 0$, since the derivative of a constant is 0, but we can now use the product rule for dot products:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{r} \cdot \mathbf{r}\} = \mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

Since $2(\mathbf{r} \cdot \mathbf{r}') = 0$, we can divide both sides by the factor of 2 and conclude that $\mathbf{r} \cdot \mathbf{r}' = 0$ and therefore the two vectors are orthogonal.